HYDRAULICS

AND THE MECHANICS OF FLUIDS

A TEXTBOOK

COVERING THE SYLLABUSES OF THE
B.Sc. (ENG.), INST.C.E., AND I.MECH.E.
EXAMINATIONS IN THIS SUBJECT

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HYDRAULICS

AND THE MECHANICS OF FLUIDS

CHAPTER I

STATIC PRESSURE OF A FLUID

1. **Introduction.** The subject of Hydraulics deals with the laws governing the pressure and flow of fluids and the application of these laws to engineering practice. The term " fluid " includes liquids and gases; but as these two physical states have widely divergent properties it is inconvenient to deal with both under the same heading. For this reason the science of Hydraulics is mainly confined to the study of liquids, and as water is the liquid usually dealt with in practice, the subject of Hydraulics is chiefly the study of water and its applications.

The subject was formerly divided into two parts: one dealing with fluids at rest, called Hydrostatics, the other dealing with fluids in motion, known as Hydrodynamics; but the great development of hydraulic machinery in recent years has rendered these sub-divisions obsolete. Hydraulics now consists of the study of water pressure, buoyancy, the flow of water, and hydraulic machinery such as pumps, turbines, and accumulators.

Mechanics of fluids includes the whole of hydraulics, together with the application of its laws to the flow and resistance of gases.

2. **Properties of Fluids.** The term fluid is applied to all substances which offer no resistance to change of shape. Fluids may be divided into two classes: liquids and gases. The former offer great resistance to compression and are not greatly affected by change of temperature; the latter are easily compressed and are more susceptible to temperature changes. Liquids have a bulk elastic modulus when under compression, and will store up energy in the same manner as a solid. The value of the bulk elastic modulus of water under compression is 300,000 lb. per sq. in. As the contraction of volume of a liquid under compression is extremely small, it is usually ignored and the liquid is assumed to be incompressible.

A liquid will withstand a slight amount of tension owing to the molecular attraction between the particles, which will

cause an apparent shear resistance between two adjacent layers; this phenomenon is known as viscosity.

The coefficients of expansion of liquids are small and, as the science of hydraulics deals with liquids under atmospheric temperature only, the effect of temperature changes may be ignored; consequently, the density of a liquid may be assumed to be constant.

The weight of water is 62·4 lb. per cu. ft.

The ratio between the density of any liquid and the density of water is known as the specific gravity of that liquid.

No liquid can exist as a liquid at zero pressure; in fact all known liquids vaporize at various pressures above zero, depending on the temperature.

Water vaporizes at a pressure of ·34 lb. per sq. in. at 20° C., below this pressure it cannot exist as a liquid. There are also dissolved gases in water which are given off at low pressures and cause great inconvenience in hydraulic problems. For this reason care must be taken to prevent the pressure of water getting below 8 ft. of water absolute, at which pressure the dissolved gases are given off and vaporization is also liable to commence.

3. Pressure of a Fluid. The intensity of pressure of a fluid is the pressure per unit area. If the pressure is measured in pounds and the area in inches, the intensity of pressure will be in pounds per square inch.

Let a fluid be under a uniform pressure and let its total pressure on an area of a sq. in. be P lb. Let p be the intensity of pressure. Then, $p = \frac{P}{a}$ lb. per sq. in.

At any point in a fluid the intensity of pressure acts equally in all directions. If a fluid is contained in a vessel and is under a uniform intensity of pressure throughout, a slight increase in the intensity of pressure at one part will be immediately transmitted to all parts of the vessel.

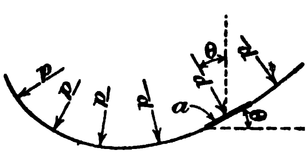


FIG. 1

The pressure of a fluid on a surface will always act normal to the surface.

Suppose a curved surface be under a uniform pressure p (Fig. 1); the direction of p at any point will be at right angles to the surface at that point. Consider a small element of the surface of area a, inclined to the horizontal at an angle θ.

Then, as p acts normal to a, its inclination to the vertical will be θ.

Total pressure on area $a = pa$.

Vertical component of total pressure on $a = pa \cos \theta$.

But, $a \cos \theta =$ horizontal projection of area a.

Therefore, vertical component of pressure on $a = p \times$ horizontal projection of area a.

From this it will be seen that if any shaped surface is under uniform pressure, the total pressure acting on it in any given direction is the intensity of pressure multiplied by the projected area normal to the given direction.

EXAMPLE.

A hemispherical dome (Fig. 2) of 2 ft. radius contains a fluid under a pressure of 120 lb. per sq. ft. Find the total force tending to lift the dome.

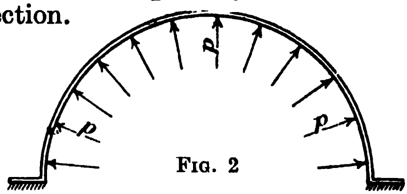


FIG. 2

$$\begin{aligned} \text{Total vertical force} &= p \times \text{horizontal projected area} \\ &= p \times \pi\,(\text{radius})^2 \\ &= 120 \times \pi\, 2^2 \\ &= 1509 \text{ lb.} \end{aligned}$$

4. **The Hydraulic Press.** The hydraulic press is a machine by which large weights may be lifted by the application of a much smaller force. A diagrammatic view of a hydraulic press is shown in Fig. 3. The weight W is lifted by the large ram R, which is raised by the pressure of the fluid. This pressure is produced by the force F acting on the plunger P.

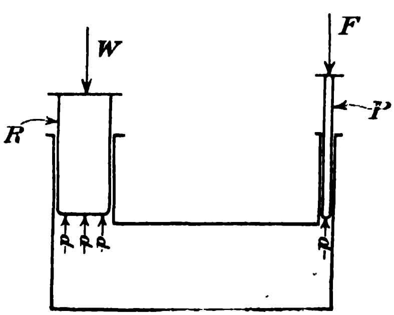


FIG. 3

Let A = area of ram

a = area of plunger

p = intensity of pressure of fluid.

As the intensity of pressure is the same throughout the chamber,

$$W = pA$$

and $$F = pa$$

Equating the values of p from these two equations,

$$\frac{W}{A} = \frac{F}{a}$$

Therefore, $W = \frac{FA}{a}\cdot$

Thus, the mechanical advantage obtained by means of this press is equal to the ratio of the areas of the ram and plunger.

This is the principle of the hydraulic lifting jack.

EXAMPLE.

A hydraulic press has a ram of 5 in. diameter and a plunger of $\frac{1}{2}$ in. diameter. What force would be required on the plunger to raise a weight of 1 ton on the ram? If the plunger had a stroke of 10 in., how many strokes would be necessary to lift the weight 3 ft.? If the time taken to lift the weight is 12 minutes, what horse-power would be required to drive the plunger? Neglect all losses, and assume the motion of the weight is continuous.

$$\text{Force on plunger} = \frac{Wa}{A}$$

$$= 2240 \times \left(\frac{\frac{1}{2}}{5}\right)^2 = 22{\cdot}4 \text{ lb.}$$

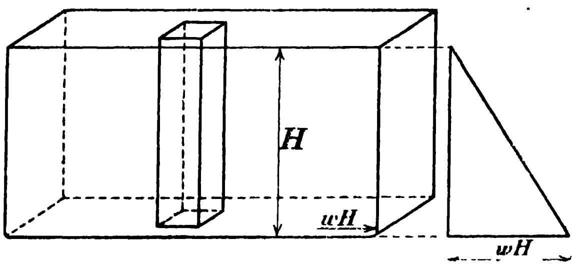


FIG. 4

As the work done by plunger equals work done by ram,

$$22{\cdot}4 \times \frac{10}{12} \times n = 2240 \times 3$$

where n = No. of strokes of plunger.

$$\text{Therefore,} \quad n = \frac{2240 \times 3}{22{\cdot}4} \times \frac{12}{10}$$

$$= 360$$

$$\text{Horse-power required} = \frac{22{\cdot}4 \times \frac{10}{12} \times 360}{12 \times 33{,}000}$$

$$= {\cdot}017$$

5. **Pressure Head of a Liquid.** A liquid is subjected to pressure due to its own weight; this pressure increases as the depth of the liquid increases. Consider a vessel containing

a liquid of a depth H ft. (Fig. 4). Let w be the weight in lb. of 1 cu. ft. of the liquid. Then the pressure at any point in the liquid will depend on the weight of liquid above that point.

Consider an area of 1 sq. ft. on the bottom of the vessel. The pressure on this square foot is equal to the weight of the column of liquid above it, which it is supporting. This column is in the shape of a square prism of a height H, standing on its end.

Then, total pressure on base of prism = weight of prism

$$= wH$$

As the base has an area of 1 sq. ft., this is the intensity of pressure p.

Therefore, $$p = wH \text{ lb. per sq. ft.} \quad . \quad . \quad . \quad (1)$$

If w were the weight of 1 cu. in. and H the height in inches, p would then be the intensity of pressure in lb. per sq. in.

As $p = wH$, the intensity of pressure in a liquid due to its depth will vary directly with the depth.

As the pressure at any point in a liquid depends on the height of the free surface above that point, it is sometimes convenient to express a liquid pressure by the height of the free surface which would cause that pressure.

Or, $$H = \frac{p}{w} \qquad \text{(From equation 1.)}$$

The height of the free surface above any point is known as the static head at that point. In this case the static head is denoted by H.

Hence, the intensity of pressure of a liquid may be expressed as a pressure in pounds per square inch, or as an equivalent static head in feet of water; and one form may be converted to the other by means of the equation

$$p = wH$$

In using this equation care should be taken that the units of one side are the same as those of the other.

When dealing with fresh water w may be taken as 62·4 lb. per cu. ft.

Referring to Fig. 4, the intensity of pressure at any point due to the weight of liquid above that point is in a vertical direction; but, as the pressure of a liquid acts equally in all directions, this pressure will cause an equal horizontal pressure

on the side of the vessel. The intensity of pressure on the sides of the vessel will, therefore, be equal to wH at the bottom and decrease uniformly to zero at the free surface, as shown by the pressure diagram to the right of the figure.

$$\left.\begin{array}{r}\text{Then, total pressure of liquid}\\ \text{on side of vessel}\end{array}\right\} = \text{average pressure} \times \text{area of side}$$

$$= \frac{wH}{2} \times \text{area of side.}$$

EXAMPLE 1.

Find the pressure in tons per sq. in. at the bottom of the sea at a point where the depth is 7 miles. The weight of 1 cu. ft. of sea water is 64 lb.

$$\begin{aligned} p &= wH \\ &= 64 \times 7 \times 5280 \text{ lb. per sq. ft.} \\ &= \frac{64 \times 7 \times 5280}{144 \times 2240} \text{ tons per sq. in.} \\ &= 7 \cdot 34 \text{ tons per sq. in.} \end{aligned}$$

EXAMPLE 2.

A rectangular tank 14 ft. long and 5 ft. wide contains water to a depth of 6 ft. Find the intensity of pressure on the base of the tank and the total pressure on the end.

$$\begin{aligned} \text{Pressure on base} &= wH \\ &= 62 \cdot 4 \times 6 \text{ lb. per sq. ft.} \\ &= \frac{62 \cdot 4 \times 6}{144} \text{ lb. per sq. in.} \\ &= 2 \cdot 6 \text{ lb. per sq. in.} \\ \text{Maximum pressure on end} &= wH \\ \text{Average pressure on end} &= \frac{wH}{2} \\ \text{Total pressure on end} &= \text{average pressure} \times \text{area} \\ &= \frac{62 \cdot 4 \times 6}{2} \times 5 \times 6 \\ &= 5620 \text{ lb.} \end{aligned}$$

6. **Pressure of Atmosphere.** The pressure of the atmosphere at the earth's surface is due to the weight of the column of air above. This cannot be calculated in the same way as a liquid because the air is compressible and, consequently, the density will vary. The pressure of the atmosphere is measured by the height of the column of liquid it will support. This will vary

slightly according to the amount of moisture in the atmosphere; the average value may be taken as 14·7 lb. per sq. in., which is equivalent to a static head of 34 ft. of water.

The pressure of water is measured by some type of gauge. A gauge registers the pressure above atmosphere, and the pressure thus measured is termed gauge pressure. To convert gauge pressure to absolute pressure the reading of the barometer must be added.

If the pressure of the water is below atmospheric pressure it is measured by means of a vacuum gauge. A vacuum gauge gives the amount the pressure is below atmosphere; this must be subtracted from the atmospheric pressure in order to obtain absolute pressure. Thus, if the reading of the vacuum gauge is 24 ft. of water, the absolute pressure will be $34 - 24 = 10$ ft. of water.

EXAMPLE.

The reading of the barometer is 76 cm. of mercury. If the specific gravity of mercury is 13·6, convert this pressure to feet of water and pounds per square inch.

Centimetres of water $= 76 \times 13{\cdot}6 = 1032$

Inches of water $= \dfrac{1032}{2{\cdot}54} = 407$

Feet of water $= \dfrac{407}{12} = 33{\cdot}85$

Pounds per sq. ft. $= wH = 62{\cdot}4 \times 33{\cdot}85 = 2112$

Pounds per sq. in. $= \dfrac{2112}{144} = 14{\cdot}67$

7. **Pressure Gauges.*** (*a*) PIEZOMETER TUBE. The pressure in a pipe or vessel, full of a liquid, may be measured by inserting a glass tube with open ends into the vessel, vertically. The liquid will rise in the tube to a height equal to the equivalent static head of the pressure in the vessel. This simple type of pressure gauge is known as a piezometer tube.

(*b*) U-TUBE. The pressure of a fluid may be measured with a glass U-tube containing a heavier fluid which does not mix with the fluid of which the pressure is required.

Let the pipe in Fig. 5 contain water under a pressure of h in. of water, and let the U-tube contain a liquid of specific gravity s. If the left limb of the U-tube be open to the atmosphere and the right limb, containing water, be connected

* For Chattock Tilting Gauge, see Art. 193.

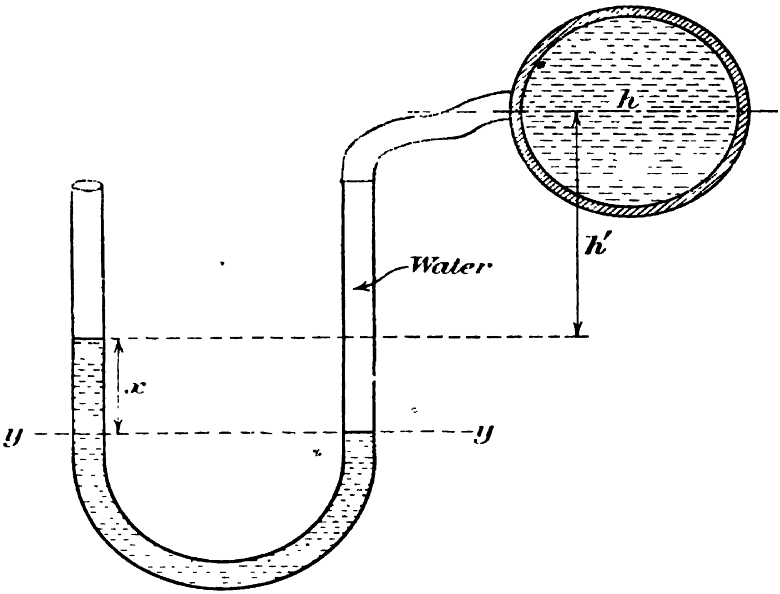


Fig. 5

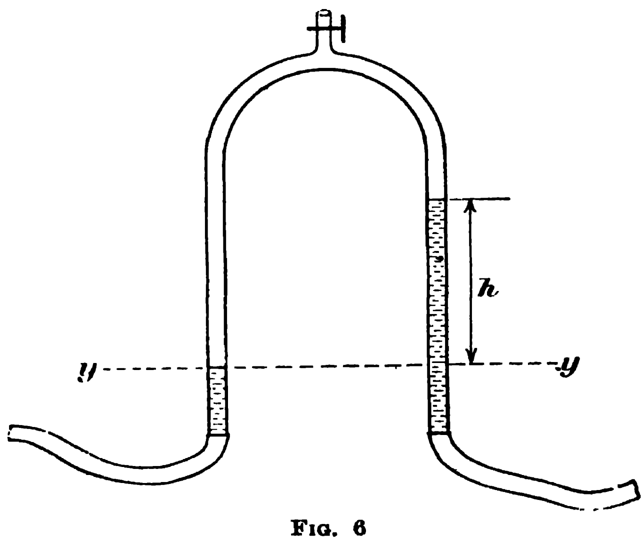


Fig. 6

to the pipe, the pressure in the pipe will force the heavy liquid in the right limb of the U-tube downwards; this will cause it to rise by a corresponding amount in the left limb. The surface of contact between the heavy liquid and the water is known as the common surface.

Consider the horizontal section yy through the common surface.

Let h' = height of centre of pipe above liquid surface in open limb in inches.

x = height of heavy liquid in left limb above yy in inches.

As the liquid below the common surface is homogeneous, the pressure at yy in left limb must equal the pressure at yy in the right limb.

Pressure in left limb at yy, above atmosphere $= xs$ in. of water.

Pressure in right limb at yy, above atmosphere $= x + h' + h$ in. of water.

Equating these pressures,

$$x + h' + h = xs$$

From which $h = x(s-1) - h'$ in. of water.

If the heavy liquid in the U-tube is mercury, $s = 13{\cdot}6$.

Then,

$$\begin{aligned} h &= x(13{\cdot}6 - 1) - h' \\ &= 12{\cdot}6x - h' \text{ in. of water.} \end{aligned}$$

If the pressure being measured is large, mercury should be used in the U-tube. For small pressures the liquid should be a little heavier than water.

(*c*) Inverted U-Tube. The difference of pressure between two sections of a pipe containing water may be measured by an inverted U-tube (Fig. 6). The upper part of the tube contains air, whilst the water from the two sections of the pipe being measured passes into the left and right limb respectively.

The heights of the water columns may be adjusted to convenient heights by letting out air through the valve at the top.

As the air trapped in the upper part of the tube is under constant pressure, the difference of pressure between the sections of the pipe is equal to the difference in the heights of the two water columns.

Let h = difference of height of water columns in inches.

Then, difference of pressure $= h$ in. of water.

(*d*) Differential Gauge. The inverted U-tube may be made very sensitive by having a liquid lighter than water in the upper part of the tube in place of the air.

Let s = specific gravity of liquid used.

Consider pressures above section yy (Fig. 6).

$$\begin{aligned} \text{Difference of pressure} &= h \text{ in. of water} - h \text{ in. of liquid} \\ &= h - hs \\ &= h(1 - s) \end{aligned}$$

The nearer s is to unity, the more sensitive the instrument becomes.

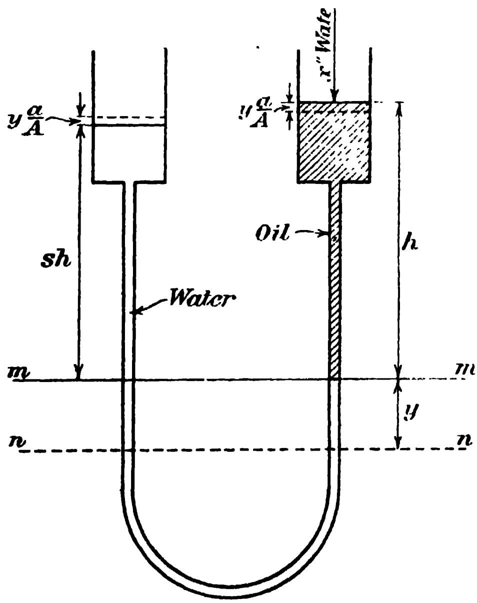


Fig. 7

All gauges which are made sensitive by using liquids of different specific gravity are known as differential gauges.

(*e*) Oil Gauge with Enlarged Ends. A sensitive type of gauge may be obtained by using a U-tube with enlarged ends (Fig. 7); this type is used for measuring small differences of pressure of gases. Water and oil are placed in the limbs, the free surface of each liquid being in the enlarged ends.

Let A = area of enlarged end
a = area of tube
s = specific gravity of oil used
mm = common surface when both limbs are subjected to equal pressures.

Assume both ends of U-tube are exposed to same pressure, and that h in. be the height of free surface of oil above the common surface mm.

Then, height of free surface of water above mm $\Big\} = sh$

Now let the surface of the oil be subjected to an additional pressure equal to x in. of water. This will cause the common surface to fall by the amount y in. to the level nn. The level of the oil in the enlarged end will consequently fall by $y\frac{a}{A}$, whilst the level of the water in the other limb will rise by the same amount.

Consider the total pressure in both limbs at the new common surface nn.

Height of oil surface above nn $= h + y - y\frac{a}{A}$

$$= h + y\left(1 - \frac{a}{A}\right)$$

Height of water surface above nn $= sh + y + y\frac{a}{A}$

$$= sh + y\left(1 + \frac{a}{A}\right)$$

Then, as pressures in both limbs at nn are equal,

$$sh + y\left(1 + \frac{a}{A}\right) = s\left[h + y\left(1 - \frac{a}{A}\right)\right] + x$$

both being in inches of water.

Therefore, $\quad y\left(1 + \frac{a}{A}\right) = sy\left(1 - \frac{a}{A}\right) + x$

Or, $\quad x = y\left[1 + \frac{a}{A} - s\left(1 - \frac{a}{A}\right)\right]$

EXAMPLE 1.

A U-tube containing mercury has its right limb open to the atmosphere. The left limb is full of water and is connected to a pipe containing water under pressure, the centre of which is level with the free surface of the mercury. Find the pressure of the water in the pipe, above atmosphere, if the difference of level of the mercury in the limbs is 2 in.

Consider a horizontal section through the common surface and consider the pressure in inches of water in each limb above this section.

Let $x =$ pressure of water in pipe above atmosphere in inches of water.

$$\begin{aligned} \text{Pressure in left limb} &= \text{pressure in right limb} \\ x + 2 &= 13{\cdot}6 \times 2 \\ x &= (13{\cdot}6 - 1)2 \\ &= 25{\cdot}2 \text{ in. of water} \end{aligned}$$

$$\begin{aligned} \text{Pressure in lb. per sq. in.} &= wH \\ &= \frac{62{\cdot}4 \times 25{\cdot}2}{144 \times 12} \\ &= {\cdot}91 \end{aligned}$$

EXAMPLE 2.

A pressure gauge consists of two cylindrical bulbs A and B, each of 1 sq. in. cross-sectional area, which are connected by a U-tube with vertical limbs, each of ·025 sq. in. cross-sectional area. A red liquid of specific gravity ·95 is filled into A and a clear liquid of specific gravity ·9 is filled into B, the surface of separation being in the limb attached to B. Find the displacement of the surface of separation when the pressure on the surface in A is greater than that in B by an amount equivalent to 1 in. head of water. (London Univ.)

Consider gauge when pressure in bulb A equals pressure in bulb B.

Let $h =$ height of liquid in A above common surface.

Then, height of liquid in B above common surface

$$= \frac{{\cdot}95}{{\cdot}9} h$$

Now let pressure of 1 in. of water act on liquid in A.

Let this cause the common surface to rise x in.

Then, surface of liquid in A will fall $\frac{x}{40}$ in. and surface of liquid in B will rise $\frac{x}{40}$ in.

Consider pressures in each limb above new common surface.

$$\left.\begin{array}{l}\text{Height of liquid in bulb } A \text{ above}\\ \quad\text{new common surface}\end{array}\right\} = h - x - \frac{x}{40} \text{ in.}$$

$$\left.\begin{array}{l}\text{Height of liquid in bulb } B \text{ above}\\ \quad\text{new common surface}\end{array}\right\} = \frac{\cdot 95}{\cdot 9} h - x + \frac{x}{40} \text{ in.}$$

Total pressure of A = total pressure of B

$$\cdot 95\left[h - x - \frac{x}{40}\right] + 1 = \cdot 9\left[\frac{\cdot 95}{\cdot 9} h - x + \frac{x}{40}\right]$$

$$-\cdot 95x\left(1 + \frac{1}{40}\right) + 1 = -\cdot 9x\left(1 - \frac{1}{40}\right)$$

$$x = 10\cdot 41 \text{ in.}$$

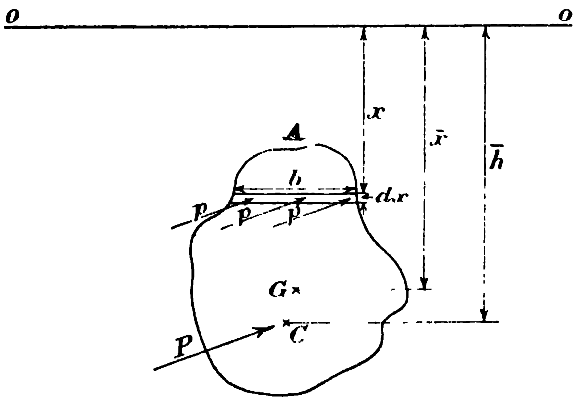


FIG. 8

8. **Total Pressure on a Surface.** As the static pressure of water varies with the depth, the intensity of pressure on a surface will not be uniform, but will vary with the depth.

Consider any vertical surface in water (Fig. 8).

Let A = area of surface
G = centre of area of surface
$\bar{x}$ = depth of centre of area
oo be the free surface of the water
P = total pressure of water on the surface

Consider a thin horizontal strip of the surface of thickness dx and breadth b. Let the depth of this strip be x.

Let intensity of pressure on strip be p; this may be taken as uniform as the strip is extremely narrow.

Then, $$p = wx$$
where w is the density of the water.

$$\begin{aligned}
\text{Area of strip} &= b.dx \\
\text{Total pressure on strip} &= p.b.dx \\
&= w.x.b.dx \\
\text{Total pressure on whole area} &= P = w\int b.dx.x \\
\text{But,} \quad \int b.dx.x &= \text{1st moment} \\
&= A\bar{x} \\
\text{Therefore,} \quad P &= wA\bar{x} \qquad \qquad (1)
\end{aligned}$$

Or, the total pressure on a surface is equal to the area multiplied by the intensity of pressure at the centre of area of the figure.

This equation will hold for all surfaces, whether flat or curved.

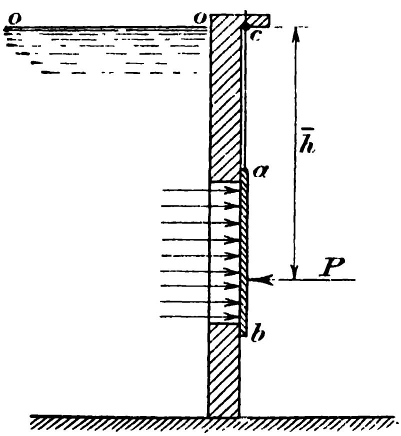


FIG. 9

EXAMPLE.

A vertical square sluice is situated with its top edge 10 ft. below the level of the water; the sluice is 3 ft. square. Find the total pressure on the sluice.

Depth of centre of area
$= 10 + 1\frac{1}{2} = 11\frac{1}{2}$ ft.

Area of sluice
$= 9$ sq. ft.

Total pressure
$= wA\bar{x}$ (From Eq. 1)
$= 62{\cdot}4 \times 9 \times 11\frac{1}{2}$
$= 6460$ lb.

9. Centre of Pressure. The intensity of pressure on a surface is not uniform but increases with the depth. As the pressure will be greatest over the lower portion of the figure it follows that the resultant pressure will act at some point towards the lower edge of the figure. The problem is to find the point of application of the resultant pressure on the surface; this point is known as the centre of pressure.

As an example of the meaning of centre of pressure, consider the diagram in Fig. 9. This represents a wall with water on the left-hand side only; the surface of the water being at *oo*. The wall contains an opening below the water level through which the water is prevented from flowing by the gate *ab*, which is freely suspended by a cord at *c*. The point *c* is on the water level. The pressure of the water is tending to swing the gate outwards about the pivot *c*; to prevent this let a force *P* be applied to the gate as shown in the figure. *P* will equal the total water pressure on the gate. There is only one point on the gate at which *P* may be applied which will keep the gate perfectly closed; that point is the centre of pressure. If *P* were applied above this point the gate would open outwards at the bottom; if *P* were applied below the centre of pressure the gate would open outwards at the top. Thus, the moment of *P* about the pivot *c* must equal the sum of all the moments of the water pressures on the gate about the water surface. Therefore, the depth of the centre of pressure may be found by taking moments about the water surface.

Referring to Fig. 8, let *C* be the centre of pressure of the immersed figure. Then the resultant pressure *P* will act through this point.

Let $\bar{h}$ = depth of centre of pressure below free surface

I_0 = moment of inertia of figure about *oo*

Consider the horizontal strip of thickness dx

Force on strip $= w.x.b.dx$ (as in Art. 8)

Moment of force on strip about free surface *oo* $= w\,x^2\,b\,dx$

Total moment of forces for whole area $= w\int b.dx.x^2$

But, $\int b.dx.x^2$ = 2nd moment $= I_0$

Therefore, total moment $= w\,I_0$

But, moment due to resultant pressure about *oo* $= Ph$

Therefore, $$P\bar{h} = w I_0$$

Or, $$\bar{h} = \frac{w I_0}{P}$$

But, $$P = wA\bar{x} \quad \text{(Eq. 1, Art. 8)}$$

Therefore, $$h = \frac{I_0}{A\bar{x}} \quad . \quad . \quad . \quad . \quad (1)$$

Or, the depth of centre of pressure $= \dfrac{\text{2nd moment}}{\text{1st moment}}$.

The value of I_0 may be obtained from the theorem :

$$I_0 = I_G + A\bar{x}^2$$

where I_G is the moment of inertia of the figure about a horizontal axis through its centre of area.

EXAMPLE.

A circular plate 6 ft. diameter is placed vertically in water so that the centre of the plate is 4 ft. below the surface. Find the depth of the centre of pressure and the total pressure on the plate.

$$I_G = \frac{\pi d^4}{64}$$

$$= \frac{\pi 6^4}{64} = 63{\cdot}6 \text{ ft.}^4$$

Area of plate $$= \frac{\pi}{4} 6^2 = 28{\cdot}22 \text{ sq. ft.}$$

$$\bar{x} = 4 \text{ ft.}$$

Also, $$I_0 = I_G + A\bar{x}^2$$

$$= 63{\cdot}6 + (28{\cdot}22 \times 16) = 515{\cdot}6 \text{ ft.}^4$$

$$\bar{h} = \frac{I_0}{A\bar{x}}$$

$$= \frac{515{\cdot}6}{28{\cdot}22 \times 4} = 4{\cdot}56 \text{ ft.}$$

Total pressure on plate $$= wA\bar{x}$$

$$= 62{\cdot}4 \times 28{\cdot}22 \times 4$$

$$= 7050 \text{ lb.}$$

10. Centre of Pressure for an Inclined Surface. The depth of the centre of pressure for a surface inclined to the water surface may be found by taking moments about the point of intersection of the plane of the inclined surface and the water surface.

Referring to Fig. 10, let mn represent the inclined surface, the view of which is shown projected.

Let B = point of intersection of surface produced with water surface

θ = Angle of inclination of immersed surface.

Consider a thin horizontal strip of area of distance x from B. Then, using same notation as Art. 8 and 9,

$p = w\,x \sin\theta$, and acts normal to surface

Area of strip $= bdx$

Force on strip $= p\,bdx$

$= wx \sin\theta\, b\, dx$

Total force on surface $= P = w\int b.dx.x \sin\theta$

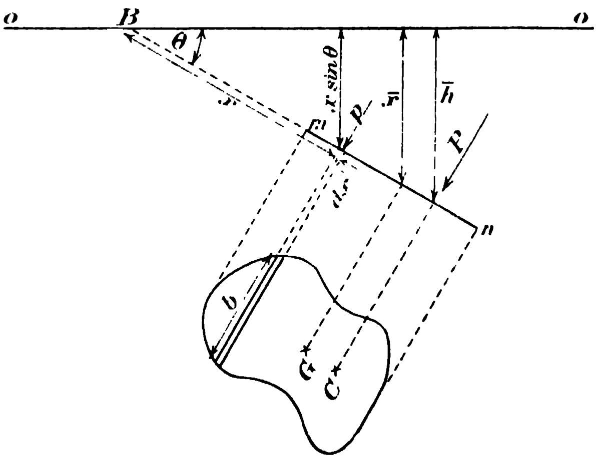


FIG. 10

But $\int b.dx.x =$ 1st moment about B

$$= \frac{A\bar{x}}{\sin\theta}$$

Therefore, $P = w\,A\bar{x}$ . . . . . . (1)

Taking moments about B,

Moment of force on strip $= w.x.\sin\theta\, b.dx.x.$

Sum of moments of forces on strip $\Big\} = w \sin\theta \int b.dx.x^2$

Let I_B = moment of inertia of surface about B

Then, $I_B = \int b.dx.x^2$

Therefore, total moment $= w \sin\theta\, I_B$

But, total moment $= \dfrac{P\bar{h}}{\sin\theta}$

Therefore, $\dfrac{P\bar{h}}{\sin\theta} = w \sin\theta\, I_B$

Or, $\bar{h} = \dfrac{w I_B \sin^2\theta}{P}$

Substituting for P from Equation (1),

$$\bar{h} = \frac{I_B \sin^2\theta}{A\bar{x}} \quad . \quad . \quad . \quad . \quad (2)$$

where $I_B = I_G + \dfrac{A\bar{x}^2}{\sin^2\theta}$

It will be noticed that if $\theta = 90°$, Equation (2) becomes the same as Equation (1) of Art. 9.

EXAMPLE.

Find (*a*) the total pressure, and (*b*) the position of the centre of pressure on one side of an immersed rectangular plate, 6 ft. long and 3 ft. wide, when the plane of the plate makes an angle of 60° with the surface of the water and the 3 ft. edge of the plate is parallel to, and at a depth of 2½ ft. below, the surface level of the water.

If you employ any formula you must prove its correctness. (London Univ.)

(*b*) Using Equation (2),

$$\bar{h} = \frac{I_B \sin^2\theta}{A\bar{x}}$$

Where $A = 18$ sq. ft.

$\theta = 60°$

$\bar{x} = 2{\cdot}5 + 3\sin 60° = 5{\cdot}1$ ft.

$$I_B = I_G + \frac{A\bar{x}^2}{\sin^2 60}$$

$$= \frac{3 \times 6^3}{12} + \frac{18 \times 5{\cdot}1^2}{{\cdot}75} = 678 \text{ ft.}^4$$

$$\bar{h} = \frac{678 \times {\cdot}75}{18 \times 5{\cdot}1} = 5{\cdot}53 \text{ ft.}$$

(*a*) Using Equation (1),

$P = w A\bar{x}$

$= 62{\cdot}4 \times 18 \times 5{\cdot}1 = 5725$ lb.

11. Fluid Pressure on a Curved Surface The total fluid pressure on a curved surface and the position of the centre of

pressure can be obtained by drawing the force polygon for the forces causing equilibrium. Consider the curved surface AB of Fig. 11. The total force on the surface and its point of application can be obtained by considering the equilibrium of the volume of water ABC. Consider unit length of the surface in a direction perpendicular to the plane of the figure.

Let P = total fluid pressure on rectangular area CB

W = weight of volume ABC of fluid acting at its centre of gravity G

R = total reaction to fluid pressure of surface AB

As the three forces P, W and R maintain the fluid ABC in equilibrium, they will intersect at a common point; this point will be "a," the point of intersection of P and W. Consider the rectangular water face CB,

$$P = wA\bar{x}$$

$$= w \times BC \times \frac{BC}{2}$$

$$= \frac{wh^2}{2}$$

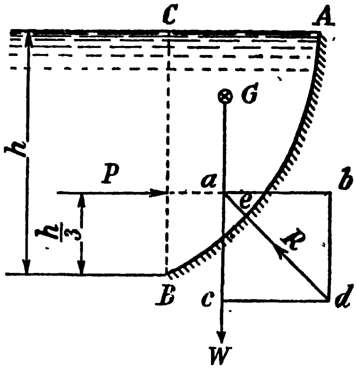


FIG. 11

The point of application of P is at $\frac{h}{3}$ from the base. Choosing a convenient scale, draw ab to represent P, and ac to represent W; then, diagonal ad gives the resultant force on the surface R. The centre of pressure on the surface AB is at e, the point at which the line of action of R cuts the surface.

12. **The Pressure on Lock Gates.** A practical problem on the centre of pressure is encountered in finding the forces on a lock gate. The plan of a pair of lock gates is shown in Fig. 12. AB and BC represent the gates which are held in contact at B by the water pressure, the water level being higher on the left-hand side of the gates. The gates are hinged at top and bottom at A and C.

Consider the forces acting on the gate AB.

The water pressure acts with a resultant force P at the

centre of the gate and normal to it. The gate BC acts on it with a pressure T which is normal to the surface of contact of the two gates. The two hinges on the side A will react with a total force R, the direction of which is not yet known. As the gate is in equilibrium under these three forces, they will all intersect at one point. Let P and T intersect at D; then R must pass through this point. Thus, the gate is in equilibrium under the action of three forces intersecting at D. Let a = angle of inclination of gate to the normal of side of lock.

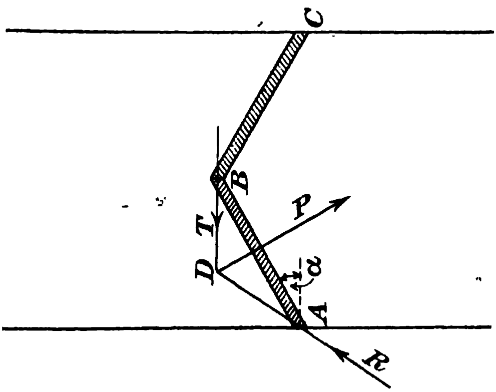


FIG. 12

Then, triangle ADB will be isosceles, as angles DBA and DAB equal a

Resolving the forces at D in a direction parallel to gate,

$$R \cos a = T \cos a$$

Therefore,

$$R = T$$

Resolving normal to gate,

$$P = (R + T) \sin a$$
$$= 2R \sin a$$

Or, $$R = \frac{P}{2 \sin a}$$

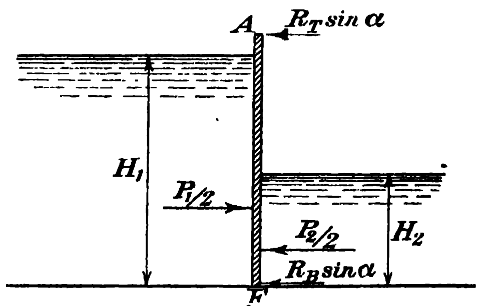


FIG. 13

Also, inclination of R to centre line of gate = a.

Next consider the water pressure on the gate. Fig. 13 is a view of the gate in the direction AB.

Let H_1 = height of water to left of gate
H_2 = height of water to right of gate
H = height of top hinge from bottom of gate

Let P_1 = total pressure of water to left of gate
P_2 = total pressure of water to right of gate
R_T = reaction of top hinge
R_B = reaction of bottom hinge

Then, $R_T + R_B = R$

Also, $P_1 = \dfrac{wH_1}{2} \times$ wetted area of gate

and, $P_2 = \dfrac{wH_2}{2} \times$ wetted area of gate

then, $P = P_1 - P_2$

P_1 will act at the centre of pressure which is $\dfrac{H_1}{3}$ from bottom.

Also, P_2 will act at $\dfrac{H_2}{3}$ from bottom.

It will be noticed that only half the water pressure may be taken as acting on the hinge edge of the gate; the remaining half will be taken by the reaction of the gate BC.

Taking moments about F (Fig. 13),

$$R_T \sin a\, H = \left(\frac{P_1}{2} \times \frac{H_1}{3}\right) - \left(\frac{P_2}{2} \times \frac{H_2}{3}\right) \quad . \quad . \quad . \quad (1)$$

Resolving horizontally,

$$\frac{P_1}{2} - \frac{P_2}{2} = R_B \sin a + R_T \sin a \quad . \quad . \quad . \quad . \quad (2)$$

Then, from Equations (1) and (2), R_T and R_B may be found.

13. **Water Pressure on Masonry Dams.** Fig. 14 shows the section of a masonry dam having a sloping back; let the height of the water be H. The total pressure P on the dam will act at the centre of pressure C, the height of C being one-third H, and will act normal to the surface. The weight of the masonry W will act at the centre of area of the cross-section of the dam.

Produce the forces P and W to intersect at a.

Let ab represent P and let ad represent W to the same scale. These are the only forces acting on the dam. Complete the parallelogram and draw the diagonal ae. Then ae gives the magnitude and direction of the resultant force R.

Let the point at which the resultant force cuts the base of

the dam be f. Then, in order to keep the stresses on the base of the dam within certain limits, f must fall within a certain distance from the centre of the base.

The above method investigates the strength of the base of the dam only; it is necessary to extend this method to other horizontal sections and so test the strength of the dam at all heights, and for all depths of water.

Consider the section of the dam in Fig. 15, and assume, in the first case, the dam to be full. Consider the horizontal section line bb as the base of the dam, as in the previous problem,

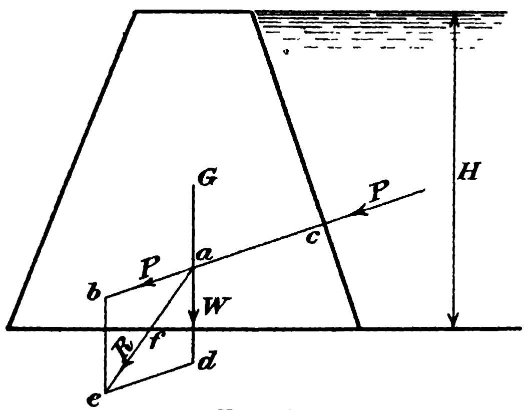


FIG. 14

and find the point at which the resultant force cuts the section line. To do this, let G_1 be the centre of area above bb and W_1 its weight; let P_1 be the water pressure above bb acting at one-third of the height above bb.

Next consider the whole section of the dam above cc. This gives a new centre of area G_2, a new weight W_2, and a new pressure P_2. Find the point on cc at which the resultant of W_2 and P_2 cuts the line. Repeat this by considering the whole section above dd and ee in turn. Mark clearly the point at which each resultant cuts its own section line and draw a smooth curve through these points. This curve is known as the line of pressure for the dam when full; and for any horizontal section line, this curve must cut the line within a given distance from the centre.

It is next required to draw the line of pressure for the dam when empty. In this case there will be no water pressure

acting, the only force being the weight of the masonry. The point at which the resultant cuts the base is now where W_1 cuts *bb*, where W_2 cuts *cc*, etc. By drawing a smooth curve to pass through these points, the line of pressure for dam when empty is obtained. This curve, also, must cut any horizontal section line within a given distance from the centre.

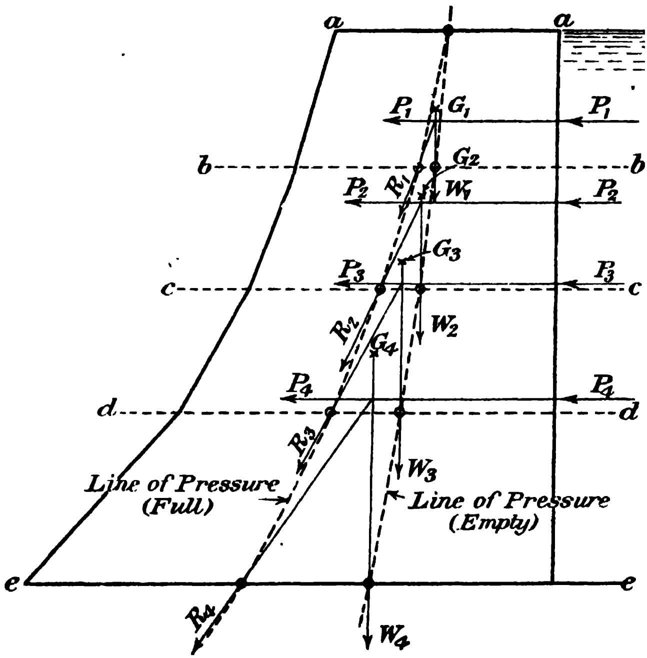


FIG. 15

It will be noticed that the centre point of the top of the dam, *aa*, will be the required point for both lines of pressure at this section.

In all problems dealing with masonry dams it is usual to calculate all the forces on a length of 1 ft.

EXAMPLES 1.

(1) A diver is working on the sea bottom at a depth of 74 ft. What is the pressure, above atmosphere, in pounds per square inch, at this depth? 1 cu. ft. sea water weights 64 lb.

Ans.—32·9 lb. per sq. in.

(2) The pressure of a gas is measured by a U-tube containing water, which has one limb open to the atmosphere, and is found to be 2·6 in. of water. The barometer reading is 76 cm. of mercury Express the pressure of this gas in pounds per square inch—(1) as gauge pressure, (2) as absolute pressure. The specific gravity of mercury is 13·6.

Ans.—(1) ·0939 lb. per sq. in. (2) 14·7639 lb. per sq. in.

(3) A hydraulic press has a ram of 4 in. diameter and a piston of $\frac{3}{8}$ in. diameter. What load on the ram can be lifted by a force of 25 lb. on the piston ?

Ans.—2,840 lb.

(4) A masonry dam of rectangular section of 20 ft. high and 10 ft. wide has the water level with its top. Find (1) the total pressure on 1 ft. length of the dam ; (2) the height of the centre of pressure ; (3) the point at which resultant cuts the base. The weight of 1 cu. ft. of masonry is 110lb.

Ans.—(1) 12,480 lb. (2) 6·67 ft. (3) 3·78 ft. from centre.

(5) A hollow triangular box, the ends of which are equilateral triangles of 4 ft. sides, is submerged in water so that one of its rectangular faces lies in the surface of the water. Find the net total pressure, and the position of the centre of pressure on one of the triangular ends, (*a*) when the inside of the box is at atmospheric pressure ; (*b*) when the inside of the box is at a pressure of 1 lb. per sq. in. above the atmospheric pressure. (London Univ.)

Ans.—(*a*) 496 lb. ; 1·735 ft. from surface. (*b*) 500 lb. ; ·585 ft. from surface.

(6) A pressure gauge, for use in a stokehold, is made of a glass U-tube with enlarged ends, one of which is exposed to the pressure in the stokehold and the other connected to the outside air. The gauge is filled with water on one side, and oil having a specific gravity of ·95 on the other—the surface of separation being in the tube below the enlarged ends. If the area of the enlarged end is 50 times that of the tube, how many inches of water pressure in the stokehold correspond to a displacement of 1 in. in the surface of separation ? (London Univ.)

Ans.—·089 in.

(7) A circular plate 5 ft. diameter is immersed in water, its greatest and least depths below the surface being 6 ft. and 3 ft. respectively ; find (*a*) the total pressure on one face of the plate ; (*b*) the position of the centre of pressure. (London Univ.)

Ans.—(*a*) 5,520 lb. (*b*) 4·63 ft. below surface.

(8) Each gate of a lock is 20 ft. high and 6 ft. wide, and is supported on pivots, situated 2 ft. from the top and bottom. The angle between the gates when they are closed is 140°. If the depths of water on the two sides are 17 ft. and 5 ft. respectively, find the magnitude and position of the resultant water pressure on each gate, the magnitude of the reaction between the gates, and the magnitude and directions of the reactions at the pivots, on the assumption that the gate reaction acts in the same horizontal plane as the resultant water pressure. (London Univ.)

Ans.—49,500 lb. ; 6·04 ft. from bottom ; 72,550 lb. ; 18,150 lb. (top) ; 54,400 lb. (bottom) ; 20° to gate.

(9) A rectangular sluice-gate 6 ft. square has its upper edge submerged to a depth of 6 ft. Determine the magnitude of the resultant pressure on one face, and the centre of pressure. (A.M.I. Mech. E.)

Ans.—20,200 lb. ; $9\frac{1}{3}$ ft.

(10). A hemispherical tank, 5 ft. in diameter, is full of water. Determine—(1) The resultant pressure on the wetted surface ; (2) the total pressure on the wetted surface ; (3) the centre of pressure on the wetted surface. (A.M.I. Mech. E.)

Ans.—(1) 2,040 lb. (2) 3,060 lb. (3) 2·5 ft.

(11) A bulkhead closing one end of a floating dock is 30 ft. wide at the bottom and 60 ft. at the top, and is 30 ft. deep. If submerged up to its upper edge, what is the pressure on the bulkhead, and what will be the depth of the centre of pressure? (A.M.Inst.C.E.)

Ans.—1,122,000 lb.; 18·75 ft.

(12) A 10 ft. length of a semicircular culvert 6 ft. in diameter has bulkheads at each end. If filled with water determine (*a*) the resultant force exerted by the water on the wetted surfaces; (*b*) the total pressure exerted on these surfaces. (A.M.I.Mech.E.)

Ans.—(*a*) 8,810 lb.; (*b*) 13,450 lb.

(13) Describe with sketches some form of differential gauge capable of enabling very small differences of head in a pipe to be measured. Explain the theory of its action. (A.M. Inst. C.E.)

(14) A circular drum, 4 ft. in diameter and 10 ft. long, rests with its axis horizontal on the bottom of a dock in which the depth of water is 10 ft. Determine: (*a*) The total pressure on the surface of the drum; (*b*) the resultant pressure on the surface of the drum; (*c*) the depth of the centre of pressure on each of the flat ends. (A.M.I. Mech. E.)

Ans.—(*a*) 74,060 lb.; (*b*) 7,820 lb.; (*c*) 8·125 ft.

(15) Water is 60 ft. deep at the face of a dam, which is vertical to 30 ft. from the water level, below which it slopes at 30° to the vertical. Specify completely the resultant pressure which acts on the face of the dam per foot run. (London Univ.)

Ans.—P = 123,000 lb. at $23\frac{1}{2}°$ to the horizontal.
$\bar{h}$ = 41·3 ft.

(16) Show that a vertical surface, subject to water pressure on one side, requires, for equilibrium, a balancing moment, about a horizontal axis through the centroid of area, which is independent of the depth of submergence.

An aperture in a vertical wall of a water-tank is closed by a circular plate 24 in. diameter. This is held in position by four stops, one at each end of the horizontal diameter and one at each lower end of the diameters at 60° to the horizontal. Determine the stop reactions when the water surface is 18 in. above the plate centre. (London Univ.)

Ans.—118·5 lb., 28·5 lb.

CHAPTER II

THE BUOYANCY OF A LIQUID

14. **Buoyancy.** If a body is floating in a fluid and is at rest, it will be in equilibrium in a vertical plane; then the total upward force must equal the total downward force. This is true whether the body be immersed in a liquid or a gas. The downward force on the body will be due to gravity, whilst the upward force will be the resultant upward pressure of the fluid in which the body is floating. This resultant upward pressure is known as the buoyancy.

Consider a body immersed in a fluid, and let *oo* be the surface of the fluid (Fig. 16). Consider a vertical prism of the body of height H and end area a. Let p be the intensity of pressure of the fluid on the upper end of the prism. Then the intensity of pressure on the lower end of the prism will be $p + wH$, the additional amount wH being due to the additional depth H of the fluid.

Total downward pressure of fluid on prism $= pa$
Total upward pressure of fluid on prism $= (p + wH)a$
Resultant upward pressure of fluid on prism $= (p + wH)a - pa$
$= wHa$
But, volume of prism $= Ha$
Therefore, resultant upward pressure $= w \times$ volume of prism
$=$ weight of fluid displaced by prism

If the whole body is imagined to be made up of similar vertical prisms, it follows that the total resultant upward pressure of the fluid will equal the weight of fluid displaced by the body. This is known as Archimedes' principle.

EXAMPLE.

The volume of the displacement of a ship in sea water is 4,000 cu. ft.; find the weight of the ship if 1 cu. ft. sea water weighs 64 lb. Find also the volume of the displacement in fresh water.

Weight of ship = weight of sea water displaced

$$= \frac{64 \times 4000}{2240} = 114 \cdot 2 \text{ tons.}$$

$$\text{Volume of fresh water displacement} = \frac{114 \cdot 2 \times 2240}{62 \cdot 4}$$

$$= 4{,}100 \text{ cu. ft.}$$

15. **Centre of Buoyancy.** When a body is floating in a liquid, a normal pressure will be exerted by the liquid at all points on the surface of the body. The resultant of all these

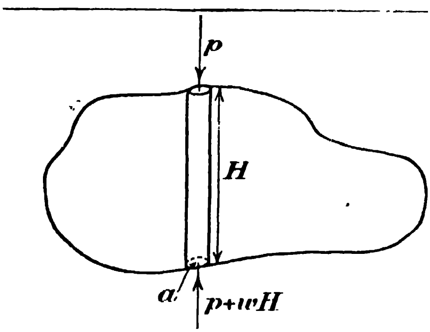


FIG. 16

normal pressures will be vertically upwards and will act through the centre of gravity of the volume of liquid displaced by the body; this point is known as the centre of buoyancy. When dealing with a transverse section of a floating body, the centre of buoyancy is at the centre of area of the immersed section.

Referring to the transverse section of the ship in Fig. 17, let *ac* be the water line; then the immersed section will be the area *acde*. The centre of buoyancy will be at the centre of area of this immersed section and is denoted by the point B_1. If the ship rolls in a clockwise direction, as shown by the dotted position (Fig. 17), the immersed section will now be acd_1e_1, and the centre of buoyancy will be at the centre of area of this new immersed section.

16. **Conditions of Equilibrium of a Floating Body.** There are three conditions of equilibrium for a floating body: stable,

unstable, and neutral. If the floating body is given a slight angular displacement, such as the rolling of a ship, after which it returns to its original position, the body is said to be stable. If, on being given a slight displacement, it heels farther over, it is said to be in unstable equilibrium. But if the body is given a small displacement into a new position and it remains at rest in that new position, the body is then said to be in neutral equilibrium.

Consider the cross-section of a ship, shown in Fig. 17, and let *oo* be the water-line, *B* the centre of buoyancy, and *G* the centre of gravity of the ship. If the ship is given a small angular

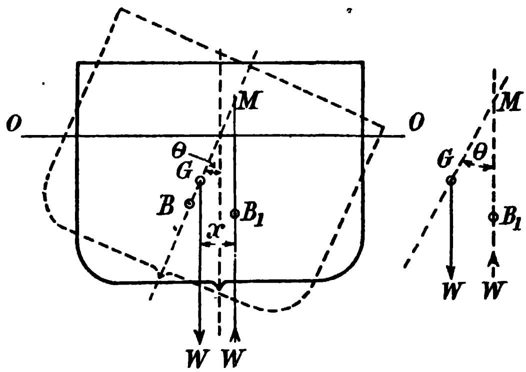


FIG. 17

displacement θ, it will rotate about some point on its vertical axis. As the rotation does not cause any alteration in the volume of water displaced, it follows that the ship must rotate about the point of intersection of the waterline with the vertical centre line. The dotted lines show the position of the ship after it has heeled through the small angle θ.

Let B_1 = position of new centre of buoyancy after heeling
G = position of centre of gravity after heeling
W = weight of ship

Draw a vertical line through B_1 to intersect the centre line of the ship at M. The point M is called the *metacentre*.

The forces now acting on the ship are shown in the right-hand diagram of Fig. 17; the upward thrust of the water is equal to W and acts vertically through B_1, whilst the weight

of the ship acts vertically downwards through G. Thus, there is a couple acting on the ship tending to restore it to its original position. This couple is known as the righting couple, or righting moment, its magnitude being $W \times$ horizontal distance between G and B_1.

$$\text{Or, righting couple} = W \times x$$
$$= W \times MG \times \tan\theta \quad (\text{as } \theta \text{ is small})$$

The distance MG is known as the *metacentric height*, when the angle θ is infinitely small.

It will be seen from this that the ship behaves as a pendulum suspended at M, the point G corresponding to the bob, as

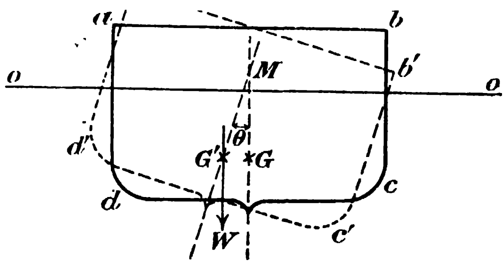


FIG. 18

shown in Fig. 18. Hence, the ship may be regarded as rotating about M, which point is considered by some authorities to be equivalent to the instantaneous centre of rotation. As M is always very close to the water-line, it does not materially affect the problem whether the ship be assumed to rotate about the water-line or about its metacentre M. M can only be regarded as a fixed point for an extremely small angle of heel.

Now, referring to Fig. 18, the righting moment is $W \times GM \tan\theta$, so that if M is above G the ship will return to its original position and is, therefore, stable. If M is below G, the moment due to W would cause the ship to turn completely over. In this case the ship would be unstable. In the case when M coincides with G the ship is in neutral equilibrium, for there would then be no moment acting on the ship. In order to ensure that a ship is perfectly stable, M should be a certain

distance above G; the distance varies with the size and type of the vessel, usually the metacentric height is between 1 ft. and 4 ft.

The term "metacentre" was first defined by Bougier* in 1746. Bougier's definition was that the metacentre is the point at which the vertical through the centre of buoyancy intersects the vertical centre line of the ship's section, after a small angle of heel.†

The metacentric height of a floating body can be determined both by calculation and by experiment.

17. **The Experimental Determination of the Metacentric Height.** The metacentric height of a ship or pontoon may be

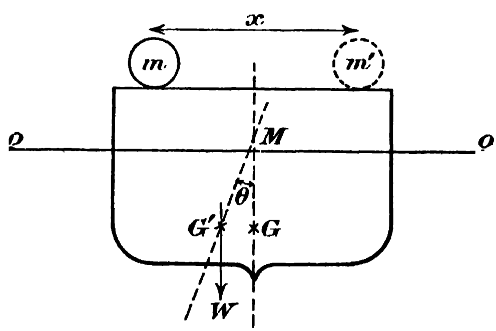


FIG. 19

found experimentally whilst the vessel is floating; the position of the centre of gravity must be known beforehand.

Let W be the weight of the ship (Fig. 19), which is known, and let G be the centre of gravity. Let a known movable weight m be placed on one side of the ship.

A pendulum consisting of a weight suspended by a long cord is placed in the ship and the position of the bob when at rest is marked. Let l be the length of the pendulum. The weight m is then moved across the deck through the distance x, the new position of m being denoted by m'. This will cause the ship to swing through a *small* angle θ about its metacentre M. Then, as the pendulum inside the ship still remains vertical, the angle θ may be measured by the apparent deflection of the pendulum.

* *Traité du navire*, by Bougier.

† For full discussion on stability of ships see Sir William White's *Naval Architecture*.

Let apparent horizontal displacement of pendulum weight $= y$.

Then, $$\tan\theta = \frac{y}{l}$$

Referring to Fig. 19, the moment caused by W about M equals the moment about M caused by moving m to m'.

Or, $$W \times GM \tan\theta = mx$$

From which $$GM = \frac{mx}{W \tan\theta} \quad . \quad . \quad . \quad . \quad (1)$$

and, as all the quantities on the right of this equation are known, the metacentric height can be calculated.

EXAMPLE.

Define the term "metacentric height" in connection with a floating body. Obtain an equation giving the metacentric height and apply it in the case of a ship which displaces 3,000 tons of sea water and which heels over $\frac{1}{30}$ when a load of 15 tons is shifted across the deck a distance of 30 ft. (London Univ.)

Taking moments about M (Fig. 19),

$$\text{Moment due to } W = \text{moment due to } m$$

$$W\,GM \tan\theta = m\,x$$

$$3000\,GM\frac{1}{30} = 15 \times 30$$

$$GM = \frac{15 \times 30 \times 30}{3000}$$

$$= 4{\cdot}5 \text{ ft.}$$

18. **Analytical Method for Metacentric Height.** An equation for the metacentric height of a floating body may be obtained if the position of the centre of gravity G is known. Consider the transverse section of the ship of Fig. 20; let the ship heel in a clockwise direction through a small angle θ (radians). The immersed section has now changed from the area of *acde* to the dotted position acd_1e_1. The new centre of buoyancy is B_1; the old centre of buoyancy, relative to the ship, is B; hence the centre of buoyancy has moved from B to B_1, relative to the ship. It will be noticed that the effect of the heeling is to move an immersed wedge from one side of

the ship to the other; that is, the immersed wedge *aom* now occupies the position *con*. The apparent movement of this wedge across the ship causes the centre of buoyancy to move from B to B_1; these movements, of course, being relative to the ship. From the effect of these two movements the required equation may be obtained.

As the volume of water displaced remains constant, the shaded area *aom* must equal the shaded area *con*; hence the old water-line *mn* will pass through the point *o*. From this it follows that the ship is rotating about the point O; but for the extreme case of θ being infinitesimally small, the same

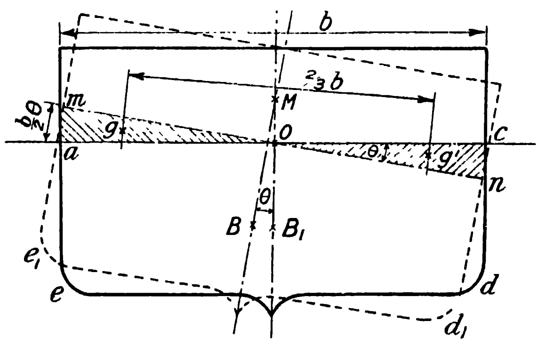


FIG. 20

result is obtained whether the ship be assumed to rotate about M or O.

Let l be the length of the ship and consider a thin transverse slice of length dl.

Let b = breadth of ship.

V = volume of water displaced by whole ship.

dV = volume of water displaced by slice considered.

I = moment of inertia of a horizontal section of ship at water line about a longitudinal axis.

dI = moment of inertia of slice considered about a longitudinal axis.

g and g' = centres of gravity of triangular prisms *aom* and *con* respectively.

Then, weight of ship $= wV$

weight of slice considered $= wdV$

distance between g and g' $= \frac{2}{3}b$

$$am = cn = \frac{b}{2}$$

volume of wedge of slice $= \frac{1}{2} \times \frac{b}{2} \times \frac{b}{2}\theta \times dl$

weight of wedge of slice $\dfrac{wb^2\theta\, dl}{8}$

Also,
$$dI = \frac{\text{breadth} \times (\text{depth})^3}{12}$$
$$\frac{dl \times b^3}{12}$$

Taking moments about M,

$$\left.\begin{array}{l}\text{moment caused by moving}\\ \text{triangular prism of water}\\ \text{from } g \text{ to } g'\end{array}\right. = \left\{\begin{array}{l}\text{moment caused by moving}\\ \text{upward thrust of water}\\ \text{from } B \text{ to } B_1.\end{array}\right.$$

That is,
$$\frac{wb^2\theta dl}{8} \times \frac{2}{3}b = wdV \times BB_1$$

Or,
$$w\left(\frac{dl \times b^3}{12}\right)\theta = wdV \times (BM \times \theta)$$

Hence,
$$dI = BM \times dV$$

Integrating for whole length of ship,
$$I = BM \times V$$

Or,
$$BM = \frac{I}{V} \quad . \quad . \quad . \quad (1)$$

Then, metacentric height $GM = BM - BG$.

Hence, as BG is known, the metacentric height can be obtained.

The moment of inertia I is actually the moment of inertia of the horizontal section of the ship at the water line. Usually the sides of a vessel are vertical at this section, so that I may be taken as the moment of inertia of the deck about a longitudinal axis. Referring to the plan of the vessel shown in Fig. 21, in order to find the moment of inertia of this figure about the axis *oo* it would be necessary to divide the section up into small

horizontal rectangles and to add together their moments of inertia. Sometimes, the moment of inertia of the deck of a ship is given as a function of the moment of inertia of the circumscribing rectangle.

Let l = length of ship (Fig. 21)

'hen, moment of inertia of circumscribing rectangle $\Big\} = \dfrac{l b^3}{12}$

And $I = k \dfrac{l b^3}{12}$

where k is a coefficient depending on the shape of the ship.

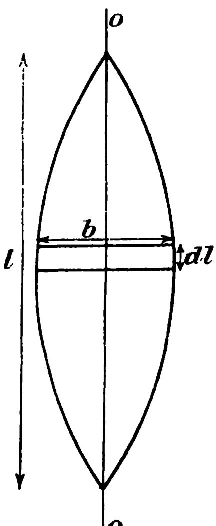


Fig. 21

In the case of a pontoon, the deck will be rectangular; then k will equal unity.

This method may be applied if the angle of heel is less than 10°. As the moment of inertia of the ship's water plane is not constant, but increases with the angle of heel, the metacentric height will increase as the angle of heel increases.

The metacentric height of large ships varies between $1\frac{1}{2}$ ft. and 4 ft.

Example 1.

A vessel has a length of 200 ft., a beam of 28 ft., and a displacement of 1,350 tons. A weight of 20 tons moved $22\frac{1}{2}$ ft. across the deck inclines the vessel 5°. The second moment of the load water-plane about its fore and aft axis is 65 per cent of the second moment of the circumscribing rectangle, and the position of the centre of buoyancy is 5 ft. below the water line. Find the position of the metacentre and the centre of gravity of the vessel. The weight of 1 cu. ft. of sea water can be taken as 64 lb. (London Univ.)

From Equation (1), Art. 16,

$$GM = \frac{mx}{W \tan\theta}$$

$$= \frac{20 \times 22{\cdot}5}{1350 \times {\cdot}0875}$$

$$= 3{\cdot}81 \text{ ft.}$$

Volume of displacement $\quad V = \dfrac{W}{w}$

$$= \frac{1350 \times 2240}{64}$$

$$= 47{,}200 \text{ cu. ft.}$$

$$I = k \cdot \frac{l\,b^3}{12}$$

$$= \frac{\cdot 65 \times 200 \times 28^3}{12} = 238{,}000 \text{ ft.}^4$$

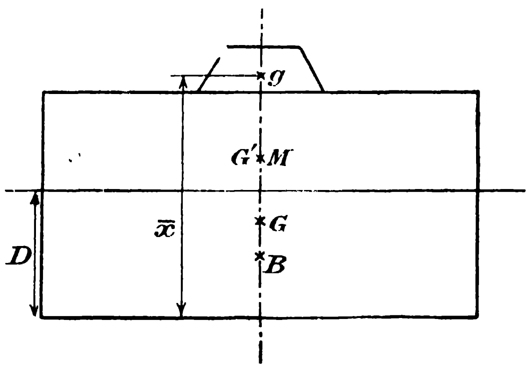


Fig. 22

From Equation (1),

$$BM = \frac{I}{V}$$

$$= \frac{238{,}000}{47{,}200}$$

$$= 5\cdot 05 \text{ ft.}$$

Position of M = $5\cdot 05 - 5 = \cdot 05$ ft. above water line

Position of G = $3\cdot 81 - \cdot 05 = 3\cdot 76$ ft. below water line

Example 2.

State the condition for the stability of a floating body; and find an expression for the distance between the centre of buoyancy and the metacentre, in terms of the second moment of the water-plane area and the volume of displacement. A cylindrical buoy floats in salt water. It is 6 ft. diameter and 4 ft. long, and weighs 2,500 lb. The C.G. is 1·5 ft. from the bottom. If a load of 500 lb. is placed on the top, find the maximum height of its C.G. above the bottom, so that the buoy may remain in stable equilibrium. [Weight of 1 cu. ft. of salt water, 64 lb.] (London Univ.)

The floating buoy is shown in Fig. 22.

Let G = centre of gravity of buoy
g = centre of gravity of weight on top
G' = centre of gravity of buoy plus weight
D = depth of buoy below water line

Then, height of B from bottom $= \dfrac{D}{2}$

Let $\bar{x}$ = required height of centre of gravity of weight.

Then, $\bar{x}$ will be a maximum when the buoy reaches the state of neutral equilibrium. That is, when G' and M coincide.

Total weight of buoy plus load $= 2500 + 500$
$= 3000$ lb.

$$\text{Volume of water displaced} = V = \frac{3000}{64} = 46{\cdot}9 \text{ cu. ft.}$$

$$D = \frac{V}{\text{area of base}}$$

$$= \frac{46{\cdot}9}{\frac{\pi}{4} \times 6^2} = 1{\cdot}66 \text{ ft.}$$

Then, height of B $= \dfrac{1{\cdot}66}{2} = {\cdot}83$ ft.

Using Equation (1),

$$BG' = BM = \frac{I}{V} = \frac{\pi\,(\text{diameter})^4}{64\ V}$$

$$= \frac{\pi \times 6^4}{64 \times 46{\cdot}9} = 1{\cdot}355 \text{ ft.}$$

Height of G' above bottom $= 1{\cdot}355 + {\cdot}83 = 2{\cdot}185$ ft.

In order to find $\bar{x}$ take moments about the bottom of cylinder.

$$500\,\bar{x} = (3000 \times 2{\cdot}185) - (2500 \times 1{\cdot}5)$$
$$x = 5{\cdot}61 \text{ ft.}$$

19. **Floating Body Anchored at Base.** Consider a floating body, such as a buoy, anchored by means of a chain from the centre of its base (Fig. 23). The tension in the anchor chain puts an additional downward force on the body causing it to displace a larger volume of water.

Let W = weight of body
T = tension in anchor cable

Then, applying Archimedes' principle,

total downward force = weight of water displaced

Or, $$W + T = wAd \quad . \quad . \quad . \quad . \quad (1)$$

where A is the area of the body in the horizontal plane and d is the draught in feet.

Also, $$BM = \frac{I}{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the body is just to float with its axis vertical, the metacentric height M must coincide with the centre of the force system acting on the body; it is then in neutral equilibrium.

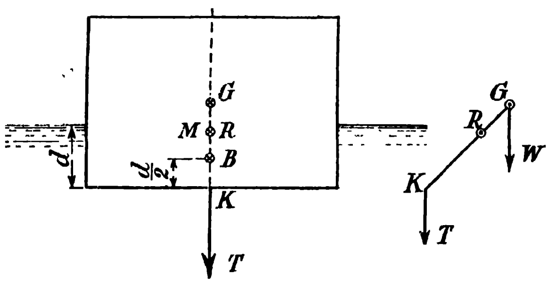


FIG. 23

Referring to Fig. 23, let R be the centre of the force system; that is, the point of application of the resultant of W and T when the body has a small angle of heel. Let K be the point of application of the tension in the anchor chain. When the body has a small angle of heel, as shown to the right of the figure, the moments due to T and W about the point R will balance; hence,

$$W \times RG = T \times RK \quad . \quad . \quad . \quad (3)$$

If the system is in neutral equilibrium, the upward thrust of the water must pass through R, which coincides with M.

EXAMPLE.

A cylindrical buoy is 5 ft. diameter and 6 ft. high. It weighs 1,500 lb., and its centre of gravity is 2·5 ft. above the base and is on the axis. Show that the buoy will not float with its axis vertical in sea water.

If one end of a vertical chain is fastened to the centre of the base, find the pull on the chain in order that the buoy may just float with its axis vertical. Density of sea water, 64 lb./ft.^{3}. (London Univ.)

When there is no Anchor Chain.

$$W = \frac{\pi}{4} D^2 \times d \times 64$$

that is, $$1500 = \frac{\pi}{4} \times 5^2 d \times 64$$

from which, $$d = 1{\cdot}192 \text{ ft.}$$

Then, $$BG = 2{\cdot}5 - \frac{d}{2}$$

$$= 2{\cdot}5 - \frac{1{\cdot}192}{2} = 1{\cdot}904 \text{ ft.}$$

$$BM = \frac{I}{V}$$

$$= \frac{\frac{\pi}{64} 5^4}{\frac{\pi}{4} 5^2 \times 1{\cdot}192}$$

$$= 1{\cdot}31 \text{ ft.}$$

$$MG = BM - BG$$

$$= 1{\cdot}31 - 1{\cdot}904$$

$$= -{\cdot}594 \text{ ft.}$$

Hence, M is below G; the body is, therefore, unstable.

When Anchor Chain is Fitted.

Applying Equation (1),

$$1500 + T = 64 \times \frac{\pi}{4} 5^2 d$$

Applying Equation (2),

$$BM = \frac{I}{V}, \text{ and } M \text{ now coincides with } R$$

that is, $$2{\cdot}5 - RG - \frac{d}{2} = \frac{\frac{\pi}{4} 5^4}{\frac{\pi}{4} 5^2 d}$$

Applying Equation (3),

$$1500 \times RG = T \times (2{\cdot}5 - RG)$$

Substituting in Equation (1) the value of T from Equation (3),

$$RG = -\frac{3}{d} + 2{\cdot}5$$

Substituting this value of RG in Equation (2),

$$d = 1{\cdot}7 \text{ ft.}$$

Then, from Equation (1),

$$T = 640 \text{ lb.}$$

20. **Floating Body with Bilge Water.** The effect of water in the bottom of a floating body, if free to move, is to reduce

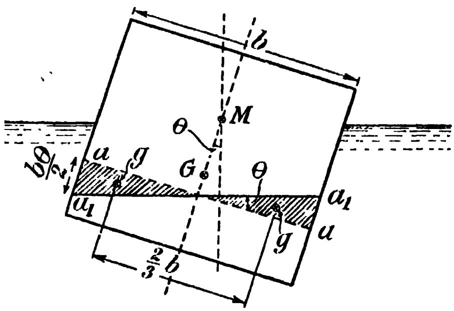


FIG. 24

the righting moment when the floating body has an angle of heel.

Consider the rectangular pontoon of Fig. 24, and let there be a quantity of free bilge water in the bottom, its surface being denoted by the line aa. Now imagine the pontoon to heel through the small angle θ radians; the surface of the bilge water will remain horizontal and is now represented by the line a_1a_1. This has the effect of moving the shaded wedge of bilge water from one side of the pontoon to the other; the centre of gravity of the pontoon and its contents are thus moved by a corresponding amount. The equation of Art. 18,

$$BM = \frac{I}{V}$$

is not affected by the relative movement of the bilge water, as this equation is based on the displacement of the external water only.

Let G = original centre of gravity of pontoon and contents before heeling

g = mass centre of wedge of bilge water

m = weight of wedge of bilge water

$$= w\left(\frac{1}{2} \times \frac{b}{2} \times \frac{b\theta}{2}\right) l$$

$$= \frac{wlb^2\theta}{8}$$

x = horizontal distance moved by mass centre of wedge

= $\frac{2}{3}b$ (approximately, if θ is small)

I = second moment of bilge water surface about longitudinal axis

$$= \frac{lb^3}{12}$$

Then, righting moment $= W\,\overline{MG}\,\theta - mx$

$$= W\,\overline{MG}\,\theta - \left(\frac{wlb^2\theta}{8} \times \frac{2}{3}b\right)$$

$$= W\,\overline{MG}\,\theta - w\theta\left(\frac{lb^3}{12}\right)$$

$$= W\,\overline{MG}\,\theta - wI\theta \quad . \quad . \qquad (1)$$

It will be noticed from Equation (1) that the effect of the free bilge water is to reduce the righting moment on a floating body.

21. **Transverse Oscillation of a Floating Body.** In Art. 16 it was shown that a floating body, when given a lateral heel, may be regarded as oscillating instantaneously about the metacentre M (Fig. 17) in the same manner as a pendulum oscillates about its point of suspension.

Let W = weight of floating body

m = metacentric height in feet

θ = angle of displacement (rads.) in t secs.

a = angular acceleration in rads. per sec.^{2}

$$= \frac{d^2\theta}{dt^2}$$

T = time of complete oscillation in secs.

I = moment of inertia of body about its centre of gravity G

k = radius of gyration about G

Then, $I = \frac{W}{g}k^2$

It is assumed that the axis of oscillation passes through G; this is approximately correct if m is small.

Assuming θ to be small,

righting moment $= Wm\theta$ (Art. 16)

and inertia torque $= -I\alpha$

then, $Wm\theta = -I\alpha$

Substituting for I and α,

$$Wm\theta = -\frac{W}{g}k^2 \times \frac{d^2\theta}{dt^2}$$

from which $$\frac{d^2\theta}{dt^2} + \frac{mg}{k^2}\theta = 0$$

The solution of this differential equation is

$$\theta = A \sin\left(\sqrt{\frac{mg}{k^2}} \times t\right) + B \cos\left(\sqrt{\frac{mg}{k^2}} \times t\right)$$

where A and B are the constants of integration.

When $t = 0$, $\theta = 0$, hence $B = 0$

then, $$\theta = A \sin\left(\sqrt{\frac{mg}{k^2}} \times t\right)$$

Also, when $t = \frac{T}{2}$, $\theta = 0$

then, $$0 = A \sin\left(\sqrt{\frac{mg}{k^2}} \times \frac{T}{2}\right)$$

As A cannot be zero, then,

$$\sin\left(\sqrt{\frac{mg}{k^2}} \times \frac{T}{2}\right) = 0$$

hence,

$$\sqrt{\frac{mg}{k^2}} \times \frac{T}{2} = \pi$$

from which $$T = 2\pi\sqrt{\frac{k^2}{mg}} \quad . \quad . \quad . \quad . \quad (1)$$

From this equation the time of oscillation of the floating body can be calculated if m and k are known.

It will be noticed from Equation (1) that if m is increased the time of oscillation is shorter. This is noticeable with a ship travelling in ballast, as the effect of the ballast is to lower the centre of gravity G and thus increase m.

EXAMPLE.

Define the metacentric height of a floating body. Derive the formula for the period of rolling about the horizontal longitudinal axis through the centre of gravity, and state the assumptions made in deriving the formula.

The metacentric height of a ship is 2 ft. and the period of rolling is 20 seconds, what is the value of the relevant radius of gyration? (London Univ.)

Assumptions made are (1) the angle θ is small; (2) the metacentric height is small so that the axis of oscillation is approximately through G.

Using Equation (1),

$$T = 2\pi\sqrt{\frac{k^2}{mg}}$$

that is, $$20 = 2\pi\sqrt{\frac{k^2}{2 \times 32\cdot2}}$$

from which $$k = 25\cdot4 \text{ ft.}$$

EXAMPLES 2.

(1) A ship has a displacement of 2,200 tons in sea water. Find the volume of the ship below the water line. 1 cu. ft. of sea water weighs 64 lb.

Ans.—77,000 cu. ft.

(2) A solid cube of wood of specific gravity of ·9 floats in water with a face parallel to water plane. If the length of one edge is 4 in., find the metacentric height.

Ans.—·17 in.

(3) A pontoon of 1,500 tons displacement floats in fresh water. A weight of 18 tons is moved 24 ft. across the deck; this causes a pendulum 10 ft. long to move 4½ in. horizontally. Find the metacentric height of the pontoon.

Ans.—7·68 ft.

(4) A rectangular pontoon weighing 240 tons has a length of 60 ft. The centre of gravity is 1 ft. above the centre of the cross-section, and the metacentric height is to be 4 ft. when the angle of heel is 10°. The freeboard must not be less than 2 ft. when the pontoon is vertical. Find the breadth and height of the pontoon, if floating in fresh water.

Ans.—21·8 ft. and 8·6 ft.

(5) State the conditions which govern the stability or instability of a floating vessel.

A buoy carrying a beacon light has the upper portion cylindrical, 7 ft. diameter and 4 ft. deep. The lower portion, which is curved, displaces a volume of 14 cu. ft., and its centre of buoyancy is situated 4 ft. 3 in. below the top of the cylinder. The centre of gravity of the whole buoy and beacon is situated 3 ft. below the top of the cylinder, and the total displacement is 2·6 tons. Find the metacentric height. [Weight of sea water, 64 lb. per cu. ft.] (London Univ.)

Ans.—1·101 ft.

(6) A rectangular pontoon, 35 ft. long, 24 ft. broad, 8 ft. deep, weighs 70 tons. It carries on its upper deck a boiler 16 ft. diameter weighing 50 tons. The centres of gravity of the boiler and pontoon may be assumed to be at their centres of figure and in the same vertical line. Find the metacentric height. [Weight of sea water, 64 lb. per cu. ft.] (London Univ.)

Ans.—3·1 ft.

(7) A cylinder has a diameter of 12 in. and a relative density of 0·8. What is the maximum permissible length in order that it may float with its axis vertical? (London Univ.)

Ans.—10·6 in.

(8) A cylindrical buoy is 6 ft. in diameter and 8 ft. high and weighs 1·8 tons. Show that it will not float with its axis vertical in sea water. If one end of a vertical chain is fastened to the centre of the base, find the pull on the chain, in order that the buoy may just float with its axis vertical. (London Univ.)

Ans.—$MG = -1{\cdot}87$ ft.; 1·15 tons.

(9) If a floating body is assumed to roll about a fixed horizontal axis through its centre of gravity, prove that the period of rolling in seconds is given by

$$T = 2\pi\sqrt{k^2/gh}$$

in which h is the metacentric height, and k is the relevant radius of gyration.

Find the value of k for a ship which has a period of rolling of 20 sec. The displacement is 10,000 tons; the second moment of the load-water-plane about its fore and aft axis is $3{\cdot}5 \times 10^6$ ft.^{4}; and the centre of buoyancy is 8 ft. below the centre of gravity. (Sea water, 64 lb./ft.^{3}) (London. Univ.)

Ans.—25·57 ft.

(10) A log of wood of square section, 14 in. × 14 in., weighing 50 lb. per cu. ft., floats in water. One edge is depressed and released, causing the log to roll. Estimate the period of a roll. (London Univ.)

Ans.—4·94 sec.

(11) Obtain a formula for the metacentric height of a floating body. A solid right cone of wood weighing 44 lb. per cu. ft. is required to float in water with the axis vertical. Determine the minimum apex angle which will enable the cone to float in stable equilibrium. (London Univ.)

Ans.—39°.

CHAPTER III

THE FLOW OF A FLUID

22. **Flow of Water.** When a liquid is flowing along a passage, such as a pipe, it will be subjected to a resistance due to viscosity, or friction. If the velocity of flow is very small, the liquid will flow in lines parallel to the sides of the passage; such a flow is called a streamline flow. If the velocity is large, cross-currents or eddies will be formed causing greater resistance to flow; such a flow is known as a turbulent or eddy flow. Also, the velocity of the liquid is not uniform over the cross-section, being slower towards the sides of the passage. In engineering problems, however, it is usual to assume the velocity to be uniform over the cross-section and equal to the mean velocity.*

Any obstruction in the passage or any change of section or direction will interfere with the steady flow. This will cause eddies or transverse motions of the particles and, consequently, there will be an additional loss of energy due to the friction caused by these transverse currents.

The streamline flow of water may be examined by inserting a coloured powder or liquid in the water at certain points of the channel and examining the path of the colour bands thus formed.

Consider a pipe of a cross-sectional area of a sq. ft. containing water which is flowing with a velocity of v ft. per sec. (Fig. 25); the pipe is running full. Consider any section of the pipe; a quantity of water in the shape of a cylinder of length v and area a will pass by this section in 1 sec.

Or, quantity of water flowing = volume of cylinder

$$= av \text{ cu. ft. per sec.}$$

23. **Flow Through Channels of Varying Section.** If water is flowing through any channel or pipe, the quantity of water passing any transverse section in a given interval of time must be equal at all such sections, providing the depth of flow at any point remains constant.

* For viscous flow of a fluid see Chapter XII.

Let Fig. 26 represent a tapering pipe through which water is flowing. Let the pipe be running full.

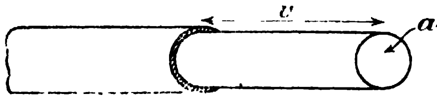


Fig. 25

Let A_a = area at section *aa*

A_b = area at section *bb*.

Then,

quantity of water passing section *aa* per sec. = quantity passing *bb* per sec.

Or, $$A_a v_a = A_b v_b$$

Therefore, $$\frac{A_b}{A_a}$$

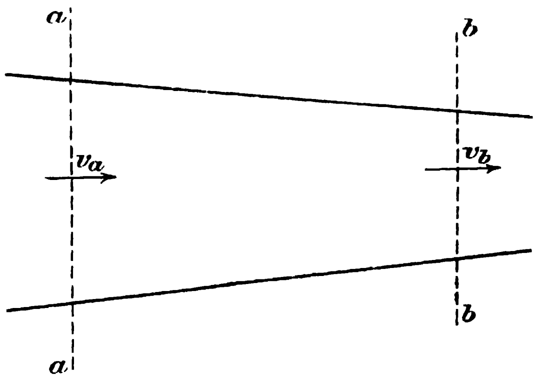


Fig. 26

24. **Work Done in Overcoming Pressure.** Let a large tank be full of water under a constant pressure of p lb. per sq. ft., and let water be forced into the tank through a small pipe of cross-sectional area of a sq. ft. (Fig. 27). Let v be the velocity in feet per second with which the water is forced through the pipe.

Then, work done per second in forcing water through pipe = Force × distance moved per second

$$= pa \times v$$

But, av = volume of water forced into tank per sec.

Therefore, work done $= p \times$ volume of flow per second

Or, work done $= wH \times$ volume

$= WH$

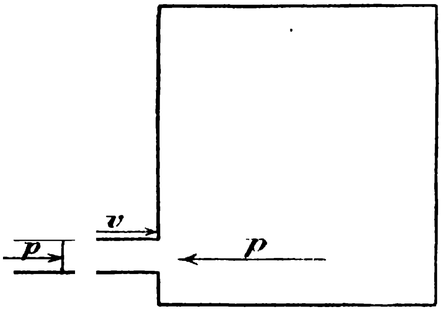


FIG. 27

where W = weight of water injected per second and H = equivalent static head in feet of water

EXAMPLE.

300 gallons of water are pumped into a tank per minute under a pressure of 20 lb. per sq. in. Find the horse-power required. 1 gallon water weighs 10 lb.

Weight of water per sec. $= \dfrac{300 \times 10}{60} = 50$ lb.

Volume of water per sec. $= \dfrac{50}{62 \cdot 4} = \cdot 802$ cu. ft.

Work done per sec. $= p \times$ volume

$= 20 \times 144 \times \cdot 802$

Horse-power required $= \dfrac{20 \times 144 \times \cdot 802}{550}$

$= 4 \cdot 2.$

ALTERNATIVE METHOD. Convert the pressure to pressure head in feet of water.

Then, horse-power $= \dfrac{WH}{550}$ where W = weight of water per sec.

Static head $= H = \dfrac{p}{w} = \dfrac{20 \times 144}{62 \cdot 4} = 46 \cdot 2$ ft. of water

Horse-power $= \dfrac{50 \times 46 \cdot 2}{550} = 4 \cdot 2$

25. Velocity Head. Consider water flowing from a tank under a constant head H (Fig. 28). Let v be the velocity of the water in feet per second. Consider a small quantity of water of weight W on the surface at the top of the tank. This quantity will have a potential energy of WH. This same

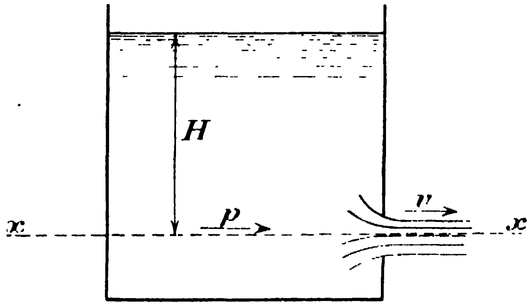


FIG. 28

quantity of water, when issuing through the orifice, may be looked upon as having fallen through the height H and converted its potential energy to kinetic energy. Then, ignoring frictional losses,

Loss of potential energy = gain of kinetic energy

Or, $$WH = \frac{Wv^2}{2g}$$

Therefore, $$H = \frac{v^2}{2g}$$

Or, $$v = \sqrt{2gH}$$

By making use of these equations, the energy of moving water may be given as a static head in feet of water; this static head is known as the velocity head.

ALTERNATIVE PROOF.

Let p = intensity of pressure of water on line xx

Then, $p = wH$

Consider the water as being forced out of orifice by pressure p.

Let a = area of jet

W = weight of water issuing per second

Work done on water $= p \times$ volume of water moved per sec.

$= p \times av$

Work done on water $=$ gain of kinetic energy

$$\text{Or,} \qquad p \times av = \frac{Wv^2}{2g}$$

$$\text{Or,} \qquad wH \times av = \frac{Wv^2}{2g}$$

$$\text{But,} \qquad W = wav$$

$$\text{Therefore,} \qquad H = \frac{v^2}{2g}$$

26. **The Total Energy of Water.** The total energy of a particle of water at any instant is the sum of its potential energy, its kinetic energy, and its pressure energy. The potential energy is reckoned above a convenient datum level. Let Z be the height of particle considered above chosen datum level.

Potential energy per lb. of water $= Z$

Kinetic ,, ,, ,, $= \dfrac{v^2}{2g}$

Pressure ,, ,, ,, $= \dfrac{p}{w}$

Then, total energy per lb. $= Z + \dfrac{p}{w} + \dfrac{v^2}{2g}$

For any mass of water in which there is a continuous connection between all the particles, the total energy of each particle is the same. This is known as Bernoulli's theorem and is very important in solving problems dealing with the flow of liquids.*

Consider the vessel in Fig. 29; let the water flow through an orifice in the side of the vessel with a velocity v under the static head H. Apply Bernoulli's theorem to the points A, B, and C.

Total energy at A = total energy at B = total energy at C

$$Z_A + \frac{p_A}{w} + \frac{v_A^2}{2g} = Z_B + \frac{p_B}{w} + \frac{v_B^2}{2g} = Z_C + \frac{p_C}{w} + \frac{v_C^2}{2g}$$

* For the application of Bernoulli's equation to gases, see Art. 185.

Assuming the line xx to be the datum level, and ignoring the atmospheric pressure, which is constant throughout, the equations become

$$H + 0 + 0 = 0 + \frac{p_B}{w} + 0 : \quad 0 + 0 + \frac{v}{2g}$$

Or, $$H = \frac{p_B}{w} \qquad - \; 2g$$

It should be noticed that no account has been taken of any frictional losses which may occur between the points chosen.

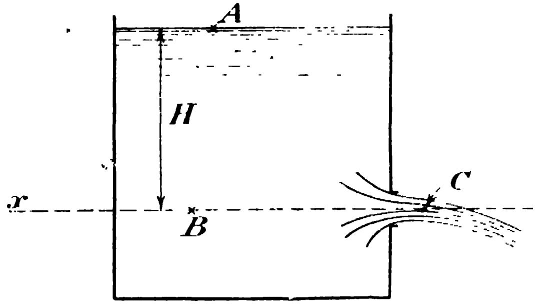


Fig. 29

Any such losses should be added or subtracted from one side of the equation.

As an example, supposing there is a loss of head between B and C equal to h ft. of water. Then,

$$Z_B + \frac{p_B}{w} + \frac{v_B^2}{2g} = Z_C + \frac{p_C}{w} + \frac{v_C^2}{2g} + h$$

Proof of Bernoulli's Theorem. Consider water flowing through the non-uniform pipe of Fig. 30. The pipe is running full and under pressure. Consider the volume of water between the two sections AA and BB.

Let Z, p, v, and a be the height above datum, pressure, velocity, and area of pipe respectively at section AA. Let Z_1, p_1, v_1, and a_1 be the corresponding values at BB. Let the whole quantity of water between AA and BB move to the position $A'A'$, $B'B'$. the movement being small.

Let distance between AA and $A'A'$ $= dl$

BB and $B'B'$ $= dl_1$

Then, $\quad a\,dl = a_1 dl_1 \quad . \quad . \quad . \quad . \quad . \quad (1)$

This movement of the whole quantity of water between AA and BB is equivalent to moving the quantity between AA and $A'A'$ to the position between BB and $B'B'$; the remainder of the water between $A'A'$ and BB may then be looked upon as stationary.

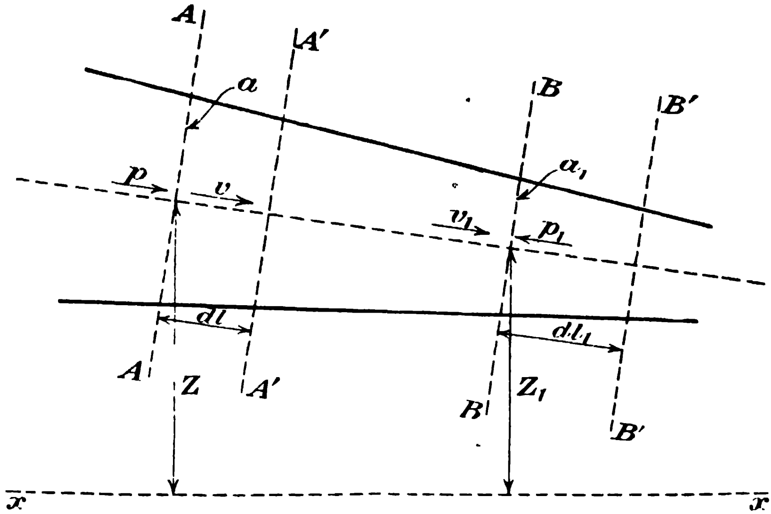


FIG. 30

Let W = weight of water between AA and $A'A'$

$$= w\,a\,dl \quad \quad (2)$$

$$= w\,a_1\,dl_1 \quad \quad (3)$$

Work done by pressure at AA $= p\,a\,dl$

BB $= -p_1\,a_1\,dl_1$

Net work done by pressure $= p\,a\,dl - p_1\,a_1\,dl_1$

" " " " $= (p - p_1)\,a\,dl$ (From Eq. 1)

$$= W\frac{(p - p_1)}{w} \quad \text{(From Eq. 2)}$$

Loss of potential energy $= W(Z - Z_1)$

Gain of kinetic energy $= \dfrac{W}{2g}(v_1^2 - v^2)$

Then,

loss of potential energy + work done by pressure = gain of kinetic energy

That is, $$W(Z - Z_1) + W\frac{(p - p_1)}{w} = \frac{W}{2g}(v_1^2 - v^2)$$

Therefore, $$Z + \frac{p}{w} + \frac{v^2}{2g} = Z_1 + \frac{p_1}{w} + \frac{v_1^2}{2g}$$

EXAMPLE.

Water is flowing down a vertical tapering pipe 6 ft. long. The top of the pipe has a diameter of 4 in., the diameter of the bottom of the pipe is 2 in. If the quantity of water flowing is 300 gallons per minute, find the difference of pressure between the top and bottom ends of the pipe.

Let v_1, p_1, Z_1, and a_1 refer to lower end of pipe.
v_2, p_2, Z_2, and a_2 refer to top end of pipe.

Quantity of water flowing per sec. $= \frac{300 \times 10}{60 \times 62{\cdot}4} = {\cdot}802$ cu. ft.

Area of lower end of pipe $= \frac{\pi}{4} \times 2^2 = 3{\cdot}14$ sq. in.

Area of top end of pipe $= \frac{\pi}{4} \times 4^2 = 12{\cdot}56$ sq. in.

$$v_1 = \frac{\text{quantity}}{\text{area in sq. ft.}} = \frac{{\cdot}802 \times 144}{3{\cdot}14} = 36{\cdot}8 \text{ ft. per sec.}$$

$$v_2 = \frac{{\cdot}802 \times 144}{12{\cdot}56} = 9{\cdot}2 \text{ ft. per sec.}$$

Applying Bernoulli's equation to both ends of pipe, and taking the datum level through the lower end,

$$Z_1 + \frac{p_1}{w} + \frac{v_1^2}{2g} = Z_2 + \frac{p_2}{w} + \frac{v_2^2}{2g}$$

$$0 + \frac{p_1}{w} + \frac{36{\cdot}8^2}{64{\cdot}4} = 6 + \frac{p_2}{w} + \frac{9{\cdot}2^2}{64{\cdot}4}$$

$$\frac{p_2 - p_1}{w} = 21{\cdot}1 - 1{\cdot}31 - 6$$

$$= 13{\cdot}79 \text{ ft. of water}$$

Or, $$p_2 - p_1 = \frac{13{\cdot}79 \times 62{\cdot}4}{144}$$

$$= 5{\cdot}97 \text{ lb. per sq. in.}$$

27. The Venturi Meter. A practical application of Bernoulli's theorem is found in the Venturi meter*, an instrument for measuring the quantity of water flowing through a pipe. The meter, in its simplest form, consists of a short length of pipe, tapering to a narrow throat in the middle (Fig. 31). Tubes enter the pipe at the enlarged end and at the throat, by means of which the pressure of the water at these sections may be measured. Piezometer tubes may be used, or the tubes may

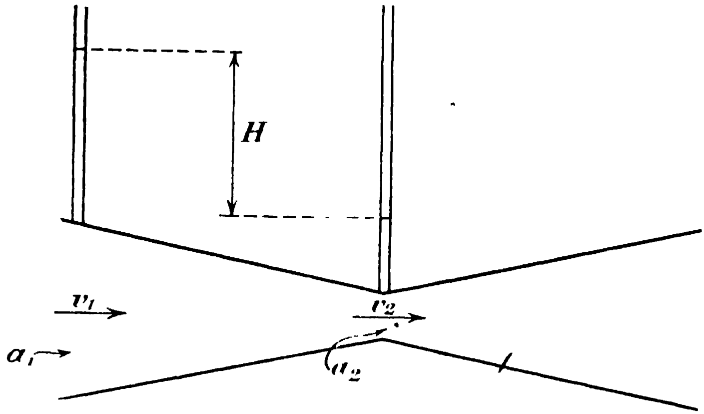


FIG. 31

be connected to a U-tube. As the water flows through the meter the velocity will increase at the throat owing to the reduction of area; consequently the pressure will be reduced. This reduction of pressure is measured by means of the piezometer tubes. Then, by applying Bernoulli's equation to the enlarged end and to the throat, the quantity of water flowing may be calculated.

Let H = difference of pressure head in feet of water in the piezometer tubes

a_1 = area of enlarged end in square feet

a_2 = area of throat in square feet

q = quantity of water flowing in cubic feet per second

v_1 = velocity of water at enlarged end

v_2 = velocity of water at throat

Then, $q = a_1v_1 = a_2v_2$

Therefore, $$v_1 = v_2\frac{a_2}{a_1} \qquad (1)$$

* For a description of an actual Venturi meter see Art. 149. For flow of gas through a Venturi meter see Art. 190.

Applying Bernoulli's equation, and assuming the meter to be horizontal,

$$\frac{p_1}{w} \quad \frac{v_1}{2g} = \frac{p_2}{w} + \frac{-z}{2g}$$

Or, $$\frac{p_1}{w} - \frac{p_2}{w} = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

But, $$\frac{p_1}{w} - \frac{p_2}{w} = H$$

Therefore, $$H = \frac{v_2^2 - v_1^2}{2g}$$

Substituting for v_1 from Eq. 1,

$$H = \frac{v_2^2}{2g}\left(1 - \frac{a_2^2}{a_1^2}\right)$$

Therefore, $$v_2 = \frac{a_1}{\sqrt{a_1^2 - a_2^2}} \sqrt{2gH}$$

$$q = a_2 v_2$$

$$= \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2g} \sqrt{H}$$

But, $\frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2g}$ is a constant for any one meter; let this constant equal c.

Then, $$q = c\sqrt{H} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In this case H is the theoretical head, as no frictional loss have been taken into account. In practice it is found tha there is a loss of head in the meter between the enlarged end and the throat; consequently the water will not rise so high in the pressure tube at the throat. This means that a larger difference of head will be measured. In order to allow for this, a coefficient k is introduced into the equation, the magnitude of k being found experimentally.

Let h = difference of head, in feet of water, actually measured

Then, $$q = k c \sqrt{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But, $\quad q \quad c\sqrt{H}$

Therefore, $k\,c\,\sqrt{h} = c\,\sqrt{H}$

From which, $k = \sqrt{\dfrac{H}{h}}$ . . . . (4)

The actual head measured, h, is known as the Venturi head.

In the converging cone of the meter, h will be larger than H; then k will be less than unity. An average value of k is ·97.

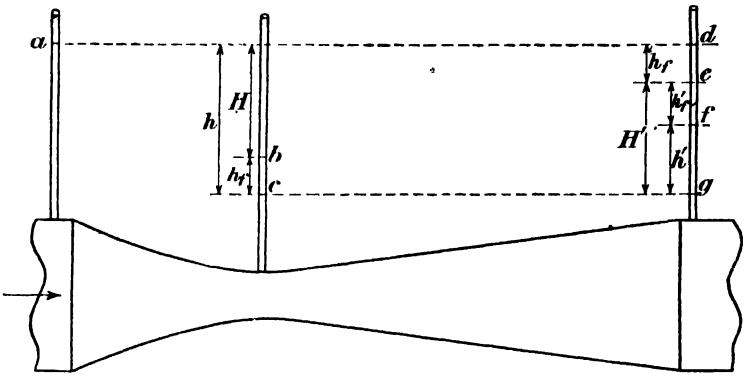


FIG. 32

The loss of head in the meter will be partly due to friction and partly due to shock caused by a change of section; consequently, k will not be truly a constant for all velocities; but the variation is slight.

The Venturi meter is not accurate for very low velocities on account of the variation of k.

It will be noticed that there is a limit to the ratio of the diameters of the throat and enlarged end. The larger this ratio is, the smaller will be the pressure in the throat; if the pressure in the throat falls below 8 ft. of water absolute, dissolved gases and vapour will be given off from the water; this will interfere with the flow (Art. 2). Hence, the limiting ratio of the diameters is reached when the throat pressure is approximately 8 ft. of water absolute.

The coefficient k will have a different value for the converging and diverging cones of the meter. In the converging cone the theoretical head is less than the actual head; whilst in the diverging cone the theoretical head is greater than the actual

head. Consider the Venturi meter shown in Fig. 32; let pressure tubes be fitted at both enlarged ends and throat. Assume the water is flowing from left to right.

Let h_f = head lost in converging cone
h'_f = head lost in diverging cone
H' = theoretical difference of head in diverging cone
h' = actual difference of head in diverging cone.

Let the water level at the left enlarged end be at *a*. Then, if there were no losses in the meter, the water level at the right enlarged end would be at the same level *d*. The friction loss in the converging cone reduces this water level to *e*; whilst the frictional loss in the diverging cone further reduces the level to *f*.

Referring to the converging cone only, from Equation (4),

$$H = k^2 h$$

But, $h_f = h - H$

Therefore, $$h_f = h(1 - k^2) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Referring to the diverging cone, let k' be the coefficient of the diverging cone. Then, if there were no frictional loss in this cone, the water level at the enlarged end would rise to *e*; the theoretical difference of head between this section and the throat would then be the height *eg*. But owing to the frictional loss the water level only reaches *f*. Then the difference of head actually measured is *fg*.

Then, quantity flowing $= c\sqrt{H'} = k'c\sqrt{h'}$

Therefore, $$H' = k'^2 h' \quad . \quad . \quad . \quad (6)$$

But, $h'_f = H' - h'$

Therefore, $$h'_f = h'(k'^2 - 1) \quad . \quad . \quad (7)$$

Also, from Fig. 32,

$$fg = dg - df$$

Or, $$h' = h - h_f - h'_f \quad . \quad . \quad (8)$$

It will be noticed that k' is greater than unity.

EXAMPLE 1.

State and prove Bernoulli's theorem. The difference of head registered in the two limbs of a mercury gauge, with water above the mercury, connected to a Venturi meter was 7 in. The diameter of the pipe and the throat of the meter are 6 in. and 3 in. respectively. The coefficient of the meter is ·97. Find the discharge through the meter. (London Univ.)

Difference of head in feet of water $= \dfrac{7(13{\cdot}6 - 1)}{12}$ (Art. 7)

$= 7{\cdot}35$ ft.

$$a_1 = \frac{\pi}{4}({\cdot}5)^2$$

$$a_2 = \frac{\pi}{4}({\cdot}25)^2$$

$$c = \frac{a_1 a_2 \sqrt{2g}}{\sqrt{a_1^2 - a_2^2}} = \frac{\pi}{4} \frac{\times {\cdot}25 \times {\cdot}0625 \sqrt{64{\cdot}4}}{\sqrt{{\cdot}0625 - {\cdot}0039}}$$

$$= {\cdot}407$$

Quantity $= k c \sqrt{h}$ (Eq. 3)

$$= {\cdot}97 \times {\cdot}407 \sqrt{7{\cdot}35}$$

$$= 1{\cdot}07 \text{ cu. ft. per sec.}$$

EXAMPLE 2.

Show that in a Venturi meter the quantity of water passing through the meter will only be proportional to the root of the " Venturi head " if the head lost in friction is proportional to the head lost due to increased velocity.

A Venturi meter placed in a 3 in. diameter pipe has a throat diameter of 1 in. The constant of the meter is ·97. Determine the number of cubic feet passing per minute when the Venturi head is 16·2 in. of water.

If the frictional loss in the diverging cone is double that in the converging cone, find the total head lost in the meter due to friction when the water is passing at the above rate. (London Univ.)

This question assumes that the whole of the head lost in the meter is due to friction.

The coefficient k can only be a constant if $h_f \propto H$; because

$$k = \sqrt{\frac{H}{h}} \qquad \text{(From Eq. 4)}$$

Also, $h = H + h_f$

Let $h_f = mH$ where m is a constant

Then, $k = \sqrt{\dfrac{H}{H + mH}} = \sqrt{\dfrac{1}{1 + m}} = a$ constant

$$\text{Quantity per sec.} = k \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2g} \sqrt{H}$$

$$= {\cdot}97 \left[\frac{\frac{\pi}{4}\left(\frac{1}{144} \times \frac{1}{16}\right)}{\sqrt{\frac{1}{256} - \frac{1}{20{,}700}}} \right] 8{\cdot}02 \sqrt{\frac{16{\cdot}}{12}}$$

$$= {\cdot}97 \times {\cdot}0055 \times 8{\cdot}02 \times 1{\cdot}16 = {\cdot}0496 \text{ cu. ft.}$$

Quantity per min. $= \cdot 0496 \times 60 = 2{\cdot}98$ cu. ft.

From Eq. (5), $h_f = h\,(1 - k^2)$

$$= \frac{16{\cdot}2}{12}(1 - \cdot 97^2) = \cdot 078 \text{ ft.}$$

Head lost in diverging cone $= 2 \times \cdot 078 = \cdot 156$ ft.

Total head lost $= \cdot 078 + \cdot 156$

$= \cdot 234$ ft.

28. **Horse-power of Jet of Water.** The horse-power of a jet of water may be obtained by dividing the kinetic energy of the jet per second by 550.

Let a = area of cross-section of jet in square feet

v = velocity of jet in feet per second

W = weight of water flowing per second

$= w\,a\,v$

Then, kinetic energy of jet $= \dfrac{Wv^2}{2g}$ ft. lb. per sec.

$$\text{Horse-power} = \frac{Wv^2}{2g\ 550}$$

$$= \frac{w\,a\,v^3}{2g\ 550}$$

EXAMPLE.

A jet of water has a velocity of 20 ft. per sec. If the diameter of the jet is 2 in., find the horse-power.

$$\text{Area of jet} = \frac{\pi}{4}\frac{2^2}{144}$$

$$= \cdot 0218 \text{ sq. ft.}$$

$$\text{Horse-power} = \frac{w\,a\,v^3}{2g\ 550}$$

$$= \frac{62{\cdot}4 \times \cdot 0218 \times 20^3}{64{\cdot}4 \times 550}$$

$$= \cdot 308$$

29. **The Radial Flow of Water.** Consider water flowing radially between two horizontal circular flat plates placed

parallel with a small distance between them (Fig. 33). The space between the plates is full of water. Let the water flow up a central pipe and then flow radially outwards between the plates. The outside of the plates is open to the atmosphere, so that the water will be discharged at atmospheric pressure.

Let v_o = velocity of water in pipe
p_o = absolute pressure of water in pipe
a_o = area of cross-section of pipe
p_a = pressure of atmosphere
v_a = velocity of water when leaving plates

As the water flows between the plates radially outwards, the area of flow will increase; therefore, the velocity will decrease. This will cause an increase in pressure.

Consider the total energy of the water at A, just inside the pipe, and at B which is at the outer edge of the plates.

Let R = radius of plates at B
and t = distance between the plates

Then,

Energy at A = Energy at B

$$\frac{p_o}{w} + \frac{v_o^2}{2g} = \frac{p_a}{w} + \frac{v_a^2}{2g} = H$$

where H is a constant.

As p_o, v_o, and p_a are known, this equation will give v_a.

Consider any point X at a radius of x from the centre of the plates. Let v_x be velocity of water at this point and p_x the pressure. Then, as quantity of water flowing is a constant at all sections,

$$v_a \times 2\pi Rt = v_x \times 2\pi xt$$

Or,

$$v_x = v_a \frac{R}{x} \qquad (1)$$

Also, total energy at $X = H$

$$= \frac{p_x}{w} + \frac{v_x^2}{2g}$$

Substituting from Eq. (1),

$$\frac{p_x}{w} = H - \frac{v_a^2 R^2}{2g\, x^2} \qquad (2)$$

Thus, the pressure at any point varies with the square of the radius at that point and will increase towards the outer edge, the increase following a parabolic law. If the pressure

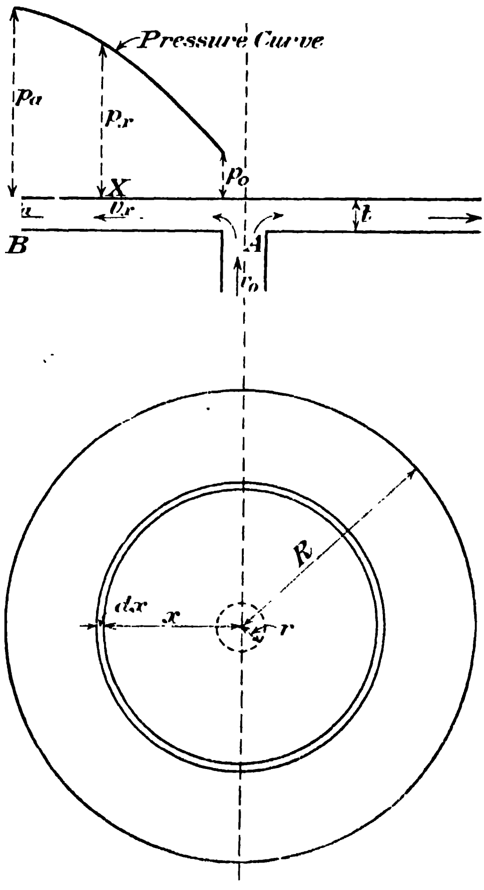


Fig. 33

at any radius is plotted as shown in Fig. 33, the parabolic curve thus obtained is known as Barlow's curve.

Having found the intensity of pressure at any radius, the total static pressure on the plate may be obtained by finding an equation for the pressure on a thin ring and integrating between the required limits.

Let r = radius of pipe.

Consider the pressure at X on a thin ring of thickness dx. From Equation (2),

$$p_x = w\left(H - \frac{k}{x^2}\right)$$

where the constant $k = \dfrac{v_a^2 R^2}{2g}$

Area of ring $= 2\pi\, x\, dx$

Total pressure on ring $= p_x\, 2\pi\, x\, dx$

$$= 2\pi\, w\left(H - \frac{k}{x^2}\right) x\, dx$$

Total static pressure due to water on upper plate $\Big\} = 2\pi w \displaystyle\int_r^R \left[H\, x\, dx - \frac{k}{x}\, dx\right] + p_o \pi r^2$

$$= 2\pi w\left[\frac{Hx^2}{2} - k \log_e x\right]_r^R + p_o \pi r^2$$

$$= 2\pi w\left\{\frac{H}{2}(R^2 - r^2) - k \log_e \frac{R}{r}\right\} + p_o \pi r^2 \quad . \quad . \quad . \quad . \quad (3)$$

This is the total upward absolute static pressure. If the atmosphere is pressing on the outside of the plate, the net static pressure will be the total atmospheric pressure on the plate minus the above water pressure.

Total atmospheric pressure on plate $= p_a\, \pi\, R^2$

It will be noticed that the dynamic force due to the entering water has not been included.

The principle is made use of in the nozzles of fire hydrants in order to produce an even distribution of flow.

The same reasoning will hold when the water is flowing radially inwards, passing away down the centre pipe.

EXAMPLE.

Water flows radially outwards between two horizontal discs which are $\frac{1}{4}$ in. apart and 12 in. diameter. The water enters at the centre of the lower disc through a 2 in. diameter pipe, with a velocity of 20 ft. per sec. Find the pressure of the water in this pipe if the pressure at the outer edge of the discs is atmospheric. Find also the resultant static pressure on the upper disc. Neglect the dynamic force of the entering water

Using the notation of Fig. 33,

$$v_a = \frac{v_o r^2 \pi}{2\pi Rt}$$

$$= \frac{20 \times 1}{2 \times 6 \times \cdot 5} = 3\cdot 33 \text{ ft. per sec.}$$

Applying Bernoulli's equation,

$$\frac{p_o}{w} + \frac{v_o^2}{2g} = \frac{p_a}{w} + \frac{v_a^2}{2g}$$

$$\frac{p_o}{w} = 34 + \frac{3\cdot 33^2}{64\cdot 4} - \frac{20^2}{64\cdot 4}$$

$$= 28 \text{ ft. of water (absolute)}$$

Let $\qquad H = \dfrac{p_a}{w} + \dfrac{v_a^2}{2g} = 34 + \cdot 173 = 34\cdot 173$

Also $\qquad k = \dfrac{v_a^2 R^2}{2g} = \cdot 173 \times \cdot 5^2 = \cdot 0432$

Using Eq. (3),

Total upward pressure on plate

$$= 2\pi w \left\{ \frac{34\cdot 173}{2}\left(\cdot 5^2 - \frac{1}{12^2}\right) - \cdot 0432 \log_e 6 \right\} + p_o \pi r^2$$

$$= 2\pi \; 62\cdot 4\{(17\cdot 086 \times \cdot 243) - (\cdot 0432 \times 2\cdot 303 \times \cdot 778)\} + 12\cdot 1\pi$$

$$= 1598 + 38$$

$$= 1636 \text{ lb.}$$

Downward pressure of atmosphere $\Big\} = p_a \pi R^2$

$$= 14\cdot 7 \times \pi \times 6^2 = 1660 \text{ lb.}$$

Net pressure on plate $= 1660 - 1636$

$$= 24 \text{ lb.}$$

30. **Centrifugal Head Impressed on Revolving Liquid.** A rotating fluid is called a vortex.* If the fluid is rotating freely without any external forces being impressed upon it, it is called a free vortex. An example of a free vortex is the whirlpool formed in the emptying of a wash basin having a

* For further work on vortices see Art. 210.

central drain. If the fluid is rotated by an external force the vortex is termed a forced vortex. A forced vortex will have a centrifugal head impressed on the liquid, caused by its rotation.

Referring to Fig. 34, imagine an arm containing water to be rotating in a horizontal plane about the centre O and with an angular velocity of ω. Let the arm be full of water between a radius of R_1 and R_2 and let the cross-sectional area of the arm be a.

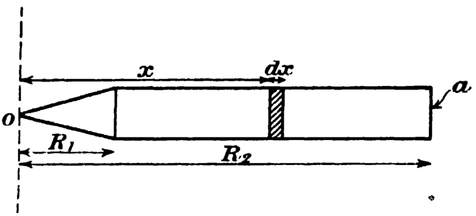


FIG. 34

Consider a small section of the water of thickness dx and at a radius of x.

Then, volume of small section of water $= a\,dx$

weight ,, ,, ,, $= w\,a\,dx$

Centrifugal force acting on water considered $= \dfrac{(w\,a\,dx)}{g}\omega^2 x$

Total centrifugal force impressed on whole of rotating water $\Big\} = \displaystyle\int_{R_1}^{R_2} \frac{w\,a\,dx}{g}\omega^2 x$

$$= \frac{w\,a\,\omega^2}{2g}\Big[x^2\Big]_{R_1}^{R_2}$$

$$= \frac{w\,a\,\omega^2}{2g}(R_2^2 - R_1^2)$$

Let $v_1 =$ tangential velocity at radius of R_1

and $v_2 =$,, ,, ,, R_2

Then, total centrifugal force impressed $= \dfrac{w\,a}{2g}(v_2^2 - v_1^2)$,

since $v_1 = \omega R_1$ and $v_2 = \omega R_2$

Intensity of pressure at end of arm due to centrifugal force $\Big\} = \dfrac{w}{2g}(v_2^2 - v_1^2)$

Centrifugal head impressed $= \dfrac{p}{w} = \dfrac{v_2^2}{2g} - \dfrac{v_1^2}{2g}$

Thus, the centrifugal head impressed on a revolving fluid is the difference between the tangential velocity heads.

This is the principle of the centrifugal pump, which obtains its lifting power from this head.

Alternative Proof. A more general proof for the centrifugal head impressed on revolving liquid may be obtained by con-

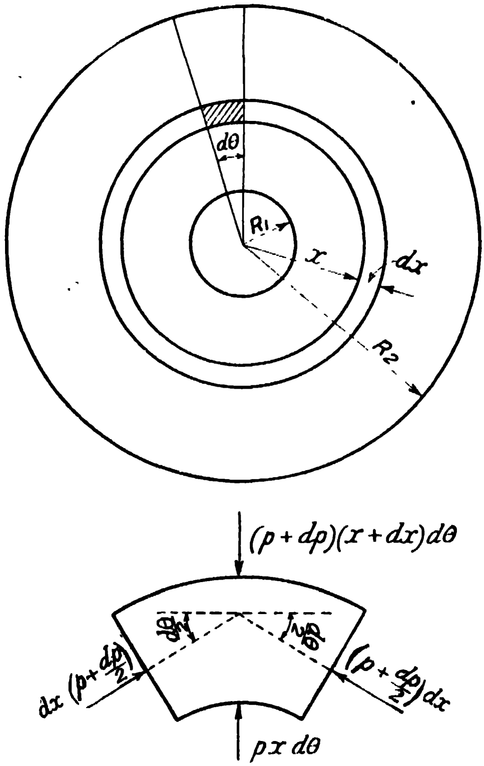


FIG. 35

sidering the annular ring of liquid revolving with an angular velocity ω (Fig. 35). Let R_1 and R_2 be the internal and external radii, and consider a thin ring of the liquid of radius x and thickness dx. Consider a portion of this thin ring subtending a small angle $d\theta$ at the centre and let p be the intensity of pressure on the inside of the element, due to the centrifugal force. Then the centrifugal pressure will increase by dp over the thickness of the ring dx. Consider the whole

annular ring to be of unit thickness in the plane of the paper; then,

area of inside of element	$= x\,d\theta$
area of outside element	$= (x + dx)d\theta$
area of sides of element	$= dx$
intensity of pressure on outside of element	$= p + dp$
intensity of pressure on sides of element	$= p + \dfrac{dp}{2}$
Weight of element	$= w(x\,d\theta)dx$
Centrifugal force on element	$= \dfrac{(w\,x\,d\theta\,dx)\,\omega^2\,x}{g}$

Consider the enlarged view of element (Fig. 35); the normal forces due to the pressure of the liquid are shown in the figure. These, together with the centrifugal force, keep the element in equilibrium. Hence, by resolving radially, the required equation may be obtained.

Resolving radially, and assuming the sine of a small angle to be equal to the angle in radians,

$$-p\,x\,d\theta - 2\left(p + \frac{dp}{2}\right)dx\frac{d\theta}{2} + (p + dp)\,(x + dx)\,d\theta = \frac{wx\,d\theta\,\omega^2 x\,dx}{g}$$

Dividing throughout by $d\theta$, and ignoring all small quantities of the second order,

$$dp = \frac{w\,\omega^2\,x\,dx}{g}$$

Integrating between R_1 and R_2,

$$\text{Centrifugal intensity of pressure} = \int dp = \int_{R_1}^{R_2} \frac{w\,\omega^2\,x\,dx}{g}$$

$$= \frac{w\,\omega^2}{2g}\,(R_2^2 - R_1^2)$$

Then, $\qquad$ centrifugal head $= \dfrac{p}{w} = \dfrac{\omega^2}{2g}\,(R_2^2 - R_1^2)$

Or, $\qquad$ as $v_1 = \omega\,R_1$ and $v_2 = \omega\,R_2$,

$$\text{centrifugal head} = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

EXAMPLE.

Water enters a revolving turbine wheel at the centre and flows through the wheel in a radial direction. The wheel is running full. If the inlet radius of the wheel is 2 ft. and the outlet radius 3·5 ft., find the centrifugal head impressed on the water when the wheel is running at 300 revs. per min.

Velocity of wheel at inlet $= v_1 = 2\pi 2 \times \dfrac{300}{60}$

$= 62{\cdot}8$ ft. per sec.

Velocity of wheel at outlet $= v_2 = 62{\cdot}8 \times \dfrac{3{\cdot}5}{2}$

$= 110$ ft. per sec.

Centrifugal head $= \dfrac{v_2^2}{2g} - \dfrac{v_1^2}{2g}$

$= \dfrac{110^2 - 62{\cdot}8^2}{64{\cdot}4}$

$= 126{\cdot}7$ ft. of water.

31. **Revolving Cylinder of Liquid.** Consider a cylinder containing a liquid to be revolved about a vertical axis OC (Fig. 36). The surface of the liquid will take the shape of a paraboloid as shown. This is another example of a forced vortex.

Consider a small particle of the liquid at the point A on the surface. Let W be the weight of the particle. It will be in equilibrium under the action of three forces: the weight, the centrifugal force, and the pressure.

Let ω = angular velocity of cylinder

x = radius of particle

Centrifugal force on particle $= \dfrac{W}{g}\omega^2 x$

The centrifugal force will act horizontally outwards, and the weight vertically downwards. The resultant of these two will be opposed by the pressure of the fluid. As the latter must act normal to the surface, it follows that a tangent to the surface at A will be at right angles to the resultant of the centrifugal force and the weight.

It follows from Fig. 36 that

$$\frac{EF}{x} = \frac{W}{\frac{W\omega^2 x}{g}} \text{(Similar triangles)}$$

Therefore, $EF = \frac{g}{\omega^2}$

and is, therefore, a constant.

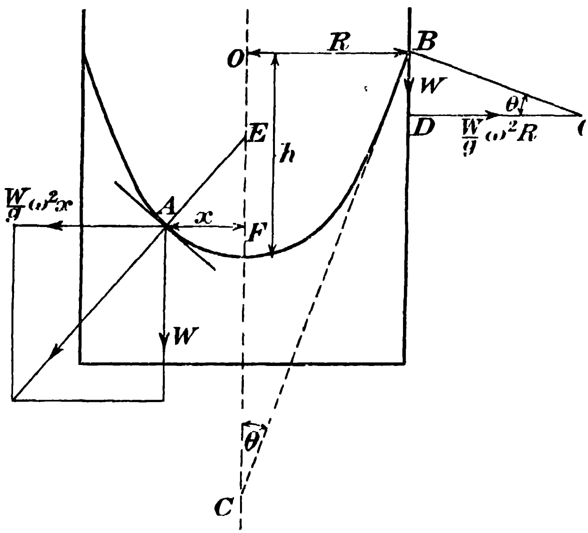


FIG. 36

As EF is the subnormal of the liquid surface, the shape of the surface is a paraboloid.

Consider the liquid at B.

Let $R =$ radius at B

$\theta =$ angle of inclination of surface at B to vertical

$h =$ height of paraboloid

Consider the similar triangles BDG and BOC,

$$\frac{BD}{DG} = \frac{BO}{OC}$$

Or, $$\frac{W}{\frac{W}{g}\omega^2 R} = \frac{R}{2h}$$

as OC will be twice the height of the paraboloid.

Therefore,

$$h = \frac{\omega^2 R^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$= \frac{v^2}{2g}$$

where v is the tangential velocity at the point B.

It will be noticed that h is a direct function of the square of the speed of rotation. This is made use of in a type of speedometer used in engine testing. A cylindrical glass vessel containing a liquid is rotated by the engine, the speed of which can be estimated by the height h of the paraboloid formed in the vessel.

EXAMPLE 1.

In order to measure the speed of a steam engine during a test, a glass cylinder containing oil is rotated on a vertical axis by the engine and is geared at twice the speed. If the paraboloid formed by the rotating liquid is 3 in. high, with a maximum radius of $1\frac{1}{2}$ in., find the number of revolutions per minute made by the engine at that instant.

Using Equation (1),

$$h = \frac{\omega^2 R^2}{2g}$$

Then, $$\frac{1}{4} = \frac{\omega^2}{64{\cdot}4}\left(\frac{1}{8}\right)^2$$

Therefore, $\omega = 32{\cdot}12$ radians per sec.

Speed of cylinder $= \dfrac{\omega}{2\pi}$

$$= \frac{32{\cdot}12}{2\pi} = 5{\cdot}11 \text{ revs. per sec.}$$

Speed of engine $= \dfrac{5{\cdot}11 \times 60}{2} = 153{\cdot}3$ revs. per min.

EXAMPLE 2.

A closed cylinder, 12 in. diameter and 0·1 in. deep, is completely filled with water. It is rotated about its axis, which is vertical, at 240 r.p.m. Calculate the total pressure of the water on each end. (A.M. Inst. C.E.)

Consider a vertical thin hollow cylinder of water of radius x and thickness dx.

From Equation (1),

$$\text{centrifugal head on thin cylinder} = \frac{\omega^2 x^2}{2g}$$

$$\text{hence, intensity of pressure at radius } x = p_x = wh$$

$$= \frac{w\,\omega^2 x^2}{2g}$$

Although this centrifugal pressure is horizontal it will also act vertically on top and bottom of the cylinder, as the pressure of water is transmitted in all directions.

Let R = radius of cylinder in question.

Then, total vertical pressure on top or bottom of cylinder due to centrifugal pressure

$$= \int_o^R p_x \times 2\pi x\,dx$$

$$= \int_o^R \frac{w\omega^2 x^2}{2g} \times 2\pi x\,dx$$

$$= \int_o^R \frac{2\pi\omega^2 x^3\,dx\,w}{2g}$$

$$= \frac{\pi\omega^2 R^4 w}{4g}$$

$$= \frac{\pi(2\pi 4)^2\,(\tfrac{1}{2})^4}{4 \times 32{\cdot}2} \times 62{\cdot}4 = 60{\cdot}9\,\text{lb.}$$

Hence, total pressure on top of cylinder = 60·9 lb.

Total pressure on bottom of cylinder

$$= \text{centrifugal pressure} + \text{weight of water}$$

$$= 60{\cdot}9 + (\pi R^2 \times \text{depth} \times w)$$

$$= 60{\cdot}9 + \left(62{\cdot}4 \times \pi(\tfrac{1}{2})^2 \times \frac{\cdot 1}{12}\right)$$

$$= 60{\cdot}9 + {\cdot}408$$

$$= 61{\cdot}308\,\text{lb.}$$

32. **Flow of Gases under Constant Head.** The velocity with which a gas will flow from one chamber to another may be

obtained in the same manner as for a liquid, providing the density in each chamber remains constant.

Let a gas flow from a chamber A through an orifice or pipe into a chamber B. Let the pressure in A remain constant and equal p_1 lb. per sq. in. Let the pressure in B also remain constant and equal p_2 lb. per sq. in. Then p_1 must be greater than p_2. Assume there is no change of temperature. Let w_1 be the density of the gas in A in pounds per cubic foot.

The head causing flow will be due to the difference of pressure in A and B. This head may be expressed as an equivalent static head in feet of gas under the same condition as the gas in A.

$$\text{Equivalent static head} \quad = H_1 = \frac{(p_1 - p_2)144}{w_1}$$

It should be noticed that such a head of gas could not actually exist under a constant density.

$$\text{Velocity of gas} \qquad = \sqrt{2gH_1}$$

If the gas being dealt with is atmospheric air, the barometer reading and temperature must be known in order to convert the standard density to the density under the required conditions. The density of air at 0° C. and 14·7 lb. per sq. in. may be taken as ·081 lb. per cu. ft. This should be converted to the required density by the law of gases

$$\frac{pv}{T} = \text{a constant},$$

where T is the absolute temperature.

33. **The Pitot Tube.** The Pitot tube is an instrument by which the velocity head of a flowing liquid may be measured. In its simplest form, it consists of a glass tube with the lower end bent through 90° (Fig. 37). It is placed in the moving liquid with the lower opening facing the direction of motion. The liquid flows up the tube until all its kinetic energy is converted to potential energy; the velocity of the liquid may then be estimated by the height of the liquid in the tube.

This instrument is often used for measuring the velocity of rivers.

Let $\quad h$ = height of liquid in tube above surface

$\qquad H$ = depth of tube in liquid

$\qquad v$ = velocity of liquid

Applying Bernoulli's equation to the points A and B, which are just outside and inside the mouth of the tube respectively,

$$\text{total energy at } A = \text{total energy at } B$$

$$H + h = H + \frac{v^2}{2g}$$

Therefore, $$h = \frac{v^2}{2g}$$

In practice, this is usually multiplied by a coefficient k then $$h = \frac{kv^2}{2g}$$

In well-formed instruments, k is equal to unity.

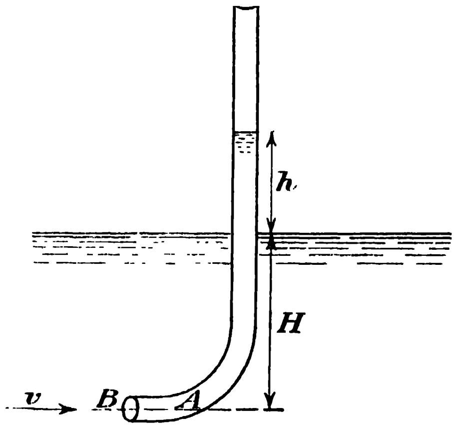


Fig. 37

Attempts have been made to deduce the value of h by considering the total force at B as equal to the rate of change of momentum; but this method gives results twice too high. This method obviously cannot be used, as there is a cone of still liquid in front of B which deviates the moving liquid from its sloping sides. This reduces the pressure on the tube, just as the windward side of a structure or building does not get the full force of the wind.

One type of Pitot tube consists of two tubes, one bent at the base, as in Fig. 37, and facing towards the motion of the water, and one straight tube open at the top end with a hole in the lower end parallel to the direction of motion. The velocity head is the difference of water level in the two tubes. The object of this is to eliminate any losses due to the tube.

If the Pitot tube is faced downstream, the water level in the tube is depressed by the amount h.

A view of an actual Pitot tube is shown in Fig. 38; this is known as the Amsler Hydrometrical Tube. It consists of two vertical tubes each having the lower end bent at right angles, one to point up-stream against the current, the other to point down-stream with the current; both lower ends are tapered to a fine nozzle. The water level in the tube facing down-stream is depressed by the amount h. In order to read the height of the water columns in the tubes a small hand pump is fitted at the top of the instrument, by means of which the water columns can be sucked up to any convenient height. The upper parts of the tubes are of glass and are fitted with a sliding graduated scale.

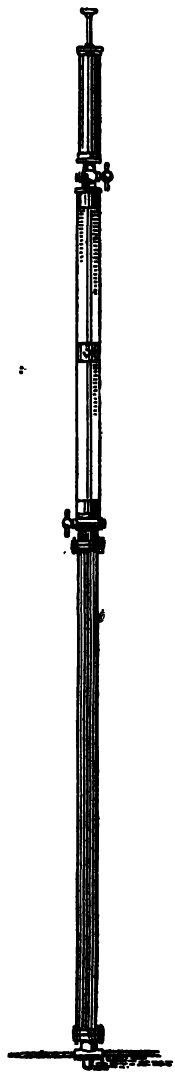


FIG. 38.—AMSLER HYDROMETRICAL TUBE

The difference of water level in the two tubes is proportional to the velocity head of the current. Let h_1 be the reading of the up-stream tube and h_2 be the reading of the down-stream tube.

Then, $$v = c\sqrt{h_1 - h_2}$$

where c is the constant of the instrument.

EXAMPLE 1.

The following observations were made for the purpose of calibrating a Pitot tube—

V = velocity of fluid	1·86	2·96	4·20	6·47	7·97	ft. per sec.
H = head	·756	1·72	3·50	9·12	14·40	in. of water

Plot V against $\sqrt{H}$ and determine the mean value of the constant for the tube. (London Univ.)

The values of V and $\sqrt{H}$ are shown plotted in Fig. 39, and a straight line is drawn a mean through the points. This line will pass through the origin as $V = 0$ when $H = 0$.

Let c = constant for the meter

Then, $$V = c\sqrt{H}$$

Therefore, $$c = \frac{V}{\sqrt{H}}$$

Using the values of V and $\sqrt{H}$ at the point B (Fig. 39),

$$c = \frac{8}{3 \cdot 7} = 2 \cdot 162$$

Then, $$V = 2 \cdot 162 \sqrt{H}$$

where H is the measured head in inches.

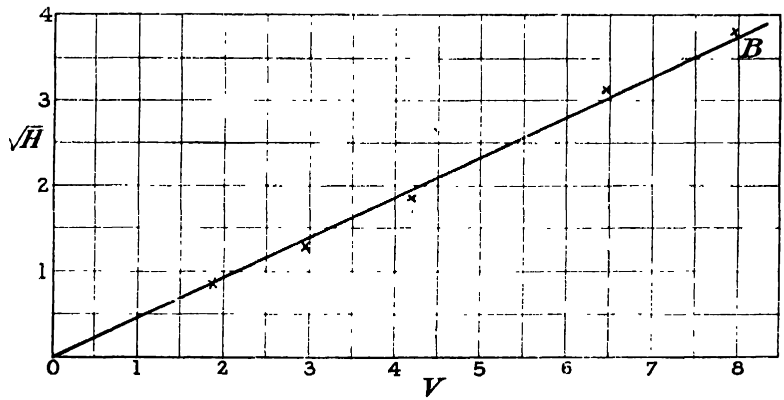


Fig. 39

EXAMPLE 2.

The velocity of water in a pipe was measured with a Pitot tube consisting of one tube with orifice facing the direction of flow and the other orifice perpendicular to the first orifice. The difference of head at the centre of pipe was 3·5 in. of water. If the mean velocity of the water is two-thirds the velocity at the centre, find the quantity of water flowing per minute. The diameter of the pipe is 10 in. Take the coefficient of the Pitot tube as unity.

Area of pipe $= \frac{\pi}{4} \frac{10^2}{144} = \cdot 545$ sq. ft.

Velocity at centre of pipe $= k \sqrt{2g\,h}$

$$= \sqrt{2g \frac{3 \cdot 5}{12}}$$

$= 4 \cdot 33$ ft. per sec.

Mean velocity in pipe $= \frac{2}{3} \times 4 \cdot 33$

$= 2 \cdot 885$ ft. per sec.

Quantity flowing per min. $= 2 \cdot 885 \times 60 \times \cdot 545$

$= 94 \cdot 3$ cu. ft.

Examples 3.

(1) Find the work done in forcing 50 gallons of water into a boiler in which the pressure is 180 lb. per sq. in. gauge. If this work is done in 5 min., what is the horse-power expended?

Ans.—208,000 ft. lb. 1·26 h.p.

(2) Water is flowing along a pipe with a velocity of 24 ft. per sec. Express this as velocity head in feet of water. What is the corresponding pressure in pounds per square inch?

Ans.—8·95 ft. 3·88 lb.

(3) Water at an altitude of 120 ft. above sea-level has a velocity of 16 ft. per sec. and a pressure of 60 lb. per sq. in. Give the total energy of 1 lb. of this water reckoned above sea-level.

Ans.—262·58 ft. lb.

(4) A pipe 1,000 ft. long has a slope of 1 in 100 and tapers from 4 ft. diameter at the high end to 2 ft. diameter at the low. The quantity of water flowing is 1,200 gallons per minute. If the pressure at the high end is 10 lb. per sq. in., find the pressure at the low end. Neglect friction.

Ans.—14·25 lb. per sq. in.

(5) Water flows from a supply tank into a chamber in which the pressure is 10 lb. per sq. in. vacuum. If the level of the water in the supply tank is 20 ft. above the vacuum chamber, find the velocity of the entering water.

Ans.—52·6 ft. per sec.

(6) A Venturi meter has an enlarged end of 2 sq. ft. area and a throat area of ·25 sq. ft. The coefficient of the meter is ·97. If the Venturi head is 9 in. of water, find the quantity of water flowing.

Ans.—1·695 cu. ft. per sec.

(7) A jet of water 1 in. diameter has a velocity of 60 ft. per sec. Find the horse-power of the jet.

Ans.—2·07.

(8) Two horizontal circular discs of 8 in. diameter are 1 in. apart. Water flows between the discs radially towards the centre and leaves by a vertical pipe of 1 in. diameter situated at the centre of the lower disc. If the pressure of the entering water is 14·7 lb. per sq. in., find the pressure inside the vertical pipe when the water is flowing at the rate of 40 gallons per minute. Find, also, the intensity of pressure of the water between the discs at a radius of 2 in.

Ans.—12·12 lb. per sq. in. 14·6924 lb. per sq. in.

(9) A cylindrical arm full of water is rotated in a horizontal plane at 100 rev. per min. about one end. The arm is 2 ft. long and its diameter is 2 in. Find the centrifugal head impressed on the water and the total pressure on the outer end of the arm.

Ans.—6·81 ft. of water. 9·28 lb.

(10) The air supply to a gas engine is measured by drawing the air into a large chamber through a small orifice. If the difference of pressure between the outside air and the air in the chamber is 16 in. of water, find the velocity with which the air flows through the orifice. Temperature of atmosphere is 18° C., reading of barometer is 29 in. of mercury. Weight of 1 cu. ft. of air at 0° C. and a pressure of 30 in. of mercury is ·081 lb.

Ans.—270 ft. per sec.

(11) A Pitot tube was placed in the centre of a pipe 8 in. diameter with one orifice facing the stream and the other perpendicular to it. The difference of pressure on the two orifices as measured by an air gauge was $1\frac{1}{2}$ in. of water. The coefficient of the tube was unity. Taking the mean velocity of the water in the pipe to be ·83 of the maximum velocity, find the discharge through the pipe. (London Univ.)

Ans.—·822 cu. ft. per sec.

(12) State Bernoulli's theorem for stream line flow of a liquid and give an elementary proof of the theorem.

A portion of a pipe for conveying water is vertical and the diameter of the upper part of the pipe is 2 in., and the section is gradually reduced to 1 in. diameter at the lower part. A pressure gauge is inserted where the diameter is 2 in., and a second gauge is placed 6 ft. below the first and where the pipe is 1 in. diameter. When the quantity of water flowing up through the pipe is 6·85 cu. ft. per min., the gauges show a pressure difference of 4·5 lb. per sq. in. Assuming that the frictional losses vary as the square of the velocity, determine the quantity of water passing through the pipe when the two gauges show no pressure difference and the water is flowing downwards. (London Univ.)

Ans.—4·05 cu. ft. per min.

(13) Find, from Bernoulli's theorem, an expression for the theoretical discharge of a horizontal Venturi meter. State how the actual discharge compares with the theoretical. A Venturi meter tapers from 12 in. diameter at the entrance to 4 in. diameter at the throat, and the discharge coefficient is ·98. The difference of pressure between entrance and throat is 2·2 in. of mercury. Calculate the discharge in gallons per minute. (London Univ.)

Ans.—409 gallons per minute.

(14) A vertical pipe of radius r_1 in. is fitted at the outlet end with a flange of radius r_2 in. A disc of the same diameter r_2 is placed above the flange, and separated from it by a narrow gap. Water from the pipe flows radially between them and is discharged into the atmosphere. Neglecting friction, find general expressions for the pressure between the surfaces at any radius, and for the resultant inward force on the disc. Sketch the curve of pressure distribution. (London Univ.)

(15) A Venturi has an entrance diameter of 6 in. and a throat diameter of 2 in. Pipes from the entrance and throat lead water to the limbs of a U-tube containing mercury, and the difference of pressure at these two places in the meter is thus recorded by a difference of mercury level. If the coefficient of the meter is ·96 draw a curve showing a relation between gallons of water passing through the meter per minute and the difference of mercury level over a range 0 to 15 in. (London Univ.)

(16) Give a proof of Bernoulli's theorem and show how this is used to determine the discharge from a Venturi meter. (A.M.I. Mech. E.)

(17). A conical tube is fixed vertically with its smaller end upwards, and forms part of a pipe line. The velocity at the smaller end is 15 ft. per sec., and at the larger end 5 ft. per sec., the tube is 5 ft. long; the pressure at the upper end is equivalent to a head of 10 ft.; the loss in the tube expressed in feet head is given by

$$\frac{\cdot 3(v_1 - v_2)^2}{2g}$$

where $v_1 = 15$ and $v_2 = 5$.

Determine the pressure at the lower end of the tube. (A.M.I. Mech. E.)

Ans.—17·64 ft. of water.

(18) Explain the theory of the Pitot tube and obtain an expression for the velocity in terms of the observed difference of level of the liquid, of specific gravity S, in the U-tube connected to the up- and down-stream orifices immersed in flowing water.

If the difference of level is 1·2 ft., the specific gravity of the liquid 1·25, and the calibration coefficient for the orifices ·865, what is the velocity in feet per second ? (A.M.I. Civil E.)

Ans.—2·68.

(19) A Venturi contraction is introduced in a 30 in. diameter horizontal pipe. The area of the pipe is six times that of the throat. The upper end of a vertical cylinder 12 in. in diameter is connected by a pipe to the throat and the lower end to the beginning of the convergence. Neglecting friction losses, and the thickness of the piston in the cylinder, determine the flow through the pipe in cusecs at which the piston begins to rise when the gross effective load—piston, piston rod, and external weight—on the piston rod is 450 lb. The piston rod is $1\frac{1}{2}$ in. diameter, and passes through both ends of the cylinder. (A.M.I. Civil E.)

Ans.—20·4.

(20) State Bernouilli's Theorem. The diameter of a pipe changes gradually from 6 in. at a point A, 20 ft. above datum, to 3 in. at B, 10 ft. above datum. The pressure at A is 15 lb. per sq. in., and the velocity of flow 12 ft. per sec. Neglecting losses between A and B, determine the pressure at B. (A.M.I. Mech. E.)

Ans.—4·82 lb. per sq. in.

(21) A closed vertical cylinder of 3 ft. internal diameter is filled with water and rotates about its axis at 950 revs. per min. Neglecting the effect of the shaft, find the total pressure of the water against the top of the cylinder. (London Univ.)

Ans.—76,500 lb.

(22) A Pitot tube used to measure the air flow in a duct 2 ft. in diameter gave the following readings—

Distance across duct, inches	0	0·25	0·50	1·00	2·00	3·6	6·0	9·0	12·0
Pitot head, inches of water	0	0·24	0·40	0·59	0·82	1·05	1·25	1·44	1·58

Find the air flow in lb. per sec. Weight of air = 0·078 lb. per cu. ft. (I. Mech. E.)

Ans.—1·035 lb. per cu. ft.

(23) A forced-draught fan takes hot flue gas of density 0·047 lb. per cu. ft. from a boiler flue and delivers it to a second flue 15 ft. above the first. A draught gauge 10 ft. above the first flue and connected to it by a length of cold pipe reads 1·8 in. of water suction. Another gauge directly mounted upon the higher flue gives the suction there as 0·5 in. of water. Find the work done by the fan per pound of flue gas. Neglect any change of speed. (I. Mech. E.)

Ans.—432 ft. lb.

CHAPTER IV

ORIFICES AND MOUTHPIECES

34. Flow Through Orifices. Supposing a tank containing water were to have a hole made in the side or base through which the water would flow, such a hole is termed an orifice, and the quantity of water which would flow through this orifice in a given time would partly depend on the shape, size, and form of the orifice. There would be a certain amount of frictional resistance at the sides of the orifice; this may be reduced by making them sharp-edged. The jet of water, in passing through the orifice, will contract in area, which will further reduce the rate of discharge. This contraction of area is caused by the water in the tank around the sides of the orifice, which, in flowing to the orifice, will have a motion parallel to it and perpendicular to that of the jet (Fig. 40). The velocity in this direction is destroyed on reaching the orifice; this causes a lateral force on the jet and a consequent reduction of area. The contraction of area will depend on the shape and size of the orifice and on the head causing flow.

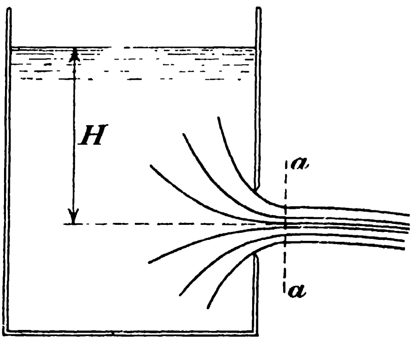


FIG. 40

The section of the jet at which the stream lines first become parallel is known as the vena contracta. This section is the line *aa* in Fig. 40. The velocity at the vena contracta has reached its maximum and there will be no further contraction of the jet beyond this section.*

35. The Coefficient of Contraction. The ratio between the area of the jet at the vena contracta and the area of the orifice is known as the coefficient of contraction.

Let C_c = coefficient of contraction

Then, $$C_c = \frac{\text{area of jet at vena contracta}}{\text{area of orifice}}$$

* For the flow of gas through an orifice see Art. 191.

This coefficient varies slightly with the head and with the size and shape of the orifice. An average value for small, sharp-edged orifices is ·64.

The coefficient of contraction may be found experimentally by direct measurement of the area of jet at the vena contracta. This may be done with the instrument shown in Fig. 41. It consists of a small collar or ring having four radial screws, equally spaced. The ring is held at the vena contracta so that the jet passes through its centre. The screws are then adjusted until all their points are in contact with the surface of the jet. The instrument is then removed and the space between the screw points measured. Micrometer screws may be used.

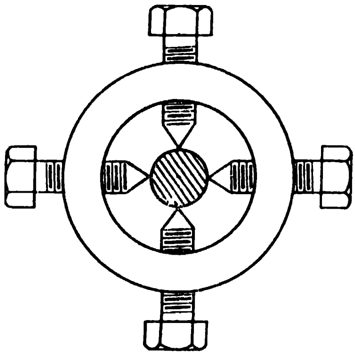


FIG. 41

This method is not very satisfactory in practice as the section of the jet is not absolutely regular; also, it is difficult to adjust the instrument so that all four screws are just in contact with the surface simultaneously.

A more accurate method of finding C_c is given in Art. 37.

36. The Coefficient of Velocity. The ratio between the theoretical velocity and the actual velocity of the jet at the vena contracta is known as the coefficient of velocity.

Let C_v = coefficient of velocity

Then, $C_v = \dfrac{\text{actual velocity at vena contracta}}{\text{theoretical velocity}}$

Let H = head causing flow

v = actual velocity

Then, $C_v = \dfrac{v}{\sqrt{2gH}}$

Or, $v = C_v \sqrt{2gH}$

The difference between the theoretical and actual velocities is due to friction at the orifice and is very small for sharp-edged orifices. The coefficient of velocity will vary slightly

for different orifices, depending on the shape and size of the orifice and on the head. An average value for C_v is about ·97.

The coefficient C_v may be found experimentally for a vertical orifice by measuring the horizontal and vertical co-ordinates of the issuing jet.

Consider the tank in Fig. 42

Let H = height of water in feet above centre of orifice

aa = vena contracta

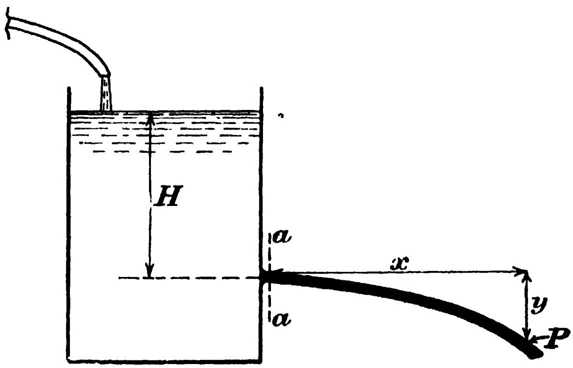


Fig. 42

The jet of water has a horizontal velocity of v but is acted upon by gravity with a downward acceleration of g. Consider a particle of water in the jet at P and let the time taken for this particle to move from aa to P be t sec.

Let x = horizontal co-ordinate of P from aa in ft.

y = vertical co-ordinate of P from aa in ft.

Then, $x = vt$

and, $y = \frac{1}{2}gt^2$

Equating the values of t^2 from these two equations,

$$\frac{x^2}{v^2} = \frac{2y}{g}$$

Or, $$v = \sqrt{\frac{g\,x^2}{2y}}$$

But, $$C_v = \frac{v}{\sqrt{2gH}}$$

Substituting for v,

$$C_v = \sqrt{\frac{x^2}{4yH}}$$

The value of C_v can be found from this equation by measuring the distances x and y for a certain point on the jet and for a known value of H.

The coefficient of velocity may also be found by measuring the actual velocity of the jet with a Pitot tube.

EXAMPLE.

In order to determine the coefficient of velocity of a small circular sharp-edged orifice under low heads, the horizontal and vertical co-ordinates of the jet were measured when the head was 8 in. The horizontal co-ordinate of a certain point of jet, from the vena contracta, was found to be 32·5 in., whilst the vertical co-ordinate for the same point was 33·7 in. Find the coefficient of velocity.

$$C_v = \sqrt{\frac{x^2}{4yH}}$$

where $H = 8$ in.

$x = 32{\cdot}5$ in.

and $y = 33{\cdot}7$ in.

$$\text{Then, } C_v = \sqrt{\frac{(32{\cdot}5)^2}{4 \times 33{\cdot}7 \times 8}}$$

$$= {\cdot}988$$

37. **The Coefficient of Discharge.** Owing to the reduction in velocity and to the contraction of the jet, the actual discharge will be much less than the theoretical; the relation between them being known as the coefficient of discharge.

Let C_d = coefficient of discharge

$$\text{Then, } C_d = \frac{\text{actual discharge}}{\text{theoretical discharge}}$$

But, actual discharge = actual velocity of jet × actual area of jet

$$= C_v \sqrt{2gH} \times C_c A$$

where A = area of orifice

Therefore, actual discharge $= C_v C_c \sqrt{2gH} \times A$

But, $A\sqrt{2gH}$ = theoretical discharge

Therefore $C_d = C_v \times C_c$

The coefficient of discharge of an orifice may therefore be found by first determining its C_v and C_c and by multiplying these together.

The coefficient of discharge will also vary with the head and type of orifice.* Usually, its value is between ·61 and ·64.

The simplest manner of determining the coefficient of discharge is by actually measuring the quantity of water discharged through the orifice in a given time under a known constant head, and by dividing this quantity by the theoretical discharge.

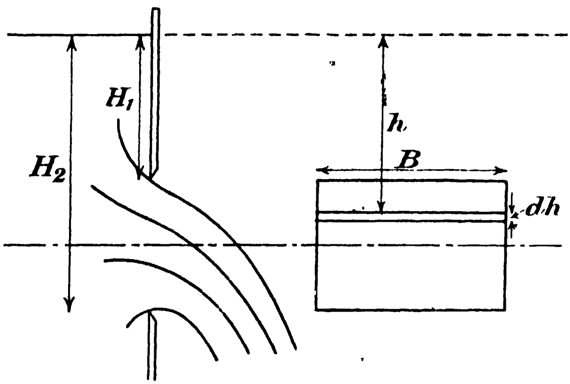


FIG. 43

Let Q be the volume of water in cubic feet actually discharged in a time t sec.

Then, $C_d = \dfrac{Q}{A\sqrt{2gH}\,t}$

A good method of finding the coefficient of contraction is to find the value of C_d by the above method, then $C_c = \dfrac{C_d}{C_v}$

38. **Large Vertical Orifices.** If a vertical orifice is large compared with the head, the velocity of the water may no longer be regarded as constant, as the variation in head at different vertical sections of the orifice will be considerable.

Consider the large orifice in Fig. 43. Let the height of the water level be H_1 above the top of the orifice and H_2 above the lower edge. Let B be the breadth of the orifice.

* For the non-dimensional factor for orifices see Art. 200.

Consider a horizontal strip of the orifice of depth h and thickness dh, and assume velocity at strip to be proportional to $\sqrt{2gh}$.

Area of strip $= B\,dh$

Velocity of water through strip $= k\sqrt{2gh}$

where k is an unknown coefficient which is included in the value of C_d.

Discharge through strip

$$= C_d \times \text{area} \times \text{velocity}$$
$$= C_d B\,dh\sqrt{2gh}$$

Total discharge

$$= C_d B\sqrt{2g}\int_{H_1}^{H_2} h^{\frac{1}{2}}\,dh$$
$$= \frac{2}{3} C_d B\sqrt{2g}\left[h^{\frac{3}{2}}\right]_{H_1}^{H_2}$$
$$= \frac{2}{3} C_d B\sqrt{2g}\left(H_2^{\frac{3}{2}} - H_1^{\frac{3}{2}}\right)$$

EXAMPLE.

A rectangular orifice in the side of a large tank is 4 ft. broad and 2 ft. deep. The level of the water in the tank is 2 ft. above the top edge of the orifice. Find the quantity of water flowing through the orifice per second if the coefficient of discharge is ·62.

$$\text{Discharge} = \frac{2}{3} C_d\sqrt{2g}\,B\left(H_2^{\frac{3}{2}} - H_1^{\frac{3}{2}}\right)$$
$$= \frac{2}{3} \times \cdot 62 \times \sqrt{64 \cdot 4} \times 4\left(4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)$$
$$= 68 \cdot 8 \text{ cu. ft. per sec.}$$

39. **Drowned Orifices.** If an orifice does not discharge into the atmosphere, but discharges into more water, the whole of the outlet side of the orifice being under water, it is known as a drowned or submerged orifice. If the outlet side of the orifice is only partly under water it is known as a partially submerged or drowned orifice.

In a drowned orifice the discharge of the jet is interfered with by the water on the outlet side. This has the effect of slightly reducing the coefficient of discharge; the discharge will, therefore, be less for a drowned orifice than for a free, assuming the net head causing flow to be the same.

The discharge through a drowned orifice may be obtained from the same equations as for an orifice running free, excepting that the head causing flow will be the difference between the heads on either side of the orifice.

The discharge through a partially drowned orifice may be found by treating the lower portion as a drowned orifice and the upper portion as an orifice running free and by adding together the two discharges thus found.

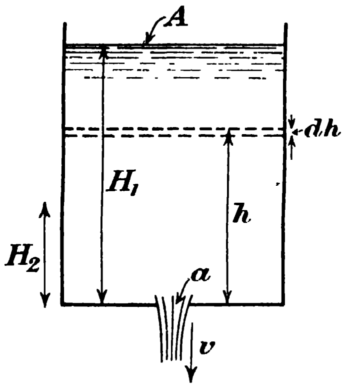


Fig. 44

EXAMPLE.

An orifice in the side of a large tank is rectangular in shape, 4 ft. broad and 2 ft. deep. The water level on one side of the orifice is 4 ft. above the top edge; the water level on the other side of the orifice is 1 ft. below the top edge. Find the discharge per second if C_d ·62.

The orifice in the question is partially drowned; the lower half may be treated as a drowned orifice and the upper half as a free orifice.

Considering upper half of orifice,

$$\text{discharge} = \frac{2}{3} C_d B \sqrt{2g} \left(H_2^{\frac{3}{2}} - H_1^{\frac{3}{2}}\right)$$

$$= \frac{2}{3} \times \cdot 62 \times \sqrt{64 \cdot 4} \times 4 \left(5^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)$$

$$= 42 \cdot 2 \text{ cu. ft. per sec.}$$

Consider lower half of orifice,

head causing flow $= 5$ ft.

$$\text{Discharge} = C_d \sqrt{2g} \times \text{area} \times \sqrt{H}$$

$$= \cdot 62 \sqrt{64 \cdot 4} \times 4 \times \sqrt{5}$$

$$= 44 \cdot 5 \text{ cu. ft. per sec.}$$

$$\text{Total discharge} = 42 \cdot 2 + 44 \cdot 5$$

$$= 86 \cdot 7 \text{ cu. ft. per sec.}$$

40. **Time of Emptying Tank.** Consider a tank of uniform cross-sectional area A (Fig. 44), then let the water be discharged through an orifice in the base of the tank so that the water level falls from a height H_1 to a height H_2 in T sec. The rate of discharge will decrease as the water level falls.

Let a = area of orifice

v = velocity of water passing through orifice at any particular instant

At any particular instant let the water level be at a height h above the orifice and let the level fall by a small amount dh in the time dt. Let the corresponding quantity of water passing through the orifice due to this small change of water level be dq. Then, as volume displaced by water level equals quantity flowing through orifice, and as dh, being measured downwards, is negative,

$$dq = -A\,dh = C_d\,a\,v\,dt$$

But, $$v = \sqrt{2g\,h}$$

Therefore, $$-A\,dh = C_d\,a\sqrt{2g\,h}\,dt$$

Or, $$dt = -\frac{A\,dh}{C_d a\sqrt{2g\,h}}$$

$$= -\frac{A\,h^{-\frac{1}{2}}\,dh}{C_d\,a\sqrt{2g}}$$

Then, total time taken $$= \int_0^T dt = -\frac{A}{C_d\,a\sqrt{2g}}\int_{H_1}^{H_2} h^{-\frac{1}{2}}\,dh$$

$$T = -\frac{2A}{C_d\,a\sqrt{2g}}\Big[h^{\frac{1}{2}}\Big]_{H_1}^{H_2}$$

$$= \frac{2A}{C_d\,a\sqrt{2g}}(H_1^{\frac{1}{2}} - H_2^{\frac{1}{2}}) \quad . \quad . \quad (1)$$

If the tank is completely emptied, $H_2 = 0$

Then, $$T = \frac{2A}{C_d\,a\sqrt{2g}}\sqrt{H_1} \quad . \quad . \quad . \quad (2)$$

41. **Time of Emptying Hemispherical Vessel.** The time taken to lower the water level in a hemispherical vessel may be found in the same manner as in Art. 40; but in this case the cross-sectional area of the vessel is not uniform (Fig. 45).

Let R be the radius of vessel and let the water level fall from H_1 to H_2 in the time T.

Consider the instant when the water level is at a height h, and let the radius of the vessel's cross-section at this level be x.

Let a small quantity dq flow through the orifice at the base in a time dt, and let the corresponding fall of water level due to this be dh.

Let a = area of orifice

and v = theoretical velocity of water passing through orifice at time considered

$= \sqrt{2gh}$

As volume displaced by water level equals volume flowing through orifice, and as dh is negative,

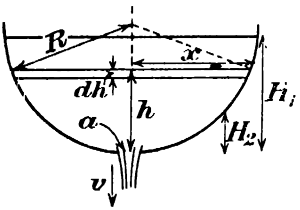


Fig. 45

$$dq = -\pi x^2\, dh = C_d\, a\, v\, dt$$
$$= C_d\, a \sqrt{2gh}\,dt$$

But, from Fig. 45,

$$x^2 = R^2 - (R-h)^2$$
$$= 2Rh - h^2$$

Therefore,
$$dt = -\frac{\pi(2Rh - h^2)dh}{C_d\, a\sqrt{2g\,h}}$$
$$= -\frac{\pi(2Rh - h^2)h^{-\frac{1}{2}}dh}{C_d\, a\sqrt{2g}}$$

$$\left.\begin{matrix}\text{tal time required}\\ \text{lower water level}\end{matrix}\right\} = -\frac{\pi}{C_d\, a\sqrt{2g}}\int_{H_1}^{H_2}(2Rh^{\frac{1}{2}} - h^{\frac{3}{2}})\,dh$$

$$= -\frac{\pi}{C_d\, a\sqrt{2g}}\left[\frac{4}{3}Rh^{\frac{3}{2}} - \frac{2}{5}h^{\frac{5}{2}}\right]_{H_1}^{H_2}$$

$$= \frac{2\pi}{C_d\, a\sqrt{2g}}\left\{\frac{2}{3}R(H_1^{\frac{3}{2}} - H_2^{\frac{3}{2}}) - \frac{1}{5}(H_1^{\frac{5}{2}} - H_2^{\frac{5}{2}})\right\} \quad . \quad (1)$$

If the vessel were full at the commencement and is completely emptied, then :

$H_1 = R$

and $H_2 = 0$

Equation (1) then becomes,

$$T = \frac{2\pi}{C_d\, a\sqrt{2g}}\left(\frac{2}{3}R^{\frac{5}{2}} - \frac{1}{5}R^{\frac{5}{2}}\right)$$

$$= \frac{14\,\pi R^{\frac{5}{2}}}{15\, C_d\, a\sqrt{2g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

EXAMPLE.

A hemispherical tank 12 ft. in diameter is emptied through a hole, 8 in. diameter, at the bottom. Assuming that the coefficient of discharge is ·6, find the time required to lower the level of the water surface from 6 ft. to 4 ft., and deduce the formula you use. (London Univ.)

Using Equation 1,

$$T = \frac{2\pi}{C_d\, a\, \sqrt{2g}}\left\{\frac{2}{3}R\left(H_1^{\frac{3}{2}} - H_2^{\frac{3}{2}}\right) - \frac{1}{5}\left(H_1^{\frac{5}{2}} - H_2^{\frac{5}{2}}\right)\right\}$$

$$= \frac{2\pi}{\cdot 6 \times \frac{\pi}{4} \times \frac{4}{9}\sqrt{64\cdot 4}}\left\{\frac{2}{3} \times 6(6^{\frac{3}{2}} - 4^{\frac{3}{2}}) - \frac{1}{5}(6^{\frac{5}{2}} - 4^{\frac{5}{2}})\right\}$$

$$= 58\cdot 2 \text{ sec.}$$

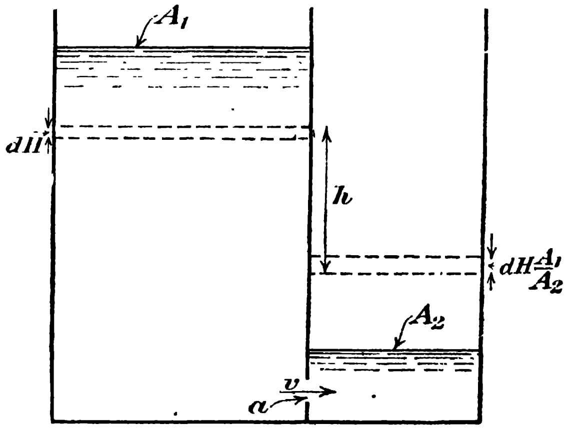


FIG. 46

42. Time of Flow from One Vessel to Another. Suppose water is flowing from one vessel into another (Fig. 46), so that as the water level falls in one vessel it will rise by a corresponding amount in the other. In this case, the orifice will be drowned and the head causing flow at any instant will be the difference between the two water levels at that instant.

Let the water flow from a vessel of area A_1 into a vessel of area A_2, and let a be the area of the orifice between the vessels. Let the difference of head between the two vessels be H_1

at the beginning ; it is required to find the time taken for the difference of head to reach H_2.

Let v = theoretical velocity of flow through orifice.

At a certain instant let the difference of head between the two vessels be h, and let a small quantity dq flow through the orifice in the time dt. This will cause the water level in A_1 to fall by the small amount dH; the water level in A_2 will rise, therefore, by the amount $dH\dfrac{A_1}{A_2}$.

$$\text{New difference of head} = h - dH - dH\frac{A_1}{A_2}$$

$$= h - dH\left(1 + \frac{A_1}{A_2}\right)$$

Therefore, change of head causing flow $= dh = dH\left(1 + \dfrac{A_1}{A_2}\right)$

Or, $$dH = \frac{dh}{\left(1 + \dfrac{A_1}{A_2}\right)} \quad . \quad . \quad . \quad . \quad (1)$$

As quantity flowing from A_1 equals quantity flowing through orifice, and as dH is negative,

$$dq = -A_1\,dH = C_d\,a\,v\,dt$$

But $$v = \sqrt{2g\,h}$$

Therefore, $$dt = -\frac{A_1\,dH}{C_d a\sqrt{2g\,h}}$$

Substituting from Equation (1),

$$dt = -\frac{A_1\,dh}{C_d\,a\left(1 + \dfrac{A_1}{A_2}\right)\sqrt{2g\,h}}$$

Or, $$dt = -\frac{A_1\,h^{-\frac{1}{2}}\,dh}{C_d\,a\left(1 + \dfrac{A_1}{A_2}\right)\sqrt{2g}}$$

Total time taken $$= \int_0^T dt = -\frac{A_1}{C_d\,a\left(1 + \dfrac{A_1}{A_2}\right)\sqrt{2g}}\int_{H_1}^{H_2} h^{-\frac{1}{2}}dh$$

hence,
$$T = -\frac{2A_1}{C_d a\left(1+\frac{A_1}{A_2}\right)\sqrt{2g}}\left[h^{\frac{1}{2}}\right]_{H_1}^{H_2}$$
$$= \frac{2A_1(H_1^{\frac{1}{2}} - H_2^{\frac{1}{2}})}{C_d a\left(1+\frac{A_1}{A_2}\right)\sqrt{2g}} \quad . \quad . \quad (2)$$

If both the vessels have the same area,
$$A_1 = A_2$$

Then,
$$T = \frac{A_1(H_1^{\frac{1}{2}} - H_2^{\frac{1}{2}})}{C_d a\sqrt{2g}} \quad . \quad . \quad . \quad (3)$$

It will be noticed that the time taken to reduce the difference of water level between two vessels of different areas is the same whether the water flows from the larger to the smaller or from the smaller to the larger, providing the reduction in water level is the same in each case.

EXAMPLE.

A tank 10 ft. long and 5 ft. wide is divided into two parts by a partition so that the area of one part is three times the area of the other. The partition contains a square orifice of 3 in. sides through which the water may flow from one part to the other. If the water level in the smaller division is 10 ft. above that of the larger, find the time taken to reduce the difference of water level to 2 ft. $C_d = \cdot 62$.

$$A_1 = 5 \times 2\tfrac{1}{2} = 12\tfrac{1}{2} \text{ sq. ft.}$$
$$A_2 = 5 \times 7\tfrac{1}{2} = 37\tfrac{1}{2} \text{ sq. ft.}$$
$$H_1 = 10 \text{ ft.}$$
$$H_2 = 2 \text{ ft.}$$
$$a = \frac{3 \times 3}{144} = \frac{1}{16} \text{ sq. ft.}$$

Using Equation (2),
$$T = \frac{2A_1(H_1^{\frac{1}{2}} - H_2^{\frac{1}{2}})}{C_d a\left(1+\frac{A_1}{A_2}\right)\sqrt{2g}}$$
$$= \frac{2 \times 12\cdot 5(\sqrt{10} - \sqrt{2})}{\cdot 62 \times \frac{1}{16}\left(1+\frac{1}{3}\right)\sqrt{64\cdot 4}}$$
$$= 105\cdot 5 \text{ sec.}$$

43. Discharge From Tank with Inflow. The problem of finding the time required to lower the water surface in a tank by means of a small orifice was dealt with in Art. 40. This problem is more complex if there is an inflow of liquid whilst the discharge occurs. Consider the tank of Fig. 47 and let its area, in plan, be A sq. ft. Let there be a constant inflow of liquid of Q cu. ft. per sec., whilst, at the same time, the tank is discharging through a small orifice at the base.

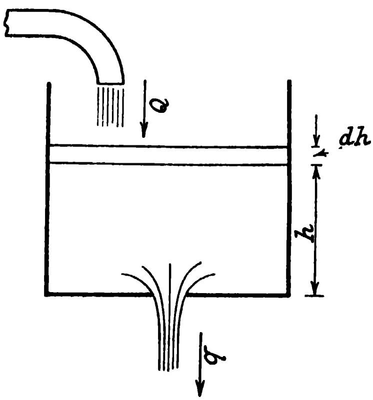


FIG. 47

Let $a =$ area of orifice in sq. ft.

$q =$ discharge from orifice in cu. ft. per sec.

Consider the tank at the instant when the liquid surface is at a height h ft. above the orifice; let this height increase by dh in a small interval of time dt. Then,

$$\text{amount of inflow} = Qdt$$

$$\begin{aligned}\text{amount of discharge} &= qdt \\ &= C_d a\sqrt{2gh}\,dt \\ &= k\sqrt{h}\,dt\end{aligned}$$

where $k = C_d a\sqrt{2g}$

$$\begin{aligned}\text{Then, increase of liquid in tank} = Adh &= Qdt - kh^{\frac{1}{2}}dt \\ &= (Q - kh^{\frac{1}{2}})dt\end{aligned}$$

from which
$$dt = \frac{Adh}{Q - kh^{\frac{1}{2}}} \quad . \quad . \quad (1)$$

The time required to raise or lower the liquid surface between the heights H_1 and H_2 can be obtained by integrating this equation. This is not a simple expression to integrate, but the

solution can be obtained by letting the denominator be represented by the symbol z. Then,

$$z = Q - kh^{\frac{1}{2}} \quad . \quad . \quad . \quad . \qquad (2)$$

then, $$h^{\frac{1}{2}} = \frac{Q - z}{k}$$

or $$h = \frac{(Q - z)^2}{k^2}$$

Differentiating with respect to z,

$$dh = -\frac{2(Q - z)dz}{k^2}$$

Substituting this value of dh in Equation (1) and substituting z for the denominator,

$$dt = -\frac{A2(Q - z)dz}{zk^2}$$

that is, $$dt = -\frac{2A}{k^2}\left(\frac{Q}{z} - 1\right)dz$$

Integrating for a total time of T sec.,

$$\int_0^T dt = -\frac{2A}{k^2}\ (Q \log_e z - z)$$

Substituting for z from Equation (2),

$$T = -\frac{2A}{k^2}\Big[Q \log_e(Q - kh^{\frac{1}{2}}) - (Q - kh^{\frac{1}{2}})\Big]_{H_1}^{H_2}$$

Hence, $$T = -\frac{2A}{k^2}\Big[\ \{Q \log_e(Q - k\sqrt{H_2}) - (Q - k\sqrt{H_2})\}$$
$$- \{Q \log_e(Q - k\sqrt{H_1}) - (Q - k\sqrt{H_1})\}\Big]$$
$$= -\frac{2A}{k^2}\left[Q \log_e\left(\frac{Q - k\sqrt{H_2}}{Q - k\sqrt{H_1}}\right) + k(\sqrt{H_2} - \sqrt{H_1})\right] \qquad (3)$$

The problem may also be approached from a graphical conception. Suppose the rise in head h is measured at various intervals of time t, commencing with the tank empty. The head h may then be plotted on a time-base, t; the curve obtained is shown in Fig. 48.

Now, as shown above, $Adh = (Q - kh^{\frac{1}{2}})dt$

hence, $$\frac{dh}{dt} = \frac{Q}{A} - \frac{k}{A}h^{\frac{1}{2}} \quad . \quad . \quad . \quad (4)$$

It will be seen from this equation that if the values of $\frac{dh}{dt}$ are plotted on a base representing $h^{\frac{1}{2}}$ a straight line is obtained;

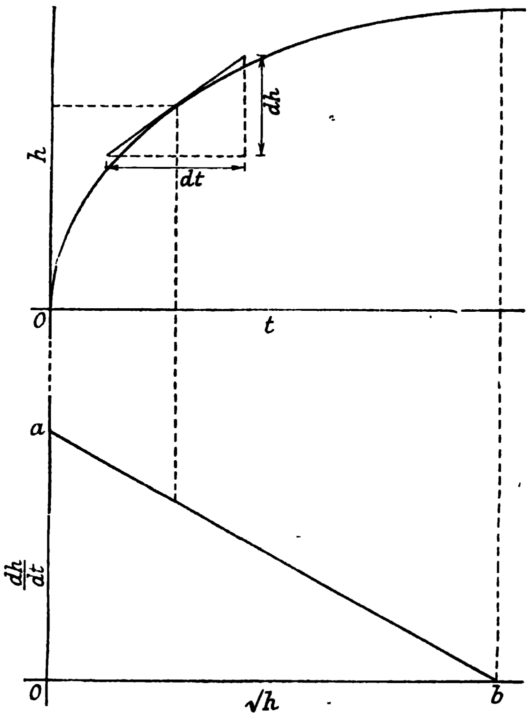


FIG. 48

this is shown plotted in Fig. 48. The values of $\frac{dh}{dt}$ for various values of h were obtained by measuring the slopes of the h/t curve; this is done by drawing tangents for several values of h.

The values of the two constants of Equation (4), $\frac{Q}{A}$ and $\frac{k}{A}$, can be obtained for any two known sets of values of $\frac{dh}{dt}$ and h.

It will be noticed from Equation (4) that when $\sqrt{h} = 0$,

$$\frac{dh}{dt} = \frac{Q}{A}$$

$$= \text{height } Oa \text{ on graph.}$$

Also, when $\frac{dh}{dt} = 0$,

$$\frac{Q}{A} = \frac{k}{A}h^{\frac{1}{2}}$$

hence, $h^{\frac{1}{2}} = \frac{Q}{k}$

$$= \text{ordinate } Ob' \text{ on graph.}$$

This latter condition occurs when the liquid surface in the tank has reached its maximum height. At this level the discharge from the orifice is at the same rate as the inflow; consequently, no further increase in head will occur.

EXAMPLE 1.

A concrete tank is 50 ft. long and 30 ft. wide, and its sides are vertical. Water enters the tank at the rate of 6 cu. ft. per sec. and is discharged from the sluices the centre line of which is 1 ft. above the bottom of the tank.

When the depth of water in the tank was 17 ft., the instantaneous rate of discharge was observed to be 12 cu. ft. per sec. How long will it take for the level in the tank to fall 10 ft.? (London Univ.)

Now, $q = kh^{\frac{1}{2}}$

that is, $12 = k\sqrt{16}$

hence, $k = 3$

Applying Equation (3), and putting $Q = 6$, $A = 1500$, $H_2 = 6$, and $H_1 = 16$,

$$T = -\frac{2 \times 1500}{3^2}\left[6 \log_e\left(\frac{6 - 3\sqrt{6}}{6 - 3\sqrt{16}}\right) + 3(\sqrt{6} - \sqrt{16})\right]$$

$$= -333{\cdot}3[6 \log_e {\cdot}225 - 4{\cdot}65]$$

$$= 4540 \text{ sec.}$$

$$= 75{\cdot}7 \text{ min.}$$

EXAMPLE 2.

A cylindrical tank is placed with its axis vertical and is provided with a circular orifice, $1\frac{1}{2}$ in. diameter, in the bottom. Water flows into the tank at a

uniform rate, and is discharged through the orifice. It is found that it takes 107 seconds for the head in the tank to rise from 2 ft. to 2 ft. 6 in. and 120 seconds for it to rise from 4 ft. to 4 ft. 3 in. Find the rate of inflow in cusecs and the cross-section of the tank, assuming a coefficient of discharge of 0·62 for the orifice. (London Univ.)

This problem is demonstrated by the curves of Fig. 48. Two values of $\frac{dh}{dt}$ are given in the question, namely,

for an average head of 2 ft. 3 in., dh = 6 in. and dt = 107 sec.
for an average head of 4 ft. $1\frac{1}{2}$ in., dh = 3 in. and dt = 120 sec.

Hence, when $h = 2{\cdot}25$,
$$\frac{dh}{dt} = \frac{\cdot 5}{107} = \cdot 00468$$

when $h = 4{\cdot}125$,
$$\frac{dh}{dt} = \frac{\cdot 25}{120} = \cdot 00208$$

Also,
$$k = \cdot 62 \times \frac{\pi}{4}\left(\frac{1}{8}\right)^2 \sqrt{2g} = \cdot 0609$$

Substituting for each value of $\frac{dh}{dt}$ in Equation (4),

$$\frac{dh}{dt} = \frac{Q}{A} - \frac{k}{A}\sqrt{h}$$

When $h = 2{\cdot}25$,
$$\cdot 00468A = Q - \cdot 0609\sqrt{2{\cdot}25} = Q - \cdot 0913 \quad . \quad . \quad . \quad (5)$$

when $h = 4{\cdot}125$,
$$\cdot 00208A = Q - \cdot 0609\sqrt{4{\cdot}125} = Q - \cdot 1233 \quad . \quad . \quad . \quad (6)$$

Solving the simultaneous equations (5) and (6),

$$A = 12{\cdot}3 \text{ sq. ft.}$$
$$Q = \cdot 1489 \text{ cusecs.}$$

44. **Losses of Head of Flowing Water.** Water flowing along a straight uniform passage with perfectly smooth walls would suffer no loss of energy except that due to viscous resistance.

It is not possible in practice to obtain this condition, on account of the roughness of the sides of the passage. The loss of energy due to such a resistance is usually expressed as a head in feet of water. Flowing water will also be subjected to losses of head due to changes of section, changes of direction, and obstructions. All such losses are expressed in terms of the velocity head.

(*a*) LOSS OF HEAD DUE TO FRICTION OF SIDES OF PASSAGE. This loss is expressed as a function of $\frac{v^2}{2g}$ and will depend on the length and diameter of the pipe, the material of which the pipe is made, and the nature of the surface. This loss is dealt with fully in a subsequent chapter.

(*b*) LOSS OF HEAD DUE TO CHANGE OF DIRECTION. This loss is due to the resistance of sharp bends and elbows, and is expressed as a function of $\frac{v^2}{2g}$.

$$\text{Or, loss of head} = k\frac{v^2}{2g},$$

where k is a coefficient found by experiment and depends on the radius of the bend and the angle of deviation. For 90° elbows, k is found to be approximately unity. The loss of energy due to a sudden change of direction is ultimately lost in the friction of the eddies formed.

(*c*) LOSS OF HEAD DUE TO CHANGE OF SECTION OF PASSAGE. Losses of head under this heading are due to a sudden enlargement of section, a sudden contraction, and the loss at entrance of a pipe. There are also losses due to a gradual enlargement or contraction of the section; but as these are extremely small, they are usually neglected.

(*d*) LOSS OF HEAD DUE TO OBSTRUCTION IN PASSAGE. Any obstruction in the passage, such as a diaphragm or a projection from the passage walls, will interfere with the steady flow of the water and form eddies, the energy of which will be ultimately lost in friction.

An obstruction will cause a contraction of the area of flow which will be followed by an enlargement when the obstruction is passed. The loss of head will be due to this sudden enlargement.

Summary of Losses of Head.

(*a*) Loss of head due to friction $= \frac{4fl}{d}\frac{v^2}{2g}$ (Art. 67)

(*b*) Loss of head due to bends and elbows $= k\frac{v^2}{2g}$ (Art. 44)

(*c*) Loss of head due to sudden enlargement $= \frac{(v_1 - v_2)^2}{2g}$ (Art. 45)

Loss of head due to sudden contraction $= \cdot 5\frac{v^2}{2g}$ Art. 46)

Loss of head at entrance to pipe $= \cdot 5\frac{v^2}{2g}$ (Art. 47)

(*d*) Loss of head due to obstruction $= \left[\frac{A}{\cdot 66(A-a)} - 1\right]^2\frac{v^2}{2g}$ (Art. 48)

45. Loss of Head Due to a Sudden Enlargement. Consider water flowing along a pipe of area a_1, with a velocity v_1 and a pressure p_1; let the pipe be suddenly enlarged to an area a_2, and let the velocity of the water in the enlarged section be v_2 and the pressure p_2 (Fig. 49). The water will flow by the enlargement as shown in the figure, and a backwash of eddies will be formed in the corner. It is the formation of these eddies which cause the loss of head.

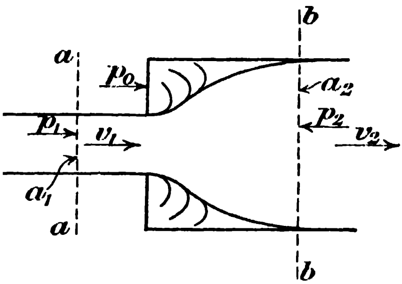


Fig. 49

The eddies press on the annular ring of area $a_2 - a_1$ with a pressure of p_o lb. per sq. in. It is found by experiment that p_o is approximately equal to p_1 and it is on this assumption that the solution is obtained.

Consider the quantity of water between *aa* and *bb*. The resultant force acting on this mass of water is :

$$p_2 a_2 - p_1 a_1 - p_o(a_2 - a_1)$$

Assuming $p_o = p_1$,

Total force $= a_2(p_2 - p_1)$

The change of momentum per second of this mass of water is

$$\frac{w a_1 v_1^2}{g} - \frac{w a_2 v_2^2}{g}$$

But, $\quad a_1 v_1 = a_2 v_2$

Therefore, change of momentum per second

$$= \frac{w a_2 v_2 v_1}{g} - \frac{w a_2 v_2^2}{g}$$

Then, as force equals change of momentum per second,

$$a_2(p_2 - p_1) = w a_2 \left(\frac{v_2 v_1}{g} - \frac{v_2^2}{g} \right)$$

Or, $$\frac{p_2}{w} - \frac{p_1}{w} = \frac{v_2 v_1}{g} - \frac{v_2^2}{g} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let h_L = loss of head due to enlargement. Applying Bernoulli's equation to sections *aa* and *bb*,

$$\frac{p_1}{w} + \frac{v_1^2}{2g} = \frac{p_2}{w} + \frac{v_2^2}{2g} + h_L$$

from which, $$\frac{p^2}{w} - \frac{p_1}{w} = \frac{v_1^2}{2g} - \frac{v_2^2}{2g} - h_L \quad . \quad . \quad . \quad . \quad (2)$$

Equating Equations (1) and (2),

$$\frac{v_2 v_1}{g} - \frac{v_2^2}{g} = \frac{v_1^2}{2g} - \frac{v_2^2}{2g} - h_L$$

from which, $$h_L = \frac{v_1^2}{2g} - \frac{2 v_2 v_1}{2g} + \frac{v_2^2}{2g}$$

$$= \frac{(v_1 - v_2)^2}{2g} \quad . \quad . \quad . \quad . \quad (3)$$

The loss given by Equation (3) occurs at any sudden enlargement of the cross-section of a passage containing a moving fluid.

EXAMPLE.

A pipe of diameter 6 in. is suddenly enlarged to a diameter of 1 ft. Find the loss of head due to this enlargement when the quantity of water flowing is 4 cu. ft. per sec.

Velocity in 6 in. pipe $= \dfrac{q}{\text{area}} = \dfrac{4}{\frac{\pi}{4} \times (\frac{1}{2})^2} = 20{\cdot}4$ cu. ft. per sec.

Velocity in 12 in. pipe $= \dfrac{4}{\frac{\pi}{4}} = 5{\cdot}1$ cu. ft. per sec.

$$\text{Loss of head} = \frac{(v_1 - v_2)^2}{2g}$$
$$= \frac{(20{\cdot}4 - 5{\cdot}1)^2}{64{\cdot}4}$$
$$= 3{\cdot}64 \text{ ft. of water.}$$

46. **Loss of Head due to a Sudden Contraction.** The loss of head due to a sudden contraction is not due to the contraction itself but to the sudden enlargement which follows the contraction.

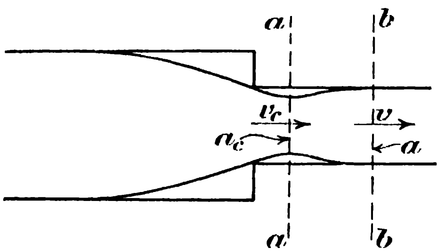


FIG. 50

Consider the pipe in Fig. 50. Let the pipe change section from an area of a_1 to an area of a. The water, in flowing into the narrow section, will be further contracted at the section *aa*, forming a vena contracta in the same way as a jet issuing from an orifice. Let the velocity at *aa* be v_c and let the contracted area be a_c.

Then, $\qquad a_c = C_c a$

where C_c is the coefficient of contraction.

Let v = velocity of water at the section *bb*.

At the section *bb* the jet of water will have expanded and filled the pipe; consequently, there will be a loss of head between *aa* and *bb* due to this expansion.

$$\text{Loss of head} = \frac{(v_c - v)^2}{2g} \qquad \text{(Art. 45)}$$

But, $\qquad av = a_c v_c$
$$= C_c a v_c$$

Therefore, $\qquad v_c = \dfrac{v}{C_c}$

$$\text{Then, loss of head} = v^2 \frac{\left(\frac{1}{C_c} - 1\right)^2}{2g}$$

Assuming C_c to be ·62 for a circular orifice,

$$\text{Loss of head} = \left(\frac{1}{\cdot 62} - 1\right)^2 \frac{v^2}{2g}$$

$$= \cdot 375 \frac{v^2}{2g}$$

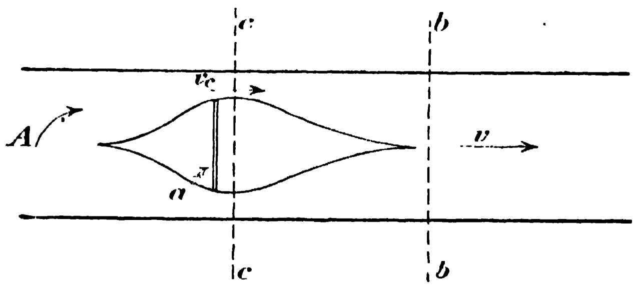


FIG. 51

It is found by experiment that the actual value of the constant is nearer ·5 than ·375; this higher value is generally used.

Then, loss of head due to sudden contraction $= \cdot 5 \frac{v^2}{2g}$

47. **Loss of Head at Entrance to Pipe.** The loss of head due to the water entering a pipe from a large container is actually a loss due to a sudden contraction.

Let $v =$ velocity in pipe.

Then, loss of head at entrance $= \cdot 5 \frac{v^2}{2g}$

In cases of water flowing along long pipes, this loss of head is very small compared with the frictional loss and may be neglected.

48. **Loss of Head due to Obstruction.** The loss of head due to an obstruction in a pipe may be looked upon as due to the sudden enlargement beyond the obstruction.

Consider a pipe of cross-sectional area A (Fig. 51), and let an obstruction of area a be placed in the pipe. The water will flow in stream lines by the obstruction, the vena contracta occurring just beyond at the section cc.

Let v = velocity of water in free section of pipe
v_c = velocity at vena contracta
bb = a section of normal flow beyond the obstruction.

There will be a loss of head due to the enlargement between the sections cc and bb equal to $\dfrac{(v_c - v)^2}{2g}$

Area of section of flow at $cc = C_c\,(A - a)$,
where C_c = coefficient of contraction

Also, $$v_c\,C_c(A - a) = v\,A$$

Therefore, $$v_c = \frac{A\,v}{C_c(A - a)}$$

Then, loss of head $$= \left[\frac{A}{C_c(A - a)} - 1\right]^2 \frac{v^2}{2g}$$

Assuming the coefficient of contraction = ·66,

loss of head due to obstruction $$= \left[\frac{A}{\cdot 66\,(A - a)} - 1\right]^2 \frac{v^2}{2g}$$

This phenomenon is made use of in an instrument known as a pipe orifice which is used for measuring the flow of water (Art. 212); the orifice forms an obstruction in the pipe and the loss of head is measured by means of pressure gauges.

EXAMPLE.

The passage of water through a 6 in. pipe is restricted by a diaphragm with a 2 in. diameter hole in its centre. The loss of head at the diaphragm when the velocity in the pipe is ·59 ft. per sec. equals 1·25 ft. Assuming the head lost $= k\dfrac{V^2}{2g}$ where V = the velocity of water in the pipe, find C_c the coefficient of contraction of the stream passing through the diaphragm. (London Univ.)

In this case the area of flow at the obstruction $= \dfrac{\pi}{4}(2)^2$

Then, loss of head $$= \left[\frac{A}{C_c(A - a)} - 1\right]^2 \frac{v^2}{2g}$$

$$1{\cdot}25 = \left[\frac{\frac{\pi}{4}(6)^2}{C_c\frac{\pi}{4}(2)^2} - 1\right]^2 \frac{(\cdot 59)^2}{2g}$$

$$\frac{9}{C_c} = 15{\cdot}22 + 1$$

$$C_c = \cdot 555$$

49. **External Mouthpiece.** The discharge through an orifice may be increased by fitting a short length of pipe to the outside. Consider the vessel in Fig. 52 to be discharging water through a short length of pipe under a head H. The jet, on entering the pipe, will at first contract and then expand and fill the pipe. Let H_a be the atmospheric pressure in feet of water. The pressure at the outlet of the pipe will be at atmospheric; but, as the velocity of the vena contracta is larger than that at outlet, the pressure at the vena contracta will be less than atmospheric.

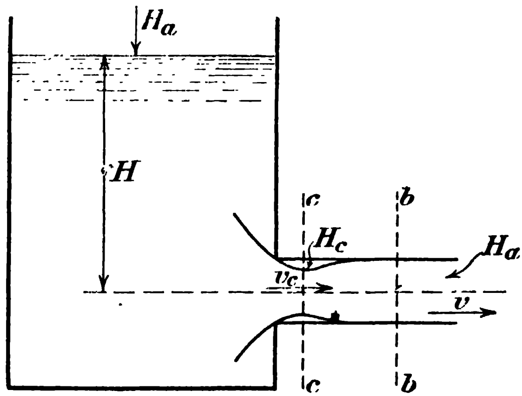


FIG. 52

As the pipe is flowing full at outlet, the coefficient of contraction will be unity. The coefficient of velocity may be calculated by applying Bernoulli's equation to certain sections of the water.

Let a = area of pipe

a_c = area of flow at vena contracta

v = velocity at outlet of pipe

v_c = velocity at vena contracta

H_c = absolute pressure in ft. of water at vena contracta

Assuming coefficient of contraction at vena contracta to be ·62,

$$a_c = C_c a$$
$$= \cdot 62a$$

As quantity flowing at section *cc* equals quantity flowing at *bb*,

$$v_c\, a_c = v\, a$$

$$v_c = v\frac{a}{a_c} \quad . \quad . \quad . \quad . \qquad (1)$$

$$= \frac{v}{\cdot 62}$$

Owing to the enlarging of the section between *cc* and *bb*, there will be a loss of head of $\frac{(v_c - v)^2}{2g}$. (Art. 45.)

Substituting the value of v_c,

$$\text{loss of head} = \frac{\left(\frac{v}{\cdot 62} - v\right)^2}{2g}$$

$$= \cdot 375\frac{v^2}{2g}$$

Applying Bernoulli's equation to free water surface in tank and *bb*,

$$H_a + H = H_a + \frac{v^2}{2g} + \text{loss of head}$$

$$\text{Therefore,} \quad H = \frac{v^2}{2g} + \cdot 375\frac{v^2}{2g}$$

$$= 1{\cdot}375\frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

$$\text{Therefore} \quad C_v = \frac{1}{\sqrt{1{\cdot}375}} = \cdot 855$$

$$\text{Then,} \quad C_d = C_v \times C_c$$

$$= \cdot 855$$

as $C_c = 1$.

The coefficient of discharge is thus considerably increased by fitting an external mouthpiece.

In order to find the pressure at the vena contracta, apply Bernoulli's equation to the water surface in the tank and to the section *cc*.

$$H_a + H = H_c + \frac{v_c^2}{2g}$$

But, from Equation (1), $v_c \;= \frac{}{\cdot 62}$

and from Equation (2), $H \;=\; 1{\cdot}375\frac{v^2}{2g}$

Then, $$H_a + 1{\cdot}375\,\frac{v^2}{2g} = H_c + 2{\cdot}6\frac{v^2}{2g}$$

Therefore, $$H_c = H_a - 1{\cdot}225\frac{v^2}{2g} \quad . \qquad (3)$$
$$= H_a - {\cdot}89H$$

Or, the pressure at the vena contracta is $1{\cdot}225\,\frac{v^2}{2g}$ or $\cdot 89H$ less than atmospheric.

The effect of the mouthpiece on the discharge is to decrease the pressure at the vena contracta and thus increase the effective head causing flow.

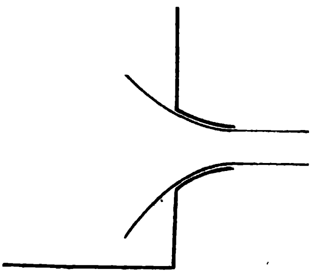


FIG. 53

It is found by experiment that the frictional resistance at the entrance to the mouthpiece reduces the coefficient of discharge from ·855 to ·813. The effect of this frictional resistance on the pressure at the vena contracta is to reduce the vacuum pressure to about $\cdot 74H$.

It will be noticed that the pressure at the vena contracta will be zero when $\cdot 74H = 34$ ft. of water. If this condition were reached, separation would take place and the flow of the water would no longer be steady. In practice this takes place before zero pressure is reached.

In this type of mouthpiece the length of pipe must be at least three diameters in order for the pipe to run full.

By making the mouthpiece to the shape of the jet up to the vena contracta, as in Fig. 53, the loss due to the enlargement is eliminated. This will make the theoretical coefficient of discharge equal to unity. Such a mouthpiece is known as a convergent mouthpiece. Actually, owing to frictional loss, the coefficient of discharge for this mouthpiece is about ·975.

By making the mouthpiece divergent, the loss due to the enlargement of the jet may be considerably reduced. In this

type the mouthpiece is sometimes made convergent up to the vena contracta and then diverges as in Fig. 54. As the divergence increases, the velocity at *cc* increases; this will cause an increase in the vacuum pressure at the vena contracta; and, as this cannot be greater than 34 ft. theoretically, or 26 ft. actually, there is a limit to the amount of divergence if a steady flow is to be maintained.

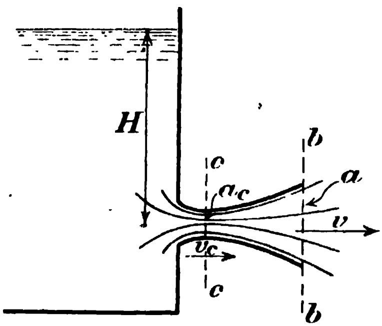


FIG. 54

Applying Bernoulli's equation to water level and to sections *cc* and *bb* (Fig. 54),

$$H_a + H = H_c + \frac{v_c^2}{2g} = H_a + \frac{v^2}{2g}$$

From which, $$\frac{v^2}{2g} = H$$

and, $$\frac{v_c^2}{2g} = H + H_a - H_c$$

But, $$\frac{a}{a_c} = \frac{v_c}{v}$$

$$= \frac{\sqrt{2g(H + H_a - H_c)}}{\sqrt{2g\,H}}$$

$$= \sqrt{1 + \frac{H_a - H_c}{H}}$$

Then, assuming the maximum vacuum pressure to be 26 ft.,

maximum ratio of $$\frac{a}{a_c} = \sqrt{1 + \frac{26}{H}}$$

EXAMPLE.

Water is discharged through an external cylindrical mouthpiece, of 4 sq. in. area, under a head of 10 ft. Find the discharge and the pressure at the vena contracta. Coefficient of contraction = ·64.

Applying Bernoulli's equation to water surface and outlet end of mouthpiece,

$$10 = \frac{v^2}{2g} + \frac{(v_c - v)^2}{2g}$$

But,
$$v_c = \frac{v}{\cdot 64}$$

Then,
$$10 = \frac{v^2}{2g} + \frac{\left(\frac{v}{\cdot 64} - v\right)^2}{2g}$$
$$= \frac{1{\cdot}316\, v^2}{2g}$$

Therefore,
$$v = 22{\cdot}18 \text{ cu. ft. per sec.}$$

$$\text{Discharge} = av = \frac{4}{144} \times 22{\cdot}18 = \cdot 616 \text{ cu. ft. per sec.}$$

$$v_c = \frac{v}{\cdot 64} = 34{\cdot}6 \text{ ft. per sec.}$$

Applying Bernoulli's equation to water surface and vena contracta,

$$34 + 10 = H_c + \frac{v_c^2}{2g}$$

Therefore,
$$H_c = 44 - \frac{(34{\cdot}6)^2}{2g} = 25{\cdot}4 \text{ ft. of water absolute.}$$

50. **Re-entrant or Borda's Mouthpiece.** An internal mouthpiece, such as shown in Fig. 55, is known as a re-entrant or Borda mouthpiece. If the jet, after contraction, does not touch the sides of the mouthpiece, as in Fig. 55, it is said to be running free. If, after contraction, the jet expands and fills the mouthpiece, as in Fig. 56, it is said to be running full.

Consider the mouthpiece of Fig. 55. In this case the mouthpiece is running free.

Let H = height of water surface above centre of mouthpiece
a = area of mouthpiece
v = velocity of flow through mouthpiece
a_c = contracted area of jet

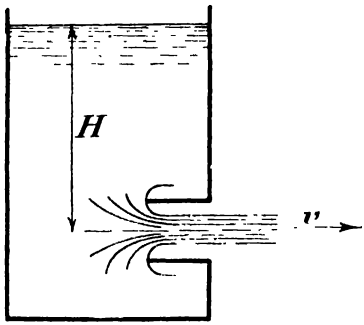


Fig. 55

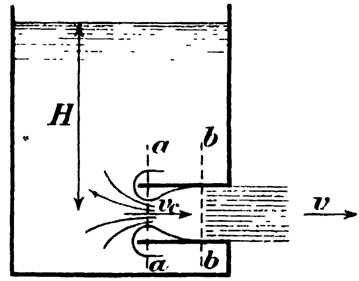


Fig. 56

As force equals rate of change of momentum, total pressure at entrance = change of momentum per second.

Or, $$p\,a = \frac{(w\,a_c\,v)}{g}v$$

Substituting $p = wH$,

$$w\,a\,H = w\,a_c\frac{v^2}{g}$$

But, $$H = \frac{v^2}{2g}$$

Therefore, $$\frac{a\,v^2}{2g} = a_c\frac{v^2}{g}$$

Or, $$a_c = \frac{a}{2}$$

That is, the coefficient of contraction = ·5.

This may be accounted for by the water surrounding the outside of the mouthpiece having to deviate through an angle of 180° in reaching the jet.

Next consider the mouthpiece running full, as in Fig. 56. This case is similar to an external mouthpiece. There will be a vacuum pressure at the vena contracta which will increase the velocity at that section. This will cause an increased discharge as the coefficient of contraction at the outlet is now unity.

Consider the sections *aa* and *bb*. There will be a loss of head due to the enlarging of the section.

Using the same notation as in Art. 49,

$$\text{loss of head due to enlargement} = \frac{(v_c - v)^2}{2g}$$

$$= \left(\frac{1}{C_c} - 1\right)^2 \frac{v^2}{2g}$$

$$= \frac{v^2}{2g}$$

as C_c for the jet. $= \cdot 5$

Applying Bernoulli's equation to the water surface and to the outlet end of the mouthpiece,

$$H_a + H = H_a + \frac{v^2}{2g} + \left(\begin{array}{l}\text{loss due to} \\ \quad\text{enlargement}\end{array}\right)$$

Or,

$$H = \frac{v^2}{2g} + \frac{v^2}{2g}$$

$$= 2\frac{v^2}{2g}$$

Then,

$$v = \sqrt{g H}$$

Discharge, when running full $= av$

$$= a\sqrt{g H}$$

Discharge, when running free $= \cdot 5a \sqrt{2g H}$

Therefore, the discharge is increased by $\dfrac{1}{\cdot 5 \sqrt{2}}$ when running full.

Coefficient of discharge when running full $= \dfrac{1}{\sqrt{2}} = \cdot 707$

In practice, the coefficient of discharge is found to be slightly greater than this amount.

The pressure at the vena contracta may be found by applying Bernoulli's equation to sections *cc* and *bb*.

$$H_c + \frac{v_c^2}{2g} = H_a + \frac{v^2}{2g} + \frac{v^2}{2g}$$

But,

$$v = 2v$$

Therefore, $$H_c + \frac{4v^2}{2g} = H_a + \frac{2v^2}{2g}$$

Or, $$H_c = H_a - H$$

as $H = \dfrac{v^2}{g}$

Thus, the pressure at the vena contracta is less than atmospheric by an amount equal to the head of water in the vessel. Assuming separation takes place at a vacuum pressure of 26 ft. of water, the maximum value of H for steady flow is when $H_c = H_a - 26$.

Then, $$H_a - 26 = H_a - H$$

Or, $$H = 26 \text{ ft. of water.}$$

EXAMPLE.

Calculate the coefficient of discharge from a projecting cylindrical mouthpiece in the side of a water tank assuming that the only loss is that due to the sudden enlargement in the mouthpiece, taking a coefficient of contraction as ·64. Compare the discharge through a Borda mouthpiece in the vertical side of a tank filled with water, and the jet running free, with that from a short cylindrical mouthpiece projecting from the vertical side of the tank if both are placed in similar positions, are 2 in. in diameter, and the constant head above the centre of each is 3 ft. Sketch the issuing jets in each case. (London Univ.)

Applying Bernoulli's equation to water surface and outlet of mouthpiece,

$$H = \frac{v^2}{2g} + \frac{(v_c - v)^2}{2g}$$

But, $$v_c = \frac{v}{\cdot 64}$$

Then, $$H = \frac{v^2}{2g} + \frac{\left(\dfrac{v}{\cdot 64} - v\right)^2}{2g}$$

$$= \frac{1{\cdot}316\, v^2}{2g}$$

And, $$v = \sqrt{\frac{2gH}{1{\cdot}316}}$$

Discharge $$= av = \frac{a}{\sqrt{1{\cdot}316}}\sqrt{2gH}$$

Theoretical discharge $= a\sqrt{2g\,H}$

Coefficient of discharge $= \dfrac{\dfrac{a}{\sqrt{1{\cdot}316}}\sqrt{2g\,H}}{a\sqrt{2g\,H}} = \dfrac{1}{\sqrt{1{\cdot}316}}$

$= {\cdot}875$

Coefficient of discharge for Borda mouthpiece running free $\Big\} = {\cdot}5$

Actual discharge for Borda mouthpiece $\Big\} = {\cdot}5\,a\sqrt{2g\,H}$

$= {\cdot}5 \times \dfrac{\pi}{4} \times \dfrac{4}{144} \times \sqrt{64{\cdot}4 \times 3}$

$= {\cdot}1518$ cu. ft. per sec.

Actual discharge for cylindrical mouthpiece $\Big\} = C_d\,a\sqrt{2g\,H}$

$= {\cdot}875 \times \dfrac{\pi}{4} \times \dfrac{4}{144} \times \sqrt{64{\cdot}4 \times 3}$

$= {\cdot}268$ cu. ft. per sec.

EXAMPLES 4.

(1) The discharge through a sharp-edged circular orifice, 1 in. diameter, under a constant head of 4 ft. is 3·24 cu. ft. per min. Find the coefficient of discharge.

Ans.—$C_d = {\cdot}615$.

(2) If the jet in Question 1, when measured with a screw gauge, is found to have a diameter of ·785 in., find the coefficient of velocity.

Ans.—$C_v = {\cdot}992$.

(3) A jet of water issues from a sharp-edged vertical orifice under a constant head of 4 in. At a certain point of the issuing jet, the horizontal and vertical co-ordinates from the vena contracta are measured and found to be 16 in. and 16·8 in. respectively. Find the coefficient of velocity of the jet.

Ans.—$C_v = {\cdot}978$.

(4) Find the discharge through a large rectangular vertical orifice, 6 ft wide and 4 ft. deep, when the water level is 10 ft. above the top edge of the orifice. $C_d = {\cdot}61$.

Ans.—403 cu. ft. per sec.

(5) Water flows from a tank at the rate of 400 gallons per minute into a horizontal pipe of 6 in. diameter. The pipe suddenly changes to 8 in. diameter at a short distance from the tank and is then suddenly reduced back to 6 in. diameter. Find the loss of head at entrance to pipe, at enlargement, and at contraction.

Ans.—·231, ·09, ·231 ft. of water.

(6) A large tank has a circular sharp-edged orifice 1·44 sq. in. in area at a depth of 9 ft. below constant water level. The jet issues horizontally, and in a horizontal distance of 7·8 ft. it falls 1·8 ft. The measured discharge is ·15 cusecs. Calculate the coefficients of velocity, contraction, and discharge. (A.M.I. Civil E.)

Ans.—·97 ; ·643 ; ·624.

(7) A reservoir is circular in plan, the diameter of the top water level is 300 ft., at a depth of 5 ft. the diameter is 250 ft. The mouth of the outlet pipe, which is 24 in. in diameter, is 12 ft. below top water level ; how long will it take to lower the depth of the water in the reservoir 5 ft. ? (Take $C = \cdot 8$.) (London Univ.)

Ans.—79·4 min.

(8) A tank 20 ft. long and 5 ft. wide is divided into two parts, by a partition, so that one part is four times the other part. The water level in the large portion is 10 ft. higher than that in the smaller. Find the time for the difference of water level in the two portions to reach 4 ft. if the water flows through an orifice in the partition 3 in. square. $C_d = \cdot 6$.

Ans.—6·2 min.

(9) A pipe 10 in. diameter has a diaphragm fitted in it, in which there is a hole 4 in. diameter concentric with the pipe. Investigate a formula for the loss of head at the diaphragm and show how the arrangement can be used to measure the flow along the pipe.

Show also how you would check experimentally the assumptions made. (London Univ.)

(10) Establish Bernoulli's equation for the stream line motion of a fluid. Show that when water is issuing steadily from a re-entrant orifice in the bottom of a tank, the area of the jet at the vena contracta is ½ of the area of the orifice. (London Univ.)

(11) Water in a tank discharges through an external divergent mouthpiece. If the outlet area of the mouthpiece is four times the minimum area, find the maximum head in the tank at which steady flow through the mouthpiece can be obtained. Assume separation takes place at an absolute pressure of 8 ft. of water.

Ans.—1·735 ft. of water.

(12) Water under a constant head of 9 ft. discharges through an external cylindrical mouthpiece of 2 in. diameter. $C_c = \cdot 6$.

Find, (1) the discharge in cubic feet per second ; (2) the coefficient of discharge ; (3) the absolute pressure at the vena contracta in feet of water.

Ans.—·437 ; ·832 ; 25·7.

(13) If the mouthpiece in Question (12) were a Borda mouthpiece running full, what would be the discharge ?

Ans.—·372 cu. ft. per sec.

(14) Compensation water is to be discharged by two circular orifices under a constant head of 2 ft. 6 in., measured to the centre of the orifices. What diameter will be required to give 3,000,000 gallons a day ? $C_c = \cdot 62$; $C_v = \cdot 97$. (A.M.I. Civil E.)

Ans.—8·18 in.

(15) A pipe increases abruptly from diameter d to diameter D. Deduce an expression for the loss of head by shock when the discharge is Q. If $d = 12$ in., $D = 18$ in., and $Q = 5$ cu. ft. per sec., what is the loss of head ? (A.M.I. Civil E.)

Ans.—·196 ft.

(16) Deduce an expression for the loss of head at a sudden enlargement in a pipe line. Using your result, determine the loss of head when a 12-in. pipe line discharges directly through the side of a reservoir, the velocity of flow being 10 ft. per sec. (A.M.I.Mech.E.) (Assume $C_c = \cdot 6$.)

Ans.—·692 ft. of water.

(17) Two vertical-sided basins, each having a surface area of 2,000 sq. ft., are connected by a sluice gate of area 2 sq. ft. The initial difference of level in the basins is 9 ft. How long will it take to reduce this to 4 ft.? The coefficient of discharge of the orifice is ·8. (A.M.Inst.C.E.)

Ans.—2 mins. 36 secs.

(18) A sharp-edged orifice 1·9 in. diameter is employed to measure the supply of air to an oil-engine. Prove that the volume in ft.^{3}/min. passing through the orifice is $13\sqrt{\frac{h}{\rho}}$, in which h is the drop of pressure between the two sides of the orifice measured in inches of water and ρ is the density of the air, assumed uniform, in lb./ft.^{3}, and the coefficient of discharge for the orifice is ·602.

Calculate the volume of air in ft.^{3}/min. at N.T.P. if $h = \cdot 85$ in. of water, the pressure of the atmosphere 30·5 in. of mercury, and the temperature 15·8° C. For air $PV = 96T$. (Lond. Univ.)

Ans.—43·0 cu. ft. per min.

(19) A tank with vertical sides and a horizontal cross-sectional area of 20 sq. ft. is provided with a notch cut at the top of one of the sides. Water flowing into the tank at a constant rate was discharged over the notch, the head over the bottom of which was 6 in.

The supply of water was suddenly stopped and it was observed that the head over the notch started to fall at the rate of 0·14 in. per sec. When the head had fallen to 3 in. it was found that it was falling at the rate of 0·05 in. per sec. Estimate the rate of inflow to the tank when there is a steady head of 5 in. over the notch. (London Univ.)

Ans.—·18 cu. ft. per sec.

(20) A cylindrical tank 5 ft. diameter and 20 ft. high discharges through a circular orifice 2 in. diameter in the base of the tank.

If there is inflow at a constant rate, and free discharge, find the rate of inflow if the head increases from 1 to 15 ft. in 15 minutes. $C_d = \cdot 62$.

Ans.—·619 cu. ft. per sec.

(21) Air freely enters a heated drying shed through many openings at the base of the wall, and leaves through a circular sharp-edged orifice, area 4 sq. ft., placed 12 ft. higher up the wall. The temperature is 25° C. inside, and 15° C. outside. Estimate the rate of flow. $C_d = \cdot 62$. (I. Mech. E.)

Ans.—13·13 cu. ft. per sec.

CHAPTER V

NOTCHES AND WEIRS

51. **Notches and Weirs.** A notch may be regarded as an orifice with the water surface below its upper edge. Notches are used for measuring the flow of water from a vessel or reservoir and are generally rectangular or triangular in shape.

A weir is the name given to a dam over which water is flowing. Theoretically, there is no difference between a simple rectangular weir and a rectangular notch, except the latter may have sharp edges.*

The sheet of water flowing through a notch or over a weir is known as the nappe or vein. The top of the weir over which the water flows is known as the sill or crest. Large weirs are sometimes divided into sections by vertical posts.†

Shallow rivers are often made navigable by building dams across the river at certain sections over which the water may flow. This has the effect of deepening the river on the up-stream side of the dam by an amount equal to the height of the dam above the original water level. During a drought, little or no water will flow past the dam; but after heavy rains the water flows over the dam, thus converting it into a weir. It is necessary to make short canals, containing locks, around these dams in order that the shipping may pass.

52. **Rectangular Notch.** If water flows from a tank or reservoir over a notch there will be a contraction of the vein and a slight frictional resistance at the sides, as in the case of an orifice. This will cause the actual discharge to be less than the theoretical discharge; the ratio between them will be the coefficient of discharge for the notch. An average value of this coefficient is about ·62.

Consider the rectangular notch in Fig. 57.

Let L = breadth of notch

H = height of water surface above sill

C_d = coefficient of discharge

Consider a horizontal strip of water of thickness dh and

* The term "weir" is sometimes loosely applied to small notches.

† For non-dimensional factors for weirs, see Art. 199.

of depth h. The theoretical velocity of the water flowing through strip will be $\sqrt{2gh}$.

$$\text{Discharge through strip} = L\,dh\,\sqrt{2gh}$$

$$\text{Total discharge} = L\sqrt{2g}\,C_d\int_0^H h^{\frac{1}{2}}\,dh$$

$$= \frac{2}{3}L\sqrt{2g}\,C_d\Big[h^{\frac{3}{2}}\Big]_0^H$$

$$= \frac{2}{3}C_d\sqrt{2g}\,L\,H^{\frac{3}{2}} \qquad . \qquad . \qquad (1)$$

This equation is not used for large weirs.

If a notch of this type is used for measuring a quantity of water flowing, it must be calibrated experimentally. The discharge for any given weir is equal to $k\,H^{\frac{3}{2}}$ where

$$k = \frac{2}{3}\,C_d\sqrt{2g}\,L.$$

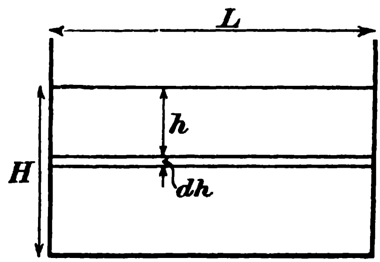


FIG. 57

Then, by measuring the discharge per second for various heads, the value of k may be obtained by plotting the discharge and $H^{\frac{3}{2}}$. A perfect straight line will not be obtained, as C_d varies slightly with the head. This method of obtaining C_d is demonstrated in Example 2.

An alternative method is to assume $Q = kH^n$; then taking logs of both sides of this equation,

$$\log Q = \log k + n\log H \qquad . \qquad . \qquad . \qquad (2)$$

which is a straight line law.

By plotting from experimental results $\log H$ as base and $\log Q$ as vertical ordinate, a straight line is obtained from which k and n can be found. For,

when $H = 1$, $\log H = O$; then $\log k = \log Q$, hence k.

Also, by choosing any convenient point on the graph,

$$n = \frac{\log Q - \log \mathrm{h}}{\log H} \qquad \text{[From Eq. (2).]}$$

This method is demonstrated in Example 1.

Example 1.

The following observations were made during measurements on a weir, whose crest "b" is 3 ft. long.

Head "*H*" ft. . .	·2	·4	·6	·8	1·0	1·2	1·5
Q in. ft. per sec. .	·846	2·34	4·24	6·48	9·00	11·78	16·35

If the discharge is given by $Q = KbH^n$, determine K and n. (A.M.I.Mech. E.)

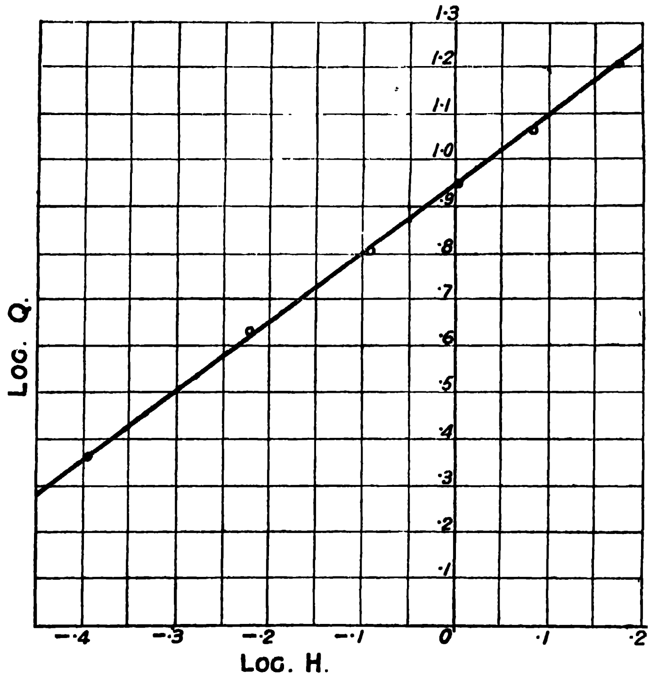


Fig. 58

First plot log H and log Q and draw a straight line a mean through the points, as shown in Fig. 58.

$$\text{Let } k = Kb$$

$$\text{When } H = 1, \log H = 0;\ \text{then}$$

$$\log k = \log Q$$

$$= \cdot 955$$

From which, $k = 9$

Then, $K = k \div b$

$= 9 \div 3 = 3$

Also, $$n = \frac{\log Q - \log k}{\log H}$$

Using the values for the point at which $\log H = \cdot 1$,

$$n = \frac{1 \cdot 105 - \cdot 955}{\cdot 1}$$

$$= 1 \cdot 5$$

Hence, equation is

$$Q = 3bH^{\frac{3}{2}}$$

EXAMPLE 2.

In order to find the coefficient of discharge for a small rectangular notch, the discharge was measured experimentally for different heads for a rectangular notch 6 in. wide. The following results were obtained—

Head in feet	·0651	·0716	·0775	·0827	·0870
Discharge in cubic feet per second . . .	·02813	·03180	·03535	·03872	·0419

Find the average value of C_d for the notch.

$$\text{Discharge} = Q = \frac{2}{3} C_d \sqrt{2g}\, L\, H^{\frac{3}{2}}$$

$$= k\, H^{\frac{3}{2}}$$

The value of the constant k may be found by plotting Q and $H^{\frac{3}{2}}$, thus obtaining a straight line. This is done in Fig. 59; a straight line is drawn a mean through the points and passing through the origin.

Taking the values of Q and $H^{\frac{3}{2}}$ from a point P on the straight line,

$$k = \frac{Q}{H^{\frac{3}{2}}} = \frac{\cdot 025}{\cdot 015} = 1 \cdot 667$$

As $$k = \frac{2}{3} C_d \sqrt{2g}\, L,$$

$$\frac{2}{3} C_d \sqrt{2g} = \frac{k}{L} = \frac{1\cdot667}{\cdot5} = 3\cdot333$$

And, $$C_d = \frac{3\cdot333}{\frac{2}{3}\sqrt{2g}} = \cdot624$$

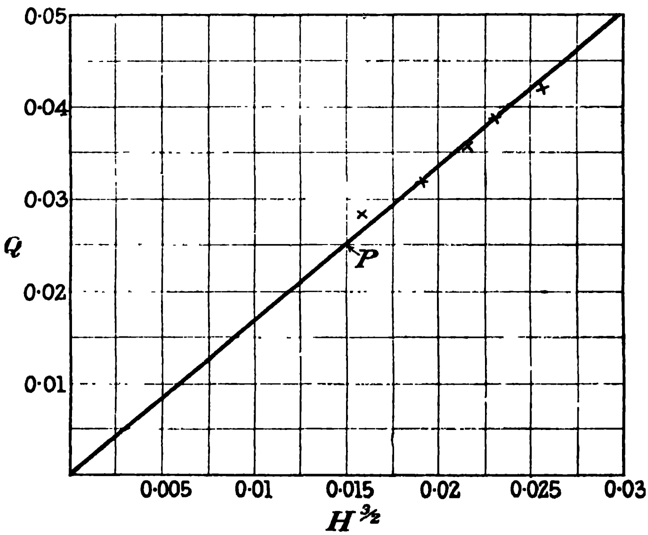


FIG. 59

53. **Triangular or V-Notch.** In the case of a rectangular notch, it will be noticed that the total wetted edge of the notch does not vary directly with the head, as the length of the base is the same for all heads. Therefore, the coefficient of contraction, which depends on the length of wetted edge, is not a constant for all heads. But in the case of a triangular notch, there is no base to cause contraction, which will be due to the sides only. The coefficient of contraction will, therefore, be a constant for all heads. For this reason, the triangular notch is the most satisfactory type for measuring the quantity of water flowing.

Consider the triangular notch in Fig. 60.

Let H = height of water surface

and θ = angle of notch

Then, width of notch at water surface $= 2H \tan\frac{\theta}{2}$

Consider a horizontal strip of the notch of thickness dh and of depth h.

Width of strip $= 2(H-h)\tan\frac{\theta}{2}$

Theoretical velocity of flow through strip $= \sqrt{2gh}$

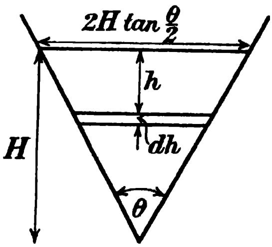


FIG. 60

Discharge through strip $= 2(H-h)\tan\frac{\theta}{2}\,dh\,\sqrt{2gh}\,C_d$

$$\left.\begin{array}{l}\text{Total discharge through}\\ \text{notch}\end{array}\right\} = 2\,C_d\sqrt{2g}\tan\frac{\theta}{2}\int_0^H (H-h)h^{\frac{1}{2}}\,dh$$

$$= 2\,C_d\sqrt{2g}\tan\frac{\theta}{2}\left[\frac{2}{3}Hh^{\frac{3}{2}} - \frac{2}{5}h^{\frac{5}{2}}\right]_0^H$$

$$= \frac{8}{15}C_d\sqrt{2g}\tan\frac{\theta}{2}H^{\frac{5}{2}}$$

Assuming $C_d = \cdot 6$,

Discharge $= 2{\cdot}56\tan\frac{\theta}{2}H^{\frac{5}{2}}$ . . . (1)

For a 90° notch, $\tan\frac{\theta}{2} = 1$

Then, discharge $= 2{\cdot}56\,H^{\frac{5}{2}}$ . . . . (2)

EXAMPLE 1.

In order to find the constant for a 90° triangular notch, the discharge through the notch was measured for different heads. The following readings were obtained—

Head in feet	**·0407**	**·0491**	**·0550**	**·0692**	**·0798**	**·0919**	**·101**
Discharge in cubic feet per second	**·00095**	**·00156**	**·00207**	**·00361**	**·00490**	**·00702**	**·00867**

Find the constant for the notch and the value of C_d.

As discharge for a 90° notch $= \frac{8}{15} C_d \sqrt{2g} H^{\frac{5}{2}}$

$$Q = k H^{\frac{5}{2}}$$

where $k = \frac{8}{15} C_d \sqrt{2g}$

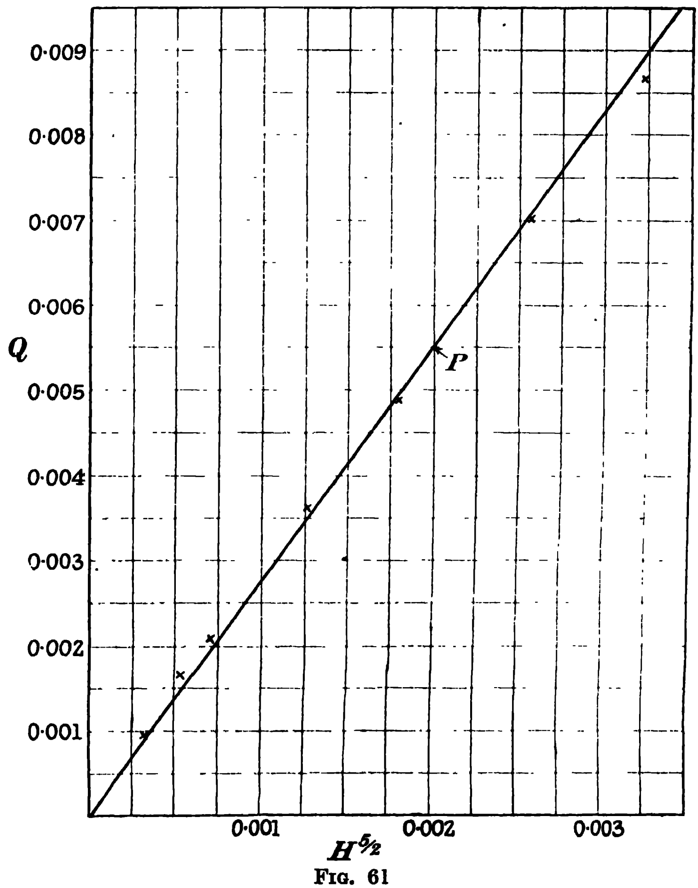


FIG. 61

A straight line should, therefore, be obtained by plotting Q and $H^{\frac{5}{2}}$. This has been done in Fig. 61 ; it will be noticed that the points lie approximately on a straight line passing through the origin.

From point P on this straight line,

$$k = \frac{Q}{H^{\frac{5}{2}}} = \frac{\cdot 0055}{\cdot 002} = 2\cdot 75$$

Then, $$Q = 2\cdot 75\, H^{\frac{5}{2}}$$

As $$k = \frac{8}{15} C_d \sqrt{2g}$$

$$C_d = \frac{2\cdot 75 \times 15}{\sqrt{2g} \times 8} = \cdot 642$$

EXAMPLE 2.

A trapezoidal notch has a base L and a head H, the sides make an angle of θ to the vertical. Deduce an expression for the discharge through the notch.

A notch of this type may be divided into a rectangular notch of breadth L, and a triangular notch subtending an angle of 2θ. Then, the total discharge may be found by adding together the discharges from these two.

Discharge through rectangular notch $\left.\right\} = \frac{2}{3} C_d \sqrt{2g}\, L\, H^{\frac{3}{2}}$

Discharge through triangular notch $\left.\right\} = \frac{8}{15} C_d \sqrt{2g} \tan \theta . H^{\frac{5}{2}}$

Total discharge $= \frac{2}{3} C_d \sqrt{2g}\, L\, H^{\frac{3}{2}} + \frac{8}{15} C_d \sqrt{2g} \tan \theta . H^{\frac{5}{2}}$

$$= C_d \sqrt{2g}\, H^{\frac{3}{2}} \left(\frac{2}{3} L + \frac{8}{15} \tan \theta . H \right)$$

This may also be obtained from first principles by producing the sloping sides to their point of intersection, and integrating between the limits of the head H.

54. **Thomson's Principle of Geometric Similarity.** Geometrically similar weirs or notches may be defined as notches which may be represented by drawings of the same notch but to a different scale. The discharge through similar notches will depend on their linear dimensions raised to power of $\frac{5}{2}$.

Consider two similar triangular notches.

$$\text{Discharge} \propto \text{area} \times \text{velocity}$$

But, $$\text{area} \propto H^2$$

and $$\text{velocity} \propto \sqrt{H}$$

Then, $$\text{discharge} \propto H^2 \times \sqrt{H}$$
$$= k\, H^{\frac{5}{2}}$$

The constant k should be the same for all similar notches.

In the case of similar rectangular notches, let the breadth of the notches be L and nL, and let the corresponding heads be H and nH.

Then,

$$\frac{\text{discharge of one weir}}{\text{discharge of other}} = \frac{(\text{area} \times \text{velocity}) \text{ of one}}{(\text{area} \times \text{velocity}) \text{ of other}}$$

$$= \frac{nL \times nH \times \sqrt{nH}}{L \times H \times \sqrt{H}}$$

$$= n^{\frac{5}{2}}$$

55. Francis' Formula for Rectangular Weirs. An empirical formula for the discharge of a rectangular weir is given by Francis as—

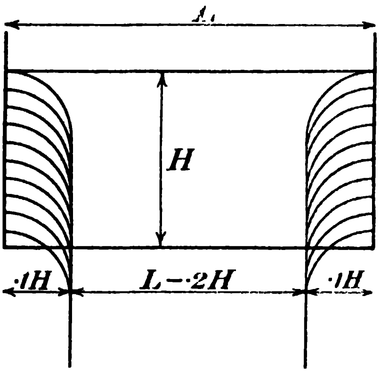


Fig. 62

$$Q = 3{\cdot}33(L - {\cdot}1\,nH)H^{\frac{3}{2}} \text{ cu. ft. per sec.}$$

where L = total breadth of weir in feet

H = head in feet

and n = number of end contractions

For a simple rectangular weir, $n = 2$. For a large weir which is split up into bays by vertical posts, n will depend on the number of bays into which the weir is divided.

Although Francis deduced this formula experimentally, it can be proved to be quite rational. As the water flows over the weir, the vein is contracted at the sides (Fig. 62) by an amount which is found, experimentally, to average $\cdot 1\,H$ for each side. If there are two contractions, as in the case of Fig. 62, the effective breadth of the weir is $(L - {\cdot}2H)$. If there were n contractions the effective length would be $(L - {\cdot}1n\,H)$.

Substituting the effective length in Equation (1), Art. 52,

$$Q = \frac{2}{3} C_d \sqrt{2g}\,(L - {\cdot}1n\,H)H^{\frac{3}{2}}$$

Assuming $C_d = {\cdot}623$,

$$Q = 3{\cdot}33\,(L - {\cdot}1n\,H)H^{\frac{3}{2}}$$

If the end contractions are suppressed, as in the case of a weir having the same width as the channel by which the water approaches, n will be zero.

Then, $Q = 3{\cdot}33\, L\, H^{\frac{3}{2}}$

EXAMPLE.

Show that a rational formula for the flow Q over a rectangular weir of width B can be expressed as

$$Q = A(B - CH)H^{\frac{3}{2}}$$

where H is the depth of water at a point near the weir which is not affected by the curvature of the surface.

In a rectangular weir, 5 ft. in breadth, the discharge is 5·91 cu. ft. per sec. when the head of water is ·51 ft., and is 14·99 cu. ft. when the head is ·96 ft. Find the values of the constants in the above expression, and estimate the discharge when the head is ·75 ft. (London Univ.)

This is Francis' formula. Substituting the values of B and H in the given two cases,

$$5{\cdot}91 = A(5 - C\,{\cdot}51)\,{\cdot}51^{\frac{3}{2}} \quad . \quad . \quad . \quad (1)$$

also,
$$14{\cdot}99 = A(5 - C\,{\cdot}96)\,{\cdot}96^{\frac{3}{2}} \quad . \quad . \quad . \quad (2)$$

From (1), $16{\cdot}2 = 5A - {\cdot}51\,AC$ . . . . (3)

From (2), $16{\cdot}0 = 5A - {\cdot}96\,AC$

Subtracting, ${\cdot}2 = {\cdot}45\,AC$

Then, $$C = \frac{.2}{{\cdot}45A}$$

Substituting in (3),

$$16{\cdot}2 = 5A - {\cdot}51A \times \frac{{\cdot}2}{{\cdot}45A}$$

$$= 5A - {\cdot}2265$$

$$A = 3{\cdot}29$$

and, $$C = \frac{{\cdot}2}{{\cdot}45 \times 3{\cdot}29} = {\cdot}135$$

Then, $Q = 3{\cdot}29\,(5 - {\cdot}135\,H)H^{\frac{3}{2}}$

When $H = {\cdot}75$ ft.,

$$Q = 3{\cdot}29\,(5 - {\cdot}101)\,{\cdot}75^{\frac{3}{2}}$$

$$= 10{\cdot}46 \text{ cu. ft. per sec.}$$

56. Bazin's Formula for Rectangular Weirs. Another type of equation used for obtaining the discharge over a rectangular weir without end contractions is known as Bazin's formula. Using Equation (1), Art. 47,

$$Q = \frac{2}{3} C_d \sqrt{2g}\, L\, H^{\frac{3}{2}}$$

$$= m \sqrt{2g}\, L\, H^{\frac{3}{2}},$$

where $$m = \frac{2}{3} C_d$$

The coefficient m was found by Bazin to vary with the head, its value being obtained from the following equation—

$$m = \cdot 405 + \frac{\cdot 00984}{H}$$

EXAMPLE.

Find the discharge, using Bazin's formula, for a rectangular weir with end contractions suppressed, when the head is 6 in. and the length 4 ft.

As $H = \cdot 5$,

$$m = \cdot 405 + \frac{\cdot 00984}{\cdot 5}$$

$$= \cdot 42468$$

Discharge $$= m \sqrt{2g}\, L\, H^{\frac{3}{2}}$$

$$= \cdot 42468 \times \sqrt{2g} \times 4 \times (\cdot 5)^{\frac{3}{2}}$$

$$= 4 \cdot 82 \text{ cu. ft. per sec.}$$

57. Velocity of Approach. If the area of the channel through which the water approaches the weir is larger than the weir itself, the water will have a velocity on reaching the weir known as the velocity of approach. This velocity may be assumed to be uniform over the whole weir.

Let A = cross-sectional area of channel behind weir

v_1 = velocity of approach

Q = discharge over weir in cu. ft. per sec.

Then, as quantity of water passing over weir per second equals quantity flowing along channel per second,

$$v_1 = \frac{Q}{A}$$

The quantity Q is determined, as a first approximation, from the ordinary weir equation, ignoring the velocity of approach.

Additional head due to velocity of approach $= \dfrac{v_1^2}{2g}$ and acts over whole of weir.

Consider the horizontal strip of the weir in Fig. 57.

Discharge through strip $= C_d \sqrt{2g\,h} \times L\,dh$

$$\text{Total discharge} = C_d \sqrt{2g}\, L \int_{\frac{v_1^2}{2g}}^{H + \frac{v_1}{2g}} h^{\frac{1}{2}}\, dh$$

$$= \frac{2}{3} C_d \sqrt{2g}\, L \Big[h^{\frac{3}{2}} \Big]_{\frac{v_1^2}{2g}}^{H + \frac{v_1}{2g}}$$

$$= \frac{2}{3} C_d \sqrt{2g}\, L \left[\left(H + \frac{v_1^2}{2g} \right)^{\frac{3}{2}} - \left(\frac{v_1^2}{2g} \right)^{\frac{3}{2}} \right] \quad . \quad . \quad . \quad . \quad (1)$$

As the value of v_1 was obtained only approximately in the first case, it should be corrected to suit the new discharge obtained from Equation (1). Then, by substituting this new value of v_1 in Equation (1), a more accurate value of the actual discharge may be obtained. If the value of v_1 is small, this correction of the first approximation will make very little difference to the discharge obtained from Equation (1).

Francis' formula for velocity of approach becomes—

$$Q = 3{\cdot}33\,(L - {\cdot}1n\,H_1) \left\{ H_1^{\frac{3}{2}} - \left(\frac{v_1^2}{2g} \right)^{\frac{3}{2}} \right\}$$

where $H_1 = \dfrac{v_1^2}{2g} + H$, and is known as the still water head.

From the results of experiments, Bazin found that the discharge could be obtained by increasing the actual measured head, H, by the amount $a\,\dfrac{v_1^2}{2g}$, where a is a constant having a mean value of 1·6.

Then, equivalent static head, or still water head

$$= H + \frac{a\,v_1^2}{2g}$$

$$= H_1$$

Bazin's formula then becomes—

$$Q = m\sqrt{2g}\,L\left(H + \frac{av_1^2}{2g}\right)^{\frac{3}{2}}$$

where $m = \cdot405 + \frac{\cdot00984}{H_1}$

EXAMPLE.

Find the discharge over a weir 10 ft. long under a measured head of 2 ft., if the channel approaching the weir is 20 ft. wide and 3 ft. deep.

First find the discharge, ignoring velocity of approach. Using Francis' formula,

$$Q = 3{\cdot}33(L - \cdot2H)H^{\frac{3}{2}}$$

$$= 3{\cdot}33\,(10 - \cdot4)2^{\frac{3}{2}}$$

$$= 90{\cdot}4 \text{ cu. ft. per sec.}$$

$$v_1 = \frac{Q}{A} = \frac{90{\cdot}4}{20 \times 3} = 1{\cdot}51 \text{ ft. per sec.}$$

$$\text{Still water head} = H_1 = H + \frac{v_1^2}{2g}$$

$$= 2 + \frac{1{\cdot}51^2}{2g}$$

$$= 2{\cdot}035 \text{ ft.}$$

Substituting in Francis' formula for velocity of approach.

$$Q = 3{\cdot}33\,(L - \cdot1nH_1)\left\{H_1^{\frac{3}{2}} - \left(\frac{v_1^2}{2g}\right)^{\frac{3}{2}}\right\}$$

$$= 3{\cdot}33\,(10 - \cdot407)\left\{2{\cdot}035^{\frac{3}{2}} - \left(\frac{1{\cdot}51^2}{2g}\right)^{\frac{3}{2}}\right\}$$

$$= 93{\cdot}0 \text{ cu. ft. per sec.}$$

In this example the value of v_1 is so small that no adjustment is necessary.

58. **Time of Emptying Reservoir with Rectangular Weir.** Consider a reservoir of area A sq. ft. in plan from which water is flowing over a rectangular weir of breadth L. It is required to find the time taken for the water level in the reservoir to fall from a height H_1 to a height H_2 above the level of the sill.

Suppose at any instant the height of water level above the sill is h. Then let a small quantity dq flow over the weir in a time dt, and let this cause the water level in reservoir to fall by amount dh. As dh is measured downwards, it is negative.

$$\text{Discharge through weir} = dq = \frac{2}{3} C_d \sqrt{2g}\, L\, h^{\frac{3}{2}}\, dt$$

$$\text{Discharge from reservoir} = dq = -A\, dh$$

$$\text{Then,} \quad \frac{2}{3} C_d \sqrt{2g} L h^{\frac{3}{2}} dt = -A\, dh$$

$$\text{And,} \quad dt = -\frac{A\, dh\, h^{-\frac{3}{2}}}{\frac{2}{3} C_d \sqrt{2g}\, L}$$

$$\text{Total time} = T = \int_0^T dt = -\frac{A}{\frac{2}{3} C_d \sqrt{2g}\, L} \int_{H_1}^{H_2} h^{-\frac{3}{2}} dh$$

$$= \frac{2A}{\frac{2}{3} C_d \sqrt{2g}\, L} \Big[h^{-\frac{1}{2}} \Big]_{H_1}^{H_2}$$

$$= \frac{2A}{\frac{2}{3} C_d \sqrt{2g}\, L} \left(\frac{1}{H_2^{\frac{1}{2}}} - \frac{1}{H_1^{\frac{1}{2}}} \right)$$

If Bazin's coefficient is used, this equation becomes

$$T = \frac{2A}{m \sqrt{2g}\, L} \left(\frac{1}{H_2^{\frac{1}{2}}} - \frac{1}{H_1^{\frac{1}{2}}} \right)$$

Using Francis' formula the equation becomes

$$T = \frac{2A}{3{\cdot}33(L - {\cdot}1nH)} \left(\frac{1}{H_2^{\frac{1}{2}}} - \frac{1}{H_1^{\frac{1}{2}}} \right)$$

the value of H being taken as a mean of H_1 and H_2.

EXAMPLE.

Show that the discharge over a sharp-edged V notch is theoretically proportional to the head raised to an index power of 2·5.

A sharp-edged V notch inserted in the side of a rectangular tank, 12 ft. long and 4 ft. broad, gives a calibration $Q = 2{\cdot}64 H^{2\cdot5}$ where Q is measured in cubic feet per second and H is measured in feet. Find how long it will take to reduce the head in the tank from 12 in. to 1 in. if the water discharges freely over the notch and there is no inflow into the tank. (London Univ.)

Consider the water level at any instant to be h ft. above bottom of notch. Let small quantity dq flow through in time dt, thereby reducing water level by dh.

Then, $$dq = 2{\cdot}64\, h^{2\cdot5}\, dt$$

Also, $$dq = -A\, dh \text{ (as } dh \text{ is negative)}$$

Therefore, $$2{\cdot}64\, h^{2\cdot5}\, dt = -A\, dh$$

Or, $$dt = -\frac{A\, dh}{2{\cdot}64\, h^{2\cdot5}}$$

$$\text{Total time} = T = \int_0^T dt = -\frac{A}{2{\cdot}64}\int_1^{\frac{1}{12}} h^{-2\cdot5}\, dh$$

$$= \frac{2}{3} \times \frac{A}{2{\cdot}64}\Big[h^{-1\cdot5}\Big]_1^{\frac{1}{12}}$$

$$= \frac{2}{3} \times \frac{A}{2{\cdot}64}\left(\frac{1}{(\frac{1}{12})^{\frac{3}{2}}} - \frac{1}{(1)^{\frac{3}{2}}}\right)$$

$$= \frac{2}{3} \times \frac{4 \times 12}{2{\cdot}64}\left(\frac{1}{{\cdot}0241} - 1\right) \text{ secs.}$$

$$= 8{\cdot}2 \text{ mins.}$$

59. **The Syphon Spillway.** It is necessary with reservoirs to have an automatic device for keeping the water level in them at a constant height. The simplest way is to fit a weir at the side, with the sill at the same height as the required water level. An increase in the water level will then cause the excess water to flow over the sill into the overflow channel below. It will thus flow away to waste. As the discharge by this method is due to the head over the sill only, the method is not sensitive unless the weir is very long, which is not a practical proposition.

A more sensitive method is to fit syphon spillways to the

reservoir; these employ the whole head between the reservoir level and the water level in the overflow channel, thus creating a high velocity and a consequently large discharge.

A cross-sectional view of an automatic syphon spillway is shown in Fig. 63. It consists of an ordinary weir sill surrounded by an air-tight cover, as shown, thus converting the discharge face of the weir into a large rectangular-sectioned pipe.

As soon as the water level in the reservoir rises above the sill *A* by a measurable amount, the water flows over the sill and strikes the inside of the cover, thus completely filling the

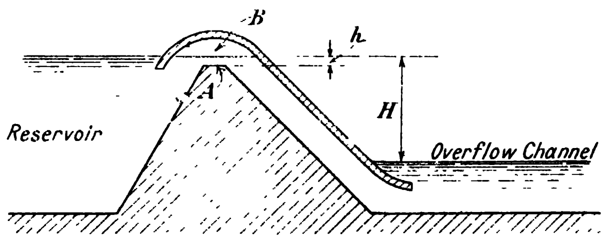


FIG. 63.—AUTOMATIC SYPHON SPILLWAY

cross-section of the pipe. The air is now trapped in the upper portion of the cover *B*, and is immediately sucked away by the stream of flowing water. This causes a negative pressure at *B* which sucks up the water from the reservoir and completely fills the pipe; the syphon action is thus started. The water will now rush down the pipe to waste, with a velocity caused by the total head *H*, thus causing a large discharge. Had this operation been performed by an ordinary open weir, the velocity of discharge would have been due to the small head *h* only, and would have been consequently very small.

It will be seen from this that the syphon spillway has a much greater discharge for a given length than an ordinary open weir, as it utilizes the whole difference of head between the reservoir and the overflow channel.

Sometimes several syphon spillways are fitted side by side with their sills at different levels. Then, for a small rise in water level, the lowest spillway only is in action. If this discharge is not sufficient to maintain the correct level, the water in the reservoir will rise higher and thus bring the next spillway into action.

Let A_1 = area of cross-section of spillway pipe, sq. ft.
H = difference in water level of reservoir and overflow channel, in feet.

Then, $Q = C_d A_1 \sqrt{2g H}$

where C_d is the coefficient of discharge, to be determined by test.

60. **Broad-crested Weir.** A weir having a broad sill is known as a broad-crested weir. The discharge of a weir of this type depends on the head H, the breadth b, the length l of the sill; it will also depend on the roughness of the sill's surface, on the viscosity, and on the temperature.

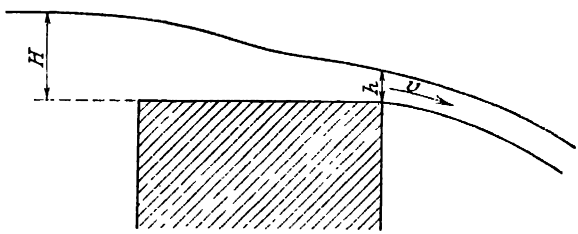


FIG. 64

As the water flows over the sill there is a loss of head due to the frictional resistance. If the sill is very long, this resistance will be similar to that of the bed of an open channel.

Let Fig. 64 represent the water flowing over a broad-crested weir and let h be the head of water at the downstream edge of the sill. Assume the sill is of sufficient length to allow the velocity of the water to be uniform throughout its depth at the downstream edge. Let this uniform velocity be v ft. per sec.

Then, neglecting losses, $H = h + \dfrac{v^2}{2g}$

from which $v = \sqrt{2g(H - h)}$

Then, discharge per sec. $= Q = C_d bhv$

$$= C_d bh\sqrt{2g(H - h)}$$

$$= C_d\sqrt{2g}\,b(Hh^2 - h^3)^{\frac{1}{2}} \quad . \quad . \quad (1)$$

where C_d is the experimental coefficient of discharge for the weir. C_d will be a function of the head, the length of sill, the

roughness of surface, the coefficient of viscosity and of the temperature. Its value can be obtained only from test.

From Equation (1) it will be noticed that the discharge depends on the head h. By differentiating this equation the value of h in terms of H can be obtained for maximum discharge. Then, differentiating Equation (1),

$$\frac{dQ}{dh} = C_d\sqrt{2g}\, b\tfrac{1}{2}(Hh^2 - h^3)^{-\frac{1}{2}}(2Hh - 3h^2) = 0$$

that is,
$$\frac{2Hh - 3h^2}{\sqrt{Hh^2 - h^3}} = 0$$

or
$$2H - 3h = 0$$

hence,
$$h = \tfrac{2}{3}H$$

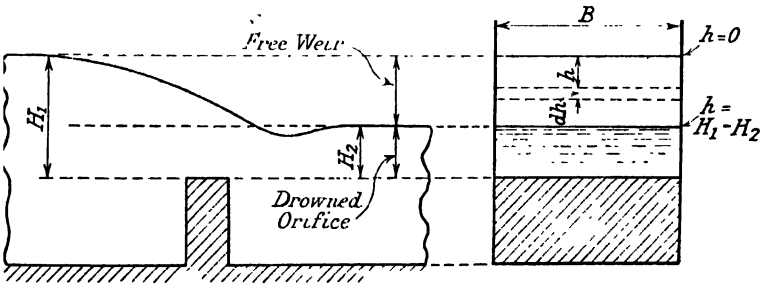


FIG. 65

Substituting this value in Equation (1),

$$Q = C_d\sqrt{2g}\, b\, \tfrac{2}{3}H\sqrt{\tfrac{1}{3}H}$$
$$= 3{\cdot}09C_d bH^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

61. **Flow over a Submerged Weir.** The discharge over a submerged weir can be obtained by dividing it into two horizontal sections, as shown in Fig. 65. The portion between the upstream and downstream water surfaces may be treated as a free weir; the portion between the downstream water surface and the top of the sill may be treated as a drowned orifice.

Let Q_1 = discharge per sec. through the free portion of the weir. Consider a horizontal strip at h ft. below the upper surface and of thickness dh. Then,

discharge through strip $= dq = C_d\, B\, \sqrt{2gh}\, dh$

Integrating between the limits of the free weir, $H_1 - H_2$ and 0,

$$Q_1 = C_d\sqrt{2g}B \int_0^{(H_1-H_2)} h^{\frac{1}{2}}dh$$

$$= \tfrac{2}{3}C_d\sqrt{2g}B\left[h^{\frac{3}{2}}\right]_0^{(H_1-H_2)}$$

$$= \tfrac{2}{3}C_d\sqrt{2g}B(H_1 - H_2)^{\frac{3}{2}}$$

Next consider the portion of weir below the downstream water surface.

Let Q_2 = discharge per sec. through this drowned portion or weir.

Treating this portion as a drowned orifice (Art. 39),

$$Q_2 = C_d \times \text{area} \times \text{velocity}$$

$$= C_d\, BH_2\sqrt{2g(H_1 - H_2)}$$

Then, total discharge $= Q_1 + Q_2$

The above simple solution can be applied if the values of C_d for the two sections of the weir are known. The problem may be complicated by the formation of a standing wave (Art. 88) on the lower water surface.

EXAMPLE.

A submerged weir spans the entire width of a rectangular channel 20 ft. wide, the sharp edge of the weir being 3 ft. above the bottom of the channel. Estimate the mean velocity of flow in the channel when the depth of water is 5 ft. on the upstream side, and 3·25 ft. on the downstream side of the weir. Allow for the velocity of approach, and take C_d = 0·62 for the weir. (London Univ.)

First the discharge over the weir is calculated, neglecting the velocity of approach, as explained in Art. 57.

$$Q_1 = \tfrac{2}{3}C_d\,\sqrt{2g}B(H_1 - H_2)^{\frac{3}{2}}$$

$$= \tfrac{2}{3} \times \cdot 62\,\sqrt{64\cdot 4} \times 20(2 - \cdot 25)^{\frac{3}{2}}$$

$$= 153 \text{ cu. ft. per sec.}$$

$$Q_2 = C_dBH_2\,\sqrt{2g(H_1 - H_2)}$$

$$= \cdot 62 \times 20 \times \cdot 25\,\sqrt{64\cdot 4(2 - \cdot 25)}$$

$$= 33 \text{ cu. ft. per sec.}$$

$$\text{Total quantity} = Q = Q_1 + Q_2$$

$$= 153 + 33$$

$$= 186 \text{ cu. ft. per sec.}$$

Let v_1 = velocity of approach

$$= \frac{Q}{\text{area of channel}}$$

$$= \frac{186}{5 \times 20} = 1 \cdot 86 \text{ ft. per sec.}$$

Now calculate the discharge over the weir using this approximate value of the velocity of approach. Then, as explained in Art. 57,

$$Q_1 = \tfrac{2}{3} C_d \sqrt{2g} B \left[\left\{ (H_1 - H_2) + \frac{v_1^2}{2g} \right\}^{\frac{3}{2}} - \left(\frac{v_1^2}{2g} \right)^{\frac{3}{2}} \right]$$

$$= \tfrac{2}{3} \times \cdot 62 \sqrt{64 \cdot 4} \times 20 \left[\left\{ (2 - \cdot 25) + \frac{1 \cdot 86^2}{2g} \right\}^{\frac{3}{2}} - \left(\frac{1 \cdot 86^2}{2g} \right)^{\frac{3}{2}} \right]$$

$$= 212 \text{ cu. ft. per sec.}$$

$$Q_2 = C_d BH \left[\sqrt{2g(H_1 - H_2)} + v_1 \right]$$

$$= \cdot 62 \times 20 \times \cdot 25 \left[\sqrt{64 \cdot 4(2 - \cdot 25)} + 1 \cdot 86 \right]$$

$$= 38 \cdot 3 \text{ cu. ft. per sec.}$$

Then, total quantity $= Q = Q_1 + Q_2$

$$= 212 + 38 \cdot 3$$

$$= 250 \cdot 3 \text{ cu. ft. per sec.}$$

Mean velocity of flow in channel

$$= \frac{Q}{\text{area of channel}}$$

$$= \frac{250 \cdot 3}{20 \times 5}$$

$$= 2 \cdot 5 \text{ ft. per sec.}$$

EXAMPLES 5.

(1) Find the discharge through a rectangular weir, 8 ft. wide, under a head of 8 in., when the side contractions are suppressed—

1. By Bazin's formula. ***Ans.*—14·7 cu. ft. per sec.**

2. By Francis' formula. ***Ans.*—14·5 cu. ft. per sec.**

(2) A rectangular weir is 6 ft. broad and has a head of 2 ft. 3 in. Find the discharge taking into account the two end contractions.

***Ans.*—62·2 cu. ft. per sec.**

(3) A rectangular weir, 20 ft. long, is divided into three bays by two vertical posts, each 1 ft. wide. Find the discharge when the head is 1 ft. 6 in.

Ans —104·7 cu. ft. per sec.

(4) Find the discharge through a triangular notch under a constant head of 10 in. if the angle of the notch is 120°. $C_d = \cdot 62$.

Ans.—2·94 cu. ft.

(5) A stream approaching a waterfall having a fall of 60 ft., is gauged by a weir. The measured head over the weir is 11 in. and the length of the weir is 10 ft. The velocity of approach u is 4 ft. per sec., and, due to this, the head may be supposed to be increased by $\frac{1\cdot 5\, u^2}{2g}$. Determine the power available from the fall, assuming that 50 per cent of the energy can be used. (London Univ.)

Ans.—165·2 h.p.

(6) Obtain a formula for the discharge over a rectangular weir, taking into account the effect of lateral contractions.

Determine the discharge over a sharp crested weir, 15 ft. long, with no lateral contractions, the measured head over the crest being 17·9 in. The width of the channel of approach is 25 ft., and its depth below the crest of the weir is 3 ft. (London Univ

Ans.—93 cu. ft. per sec.

(7) During a test in a laboratory, the water which has passed through a Venturi meter flows over a right-angled V notch, the head at the notch being registered. The larger diameter of the Venturi meter is 10 in., and the diameter of the throat is 4 in. When a steady head over the V notch of ·604 ft. is maintained, the difference of pressure head at the Venturi meter is found to be 1·075 ft. of water. Determine the coefficient of this Venturi meter on the assumption that the V notch results are correct, the coefficient being ·60. (London Univ.)

Ans.—$k = \cdot 99$.

(8) A reservoir has an area of 100,000 sq. yd. and is provided with a weir 15 ft. long; find how long it will take for the level at the sill to fall from 2 ft. to 1 ft.

Deduce the formula you use and note any assumptions made. (London Univ.)

Ans.—179·5 min.

(9) State the principle of similarity and show how it can be used to prove that the discharge from a triangular notch is

$$Q = C H^{\frac{5}{2}}$$

The compensation water from a waterworks of 12,000,000 gallons per day is discharged over a rectangular weir. Find the length of the weir if the head is not to be more than 15 in. (London Univ.)

Ans.—5·05 ft.

(10) Explain why a sharp-edged V notch gives a coefficient of discharge which is practically independent of the head. (London Univ.)

(11) Deduce an expression for the discharge over a triangular notch. What does this become if the angle of the notch is 90° ? (A.M.I. Mech. E.)

(12) Find the depth and top width of a triangular notch capable of discharging a maximum quantity of 25 cusecs and such that the head shall be 3 in. when the discharge is ·2 cusecs. For a right-angled notch, $c = 2{\cdot}54$. (A.M.I. Civil E.)

Ans.—1·725 ft. 8·7 ft.

(13) The following observations were made during measurements on a weir, whose crest " b " is 3 ft. long.

Head " H " ft.	0·2	0·4	0·6	0·8	1·0	1·2	1·5
Q cub. ft. per sec	0·846	2·34	4·24	6·48	9·00	11·78	16·35

If the discharge is given by $Q = K\,b\,H^n$, determine K and n. (A.M.I. Mech.E.)

Ans.—$K = 3{\cdot}01$; $n = 1{\cdot}49$.

(14) Deduce an expression for the discharge over a right-angled triangular notch. If the coefficient of discharge is 0·61, what will be the discharge under a head of 12 in. ? (A.M.I. Mech. E.)

Ans.—2·6 cu. ft. per sec.

(15) Deduce an expression for the discharge over a right-angled triangular notch. If the coefficient of discharge is 0·61, what will be the discharge if the head is 18 in. ? (A.M.Inst. C.E.)

Ans.—7·16 cu. ft. per sec.

(16) Calculate the maximum discharge over a broad-crested weir of length 12 ft. having no end contractions. The head above the sill of the weir is 1·6 ft. and C_d is ·97.

Ans.—72·8 cu. ft. per sec.

(17) Find the discharge over a flat-topped broad-crested weir 100 ft. long with rounded entrance, when the up-stream level is 2·25 ft. above the crest. Deduce any formula used. Point out the limitations of the treatment. [Assume C_d = ·66.] (London Univ.)

Ans.—687·0 cu. ft. per sec.

(18) Explain why the depth of the stream flowing over the top of a broad-crested weir is theoretically two-thirds of the height of the level of the water in the reservoir above the sill.

If the ordinary formula for a rectangular notch, without end contractions, is used to give the flow across a broad-crested weir, what should be the value of the coefficient of discharge in this formula ? (London Univ.)

Ans.—$\sqrt{\frac{1}{3}}$.

(19) Show that the flow over a submerged weir is given approximately by the expressions,

$$Q = \tfrac{1}{3}\,Cb\sqrt{2g(H_1 - H_2 + h)}\cdot[2(H_1 + h) + H_2],$$

in which b is the width, H_1 and H_2 are the heights of the free surfaces above the sill, and h is a supplementary head which takes into account the velocity of approach, C is a coefficient of discharge. State the assumptions made in obtaining the formula. (London Univ.) ✓

(20) Find the discharge over 100 ft. of broad-crested weir when the head is 3 ft. Prove any formula used. Assume C_d = ·61. (I. Mech. E.)

Ans.—980 cu. ft. per sec.

CHAPTER VI

FRICTION AND FLOW THROUGH PIPES

62. Fluid Friction. Fluids in motion are subjected to certain resistances which are assumed to be due to friction. For convenience, they will be known, in what follows, as frictional resistances. Actually, these resistances are mainly due to viscosity; that is, to the resistance to sliding between two adjacent layers of the fluid.

Viscous resistance is a shear resistance and is probably due to overcoming the tension between the particles on a plane inclined to the plane of shear. The resistance would then be equal to the components of this tension, in the plane of shear. It has been suggested that the resistance of a fluid to tension is due to molecular attraction; in which case, the apparent frictional resistance of a fluid is primarily due to this cause.

It is found that the motion of a liquid is a steady stream line flow for low velocities only. After a certain velocity is reached the motion is no longer steady and eddy currents appear. The velocity at which the flow changes from steady flow to eddy flow is known as the critical velocity.

For liquids moving with a steady stream line motion, that is, before the critical velocity is reached, the frictional resistance obeys certain laws; this is known as a stream line flow. But once the critical velocity is passed, there is a distinct change in many of these laws.

For steady stream line flow the frictional resistance is—

(1) Proportional to the velocity.
(2) Independent of the pressure.
(3) Proportional to the area of surface in contact.
(4) Independent of nature of surface in contact.
(5) Varies greatly with the temperature.

On account of (4), it is inferred that when a liquid is flowing past a surface with a velocity less than the critical velocity, a film of stationary liquid is formed over the surface; the resistance is then due to viscosity only.

For unsteady flow beyond the critical velocity, the frictional resistance is—

(1) Proportional to the square of velocity.
(2) Independent of pressure.
(3) Proportional to density of fluid.
(4) Varies only slightly with the temperature.
(5) Proportional to area of surface in contact.
(6) Depends on nature of surface in contact.

This type of flow is known as a turbulent flow.

63. **Froude's Experiments.** The frictional resistances of surfaces moving in water were investigated by Froude.* An experimental tank, about 300 ft. long, containing water was used. Thin wooden boards were towed endwise in this tank by connecting them to a carriage running on rails at the side. The carriage was hauled along at various speeds by means of a wire rope passing around a drum, the force required to tow the boards being measured. Boards of lengths varying from 2 ft. to 50 ft. were used, their surfaces being covered with varnish, tinfoil, calico, and sand, in turn.

From the results of these experiments Froude concluded—

(1) The frictional resistance varies approximately with the square of the velocity.†

(2) The frictional resistance varies with the nature of the surface.

(3) The frictional resistance per square foot of surface decreases as the length of the board increases, but is constant for long lengths.

The explanation of this last conclusion is that the relative velocity between the water and the board is greater at the front edge. The water is at rest when cut by the front edge, but is dragged along with the board once the front edge is passed. This causes the mean relative velocity to be greater with a short board than with a long one. For this reason, the frictional resistance per square foot may be taken as constant.

Let f' = frictional resistance per sq. ft. of a given surface at unit velocity†
A = area of wetted surface in sq. ft.
V = velocity of surface in ft. per sec.

* *British Association Reports*, 1872–1874.

† Actually f' will vary with the temperature, the velocity, and the length as shown in Chapter XII.

Then, total frictional resistance $= f' A V^n$

Assuming the index $n = 2$,

total frictional resistance $= f' A V^2$. . . (1)

64. **Resistance of Ships.** The resistance of a ship to motion is due to the frictional resistance of its wetted surface and to head resistance. The latter will depend on the shape of ship and can be reduced by making the immersed portion of the ship a stream line form. The energy utilized in overcoming the head resistance is wasted in the formation of waves, known as the wash, and is eventually lost in friction. The best stream line form for a ship will depend on the speed and on the density of the fluid in which the ship is immersed. The stern should be more tapered than the bow, fast ships should be more tapered than slow ones, whilst an airship, which travels in a much lighter fluid, need not be tapered as much as a sea-going vessel of the same speed. The best form of ship can only be determined by experiment, and it is usual, before building a ship, to make a small model of the same proportions and to measure its resistance in an experimental tank. By so doing, the head or wave resistance of the proposed ship may be calculated from that of the model.*

Total resistance of ship=frictional resistance+wave resistance.

Let R_f = frictional resistance of ship

R_w = wave resistance of ship

and R = total resistance of ship

Then, $R = R_f + R_w$

Let f_s = frictional resistance per sq. ft. of ship at unit velocity

A_s = area of wetted surface of ship

and V_s = speed of ship

From Equation (1), Art. 50,

$$R_f = f_s A_s V_s^2$$

then $$R_w = R - f_s A_s V_s^2$$

It is known from experiments with ships that the wave resistance of a ship is in proportion to the square of the speed and to the wetted surface area, and may be calculated from the wave resistance of the model.†

* For a fuller account of the resistance of ships see Sir William White's *Naval Architecture.*

† See also Art. 146.

Let f_m = frictional resistance per sq. ft. of model at unit velocity

A_m = area of wetted surface of model

V_m = speed of model

r = total resistance of model

r_f = frictional resistance of model

r_w = wave resistance of model

Then,
$$r_w = r - r_f$$
$$= r - f_m A_m V_m^2$$

The model is made the same form as the ship and, therefore, represents the ship to a given scale. Let n be the ratio between the linear dimensions of the ship and model.

Then, $A_s = n^2 A_m$

The speed of the model should be $\frac{1}{\sqrt{n}} V_s$, in order to compare with that of the ship. This is known as the corresponding speed.*

Or, $V_s = \sqrt{n}\, V_m$

As the wave resistance is in proportion to the area and to the velocity squared,

$$\frac{R - R_f}{r - r_f} = \left(\frac{V_s}{V_m}\right)^2 \frac{A_s}{A_m} = \frac{n V^2_m n^2 A_m}{V^2_m A_m}$$
$$= n^3 \qquad \qquad (1)$$

But,
$$\frac{R_f}{r_f} = \frac{f_s A_s V^2_s}{f_m A_m V^2_m} = \frac{f_s n^2 A_m n V^2}{f_m A_m V^2_m}$$
$$= \frac{f_s}{f_m} n^3$$

Substituting in Equation (1),

$$R - r_f \frac{f_s}{f_m} n^3 = n^3 (r - r_f)$$

Then,
$$R = n^3 \left\{ r + r_f \left(\frac{f_s}{f_m} - 1 \right) \right\} \qquad \qquad (2)$$

If the surface of the model is the same as that of the ship, $f_s = f_m$. Equation (2) then becomes

$$R = n^3 r$$

* For explanation of this see Art. 146.

With ships of similar form and of the same surface, and using notation of Art. 63,

total resistance = frictional resistance + wave resistance

$$= f' L^2 V^2 + c L^2 V^2$$

where c is a coefficient depending on the form and L is the linear dimension.

Then, total resistance $= (f' + c) L^2 V^2$

$$\text{Horse-power} = \frac{\text{resistance} \times V}{550}$$

where resistance is in pounds and velocity in feet per second.

$$\text{Then,}\quad \text{horse-power} = \frac{(f' + c)}{550} L^2 V^3$$

$$= k L^2 V^3$$

where k is a constant for the type of ship considered.

The same laws will hold for any other fluid; this will be seen from the curve in Fig. 66. This curve was plotted from the maximum speed and maximum horse-power of nine rigid air-ships, covered with the same material and of almost similar form. There was a considerable variation in their sizes. As the horse-power equals $k L^2 V^3$, a straight line passing through the origin should be obtained if the horse-power is plotted against $L^2 V^3$. The slope of this line gives the constant k. This has been done in Fig. 66; it will be noticed that the points lie approximately on a straight line, thus proving the above laws to hold.

EXAMPLE.

Show how the total resistance and the power required for propulsion of a ship can be deduced from experiments on a scale model. Determine the indicated horse-power to drive a ship 300 ft. long, having 13,500 sq. ft. wetted surface, at 20 knots, if the resistance of the model, one-sixteenth the size of the ship, is 20 lb. at the corresponding speed. Take f for the ship as ·0091, and for the model ·0094, and assume that 60 per cent of the I.h.p. is available for propulsion. [1 knot = 1·69 ft. per sec.] (London Univ.)

$$\text{Area of wetted surface of model} = A_m = \frac{A_s}{n^2} = \frac{13{,}500}{(16)^2}$$

$$\text{Corresponding speed of model} = \frac{V_s}{\sqrt{n}} = \frac{20}{4}\ \text{knots}$$

Assuming the units of f' are for velocities in knots,

frictional resistance of model $= r_f = f_m A_m V_m^2$

$$= \cdot 0094 \times \frac{13{,}500}{256} \times \left(\frac{20}{4}\right)^2$$

$$= 12{\cdot}4 \text{ lb.}$$

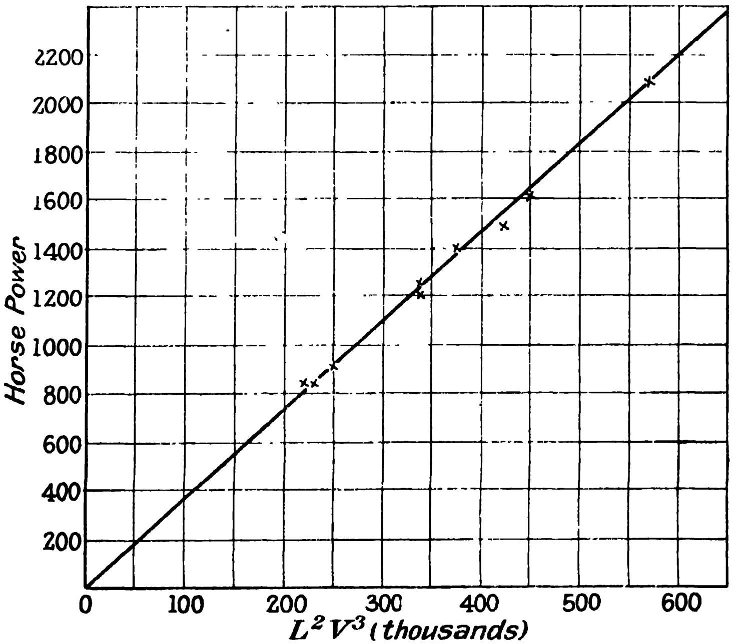


FIG. 66

Then, $\quad r_w = 20 - 12{\cdot}4 = 7{\cdot}6$ lb.

Using Equation (2),

Total resistance of ship $= n^3 \left\{r + r_f\left(\frac{f_s}{f_m} - 1\right)\right\}$

$$= 16^3\left\{20 + 12{\cdot}4\left(\frac{\cdot 0091}{\cdot 0094} - 1\right)\right\}$$

$$= 80{,}100 \text{ lb.}$$

Horse-power $= \dfrac{RV}{550} \times \dfrac{100}{60}$

$$= \frac{80{,}100 \times 20 \times 1{\cdot}69 \times 100}{550 \times 60}$$

$$= 8{,}200$$

65. **Friction of Revolving Disc.** Froude's experiments gave the true coefficient of friction only when very long boards were used. Professor Unwin overcame this difficulty by revolving a disc at a known speed in the liquid and obtained the coefficient of friction of the disc's surface by measuring the work done.

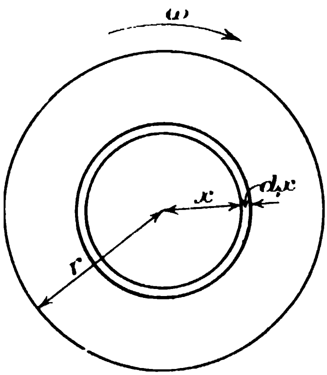


FIG. 67

Consider the disc in Fig. 67.

Let ω = angular velocity of disc in radians per second

r = radius of disc

and μ = coefficient of friction at unit velocity*

Then, frictional force = $\mu \times$ area $\times$ (velocity)^2

Consider a thin ring of the disc of a radius x and let thickness of ring be dx.

Area of ring (both sides) $= 4\pi\, x\, dx$

Tangential velocity of ring $= \omega\, x$

Frictional resistance of ring $= \mu \times 4\pi\, x\, dx \times \omega^2 x^2$

Moment of resistance about centre $= 4\pi\, \mu\, \omega^2\, x^3\, dx \times x$

Total moment of disc $= 4\pi\, \mu\, \omega^2 \displaystyle\int_0^r x^4\, dx$

$$= \frac{4}{5}\pi\, \mu\, \omega^2\, r^5$$

Work done per second = Moment $\times$ angle turned through

$$= \frac{4}{5}\pi\, \mu\, \omega^3\, r^5$$

* Actually μ will vary with the temperature and velocity; see Chapter XII on Viscous Flow.

If the frictional resistance is assumed to vary with (velocity)^n, this expression becomes

$$\text{Work done per second} \quad = \frac{4\pi\,\mu\,\omega^{n+1}\,r^{n+3}}{n+3}$$

66. **Friction in Pipes—Hydraulic Gradient.** Fluids flowing through pipes are subjected to a frictional resistance depending on the velocity, the area of the wetted surface, and the nature of the surface. In long pipes the frictional resistance is so large that all other resistances are rendered insignificant in comparison, and the total energy of the fluid is absorbed in overcoming it. The energy lost in overcoming the frictional resistance is expressed in feet of water and is known as the head lost in friction.

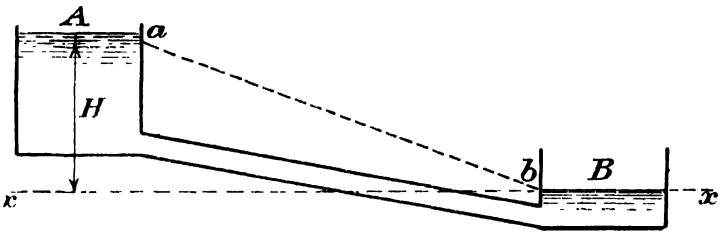


FIG. 68

Consider water to flow through a long pipe from a reservoir A into a reservoir B (Fig. 68). Take the line xx through the water level in B as the datum line. Let H be the height of water in A above datum, and let l be the length of the pipe. If p is the intensity of pressure of the water at any section of the pipe, the pressure energy at that section will be $\frac{p}{w}$. Supposing the pressure energy of the water at all sections of the pipe are plotted as vertical ordinates, using the centre line of the pipe as a base line, a straight sloping line $a\,b$ will be obtained. This line falls off uniformly from A, as there is a uniform loss of head due to friction as the water flows along the pipe. This line is called the hydraulic gradient, and its slope is equal to the total loss of head divided by length of pipe.

In practice the slope is small, so that either the sine or the tangent may be used as the slope of the hydraulic gradient.

Let i = slope of pressure energy line $a\,b$

and h = total head lost

Then, slope of hydraulic gradient $= i = \frac{h}{l}$ and is known as the vertical slope.

It will be noticed that if the pipe is uniform and if the whole of the available head is lost in friction, the slope of the hydraulic gradient will be the difference of level of water surfaces divided by length of pipe.

Next consider the pipe line shown in Fig. 69. A and B are two reservoirs separated by a hill; a uniform pipe is laid over the

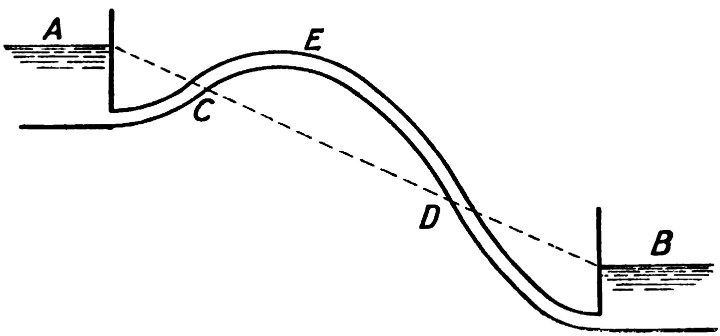
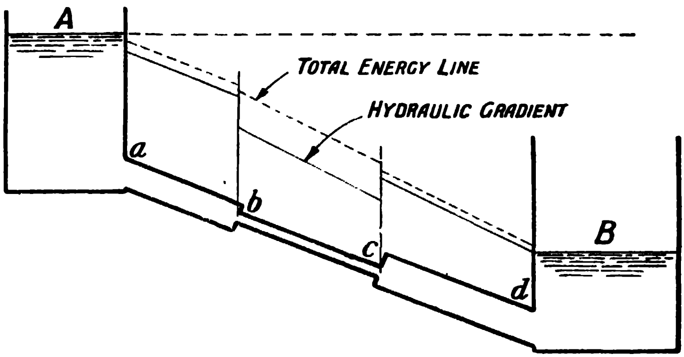


FIG. 69

hill so that water from A may flow into B. Fig. 69 is drawn to greatly enlarged vertical scale; actually, the length of the pipe may be taken as the length of its horizontal projection. As the pipe is long, the loss of head due to friction will be very large and all other losses may be neglected; hence, taking the water level at B as datum, the water will lose energy at a uniform rate from the water level in A to the water level in B. From this it follows that the hydraulic gradient will be a straight line joining the water surface in A and B.

The pressure energy at any section of the pipe will be represented by the vertical distance between the hydraulic gradient and the pipe centre line at that section. If the hydraulic gradient is above the centre line of pipe the pressure is above atmospheric; if below the centre line of pipe the pressure is below atmospheric.

It will be seen from Fig. 69 that at C and D the water pressure is atmospheric, whilst between C and D it is less than atmospheric. The highest point of the pipe above the hydraulic gradient is E; at this point the water pressure is least. If the absolute pressure at E is less than 8 ft. of water, or 26 ft. vacuum, separation will occur, for at this pressure the water commences to vaporize, large bubbles of gas will occur causing the flow to cease. It follows from this that engineers must lay their pipe lines so that no section of the pipe will



Datum

FIG. 70

be more than 26 ft. above the hydraulic gradient at that section.

A pipe which rises above its hydraulic gradient is known as a syphon. It will be noticed from Fig. 69 that a pipe may be above the hydraulic gradient and yet be below the water surface at A; such a pipe would still be a syphon.

Consider next the short pipe line shown in Fig. 70. Let the water flow from A to B along a pipe of varying section $a\ b\ c\ d$. At any section of the pipe the total energy of the water will be the datum head + the velocity head + the pressure head. Choose any horizontal line as the datum line, and starting from the water level in A, mark off the losses of head in the pipe from all sources, to the same vertical scale as the figure. The line thus obtained is the total energy line, and is shown dotted. The height of this line above the datum line, at any section, will give the total energy of the water at that section.

Let v_1 = velocity of flow in $a\,b$
v_2 = velocity of flow in $b\,c$
v_3 = velocity of flow in $c\,d$.

The following are the losses to be taken into account—

At a, a loss due to entrance to pipe $= \cdot 5\,\dfrac{v_1^2}{2g}$

Between a and b a uniform loss due to friction

$$= \frac{4\,f\,l\,v_1^2}{2g\,d} \qquad \text{(Art. 67)}$$

At b, a loss due to sudden contraction $= \cdot 5\,\dfrac{v_2^2}{2g}$

Between b and c, a uniform loss due to friction.

At c, a loss due to sudden enlargement $= \dfrac{(v_2 - v_3)^2}{2g}$

Between c and d, a uniform loss due to friction.

At d, a loss due to velocity head being destroyed $= \dfrac{v_3^2}{2g}$

The sum of all these losses will equal the difference of level between the water surfaces in A and B.

As the height of the hydraulic gradient above the centre line of pipe represents the pressure head of the water, it follows that if the velocity head is deducted from the total energy line the hydraulic gradient will be obtained. For,

$$\left.\begin{array}{l}\text{pressure head above}\\ \text{centre line of pipe}\end{array}\right\} = \begin{array}{l}\text{total energy above datum} -\\ \quad\text{velocity head.}\end{array}$$

Velocity head between a and $b = \dfrac{v_1^2}{2g}$

Velocity head between b and $c = \dfrac{v_2^2}{2g}$

Velocity head between c and $d = \dfrac{v_3^2}{2g}$

These amounts have been subtracted from the total energy line of Fig. 70 and the full line representing the hydraulic gradient is obtained.

It will be noticed that if the diameter of the pipe $b\,c$ is much smaller than that of $c\,d$, the velocity v_2 will be large compared with v_3; hence the hydraulic gradient of the pipe $c\,d$ may be higher than that of $b\,c$.

67. **Loss of Head Due to Friction in Pipes.** A rational formula for the loss of head in a pipe due to friction may be obtained by assuming the experimental results of fluid friction to hold.

Consider water flowing along a uniform horizontal pipe, of cross-sectional area A, with a velocity v. Let l be the length of the pipe and let the intensity of pressure be reduced by the frictional resistance from p_1 to p_2 over the length l.

Let f' = frictional resistance per unit area at unit velocity

P = wetted perimeter of pipe

Resolving horizontally,

$$p_1 A = p_2 A + \text{frictional resistance}$$

But, frictional resistance $= f' \times \text{area} \times v^n$

$$= f' P\, l\, v^n$$

Therefore, $(p_1 - p_2)\,A = f' P\, l\, v^n$

Dividing through by the density of water w.

$$\frac{p_1}{w} - \frac{p_2}{w} = f' \frac{P}{A}\frac{lv^n}{w}$$

Let h_f = head lost due to friction

$$= \frac{p_1}{w} - \frac{p_2}{w}$$

Then, $h_f = f' \dfrac{P}{A}\dfrac{lv^n}{w}$

The ratio $\dfrac{A}{P}$ is called the hydraulic mean depth and is represented by m.

Then, $h_f = \dfrac{f'\,lv^n}{m\,w}$ . . . . . . (1)

Assuming $n = 2$, and as $i = \dfrac{h_f}{l}$,

$$i = \frac{f' v^2}{m w}$$

Or, $$v = C\sqrt{m i} \qquad \text{. . . . . . (2)}$$

where $C = \sqrt{\dfrac{w}{f'}}$.

This form is known as the Chezy formula, the constant C being found experimentally.

Another useful form of this formula is obtained by expressing the head lost in terms of the velocity head.

For a pipe flowing full,

$$m = \frac{A}{P} = \frac{\frac{\pi}{4}d^2}{\pi d} = \frac{d}{4}$$

Substituting this in Equation (1), and assuming $n = 2$,

$$h_f = \frac{4f'}{wd} l v^2$$

Putting $$f' = \frac{fw}{2g},$$

$$h_f = \frac{4 f l v^2}{d\, 2g} \qquad \text{. . . . . . (3)}$$

where f is a constant found experimentally.*

This modified form of the Chezy formula is usually used for pipes running full, as it is convenient to have the frictional head lost in terms of the velocity head.

Darcy found that the coefficient f of Equation (3) varied with the surface of the pipe and with the diameter, and gives the following formula for f—

For new pipes,

$$f = \cdot 005\left(1 + \frac{1}{12d}\right)$$

For old pipes,

$$f = \cdot 01\left(1 + \frac{1}{12d}\right)$$

where d is the diameter of the pipe.

* Actually f varies with the temperature, velocity, and diameter of pipe; see Chapter XII on Viscous Flow.

In using Equation (3), care should be taken that all the dimensions are in feet and seconds units.

Although Equation (3) is used by all engineers for calculations on pipe flow, the results obtained can only be very approximate. As f varies greatly with the temperature it follows that there will be a large variation in the flow during the year, a greater flow being obtained in summer.*

EXAMPLE.

Water flows through a pipe, 8 in. diameter, 150 ft. long, with a velocity of 8 ft. per sec. Find the head lost in friction—(*a*) using the formula

$$h_f = \frac{4\,f\,l\,v^2}{d\ 2g}$$

assuming f to be ·0056 ; (*b*) using the formula $v = C\sqrt{mi}$, assuming $C = 106$.

(*a*) $$h_f = \frac{4 \times \cdot 0056 \times 150 \times 8^2}{\frac{8}{12} \times 64 \cdot 4}$$

$$= 5 \cdot 0 \text{ ft. of water.}$$

(*b*) $$m = \frac{d}{4} \text{ for a circular pipe running full}$$

$$i = \frac{h_f}{l}$$

Then, $$v = 106\sqrt{\frac{d}{4} \times \frac{h_f}{150}}$$

Squaring both sides,

$$8^2 = 11{,}200 \times \frac{8}{12 \times 4} \times \frac{h_f}{150}$$

Therefore, $$h_f = \frac{12 \times 4 \times 64 \times 150}{8 \times 11{,}200}$$

$$= 5 \cdot 15 \text{ ft. of water.}$$

68. Reynolds' Experiments on Flow Through Pipes.

Reynolds† measured the loss of head in a pipe by measuring the fall of pressure over a known length of the pipe ; from this "*i*," the slope of the hydraulic gradient, was obtained. For,

$$i = \frac{h_f}{l}$$

* For results of experiments on pipe flow made by Darcy and others, see Barnes' *Hydraulic Flow Reviewed.*

† For complete account of Reynolds' experiments, see *Phil. Trans.*, 1883.

Reynolds' apparatus is shown in Fig. 71.

The velocity of the water in the pipe was obtained by measuring the discharge over a known time; then

$$v = \frac{\text{discharge per sec.}}{\text{area of cross-section of pipe}}$$

This was repeated for several velocities, and the results were then plotted as shown in Fig. 72, the base of the graph representing v and the ordinate representing i. The graph obtained

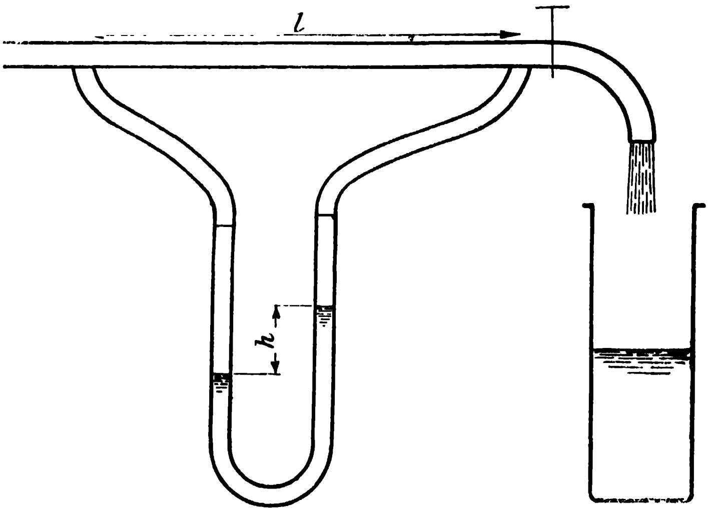


FIG. 71

was found to be a straight line up to a certain velocity, beyond this velocity the graph was curved.

The graph is evidently following a law of the type

$$i = k\,v^n$$

where k and n are constants. For the straight line portion of the graph, n equals unity. The value of n for the curved portion of the graph can be found by plotting log i and log v.

For, $\quad i = k\,v^n$

Then, $\log i = \log k + n \log v \quad . \quad . \quad . \quad . \quad . \quad (1)$

When $v = 1$, $\log v = 0$, then $\log i = \log k$, from which the value of k can be found.

Also, from Equation (1), $n = \dfrac{\log i - \log k}{\log v}$

These logs are shown plotted in Fig. 73. For the portion of the graph over which n is unity the straight line AB was obtained; the remaining portion of the graph gave the straight line CD. The line BC, which joins the other two lines, follows no defined law and is due to the changing from one type of flow to the other.

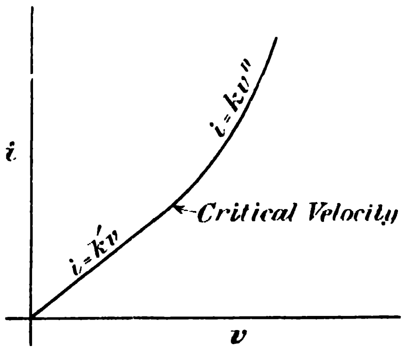


FIG. 72

It follows from this graph that the flow of the water consists of two types—

(1) a steady or stream line flow up to the point B;

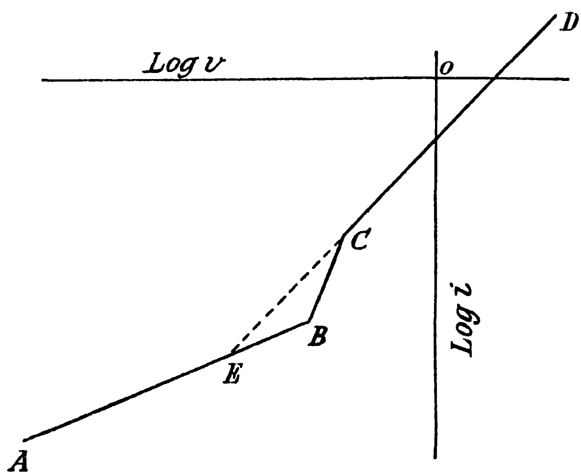


FIG. 73

(2) an unsteady or eddy flow for the higher velocities beyond B. This is sometimes known as a turbulent flow.

The point B, at which point the change from steady to turbulent flow takes place, is known as the critical velocity.

After the highest velocity had been reached the experiment was continued by gradually reducing the velocities and again measuring the loss of head; the points on the curve then retraced the line *DC*. On reaching *C* the points continued in the same straight line to *E*, and finally retraced the line *EA*. Hence, the path *EBC* was only followed when the velocities

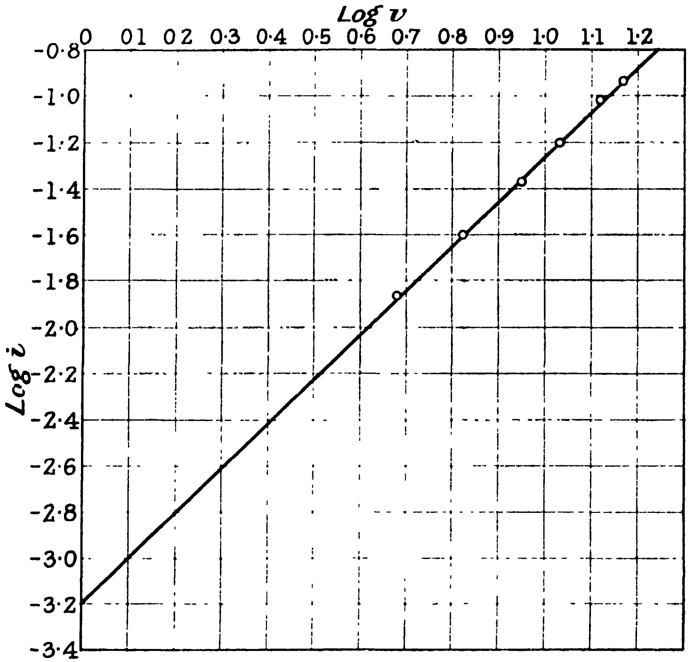


FIG. 74

were increasing. Reynolds concluded from this that the path *EBC* was due to the inertia of the water in changing from steady flow to turbulent flow and that the point *E* is the true critical velocity. The point *E* is known as the lower critical velocity, and is assumed to be the true critical velocity.

Reynolds repeated these experiments with pipes of different diameters and with water at different temperatures. From these results he found that the value of the critical velocity varies inversely with the diameter of the pipe and inversely with the temperature of the water.

These results hold for all other liquids; the value of the

critical velocity of any liquid will also depend on the density of the liquid and on its viscosity.*

In all civil engineering problems on flow of water it is found that the velocities used are all above the critical velocity; and in all large pipes, such as used in practice, the suffix "n" approximates to 2 which agrees with the practical friction formula given in Art. 67.

EXAMPLE.

An experiment was carried out on an 8 in. diameter wrought iron pipe over a length of 8 ft. The velocity of flow through the pipe was varied and the loss of head for each velocity was measured. The following values of i were obtained—

v (ft. per sec.) . .	4·7	6·5	8·72	10·6	12·8	14·6
i	·0134	·0250	·0425	·0629	·0975	·1171

Find the values of k and n in the formula $i = kv^n$

First plot log i and log v.

log v . . .	·672	·813	·941	1·025	1·107	1·164
log i . . .	– 1·873	– 1·602	– 1·371	– 1·201	– 1·013	– ·931

These are shown plotted in Fig. 74.

When $\log v = 0$, $\log k = \log i$

$$= -3{\cdot}19$$

$$= \bar{4}{\cdot}81$$

Therefore, $k = {\cdot}000645$

$$n = \frac{\log i - \log k}{\log v}$$

$$= \frac{-2{\cdot}6 + 3{\cdot}19}{{\cdot}302}$$

$$= 1{\cdot}955$$

Then, $i = {\cdot}000645\, v^{1{\cdot}955}$

69. **Determination of Critical Velocity.** Besides the method given in Art. 68 there are two other methods of obtaining the critical velocity of water.

(*a*) COLOUR BANDS (REYNOLDS' METHOD). The critical velocity may be determined by allowing water to flow through a glass tube and injecting a thin stream of coloured liquid into the centre of the stream (Fig. 75). As long as the velocity in

* See Chapter XII on Viscous Flow.

the glass tube is below the critical velocity, the colour band will remain a thin straight line flowing along the centre of the stream. But for velocities above the critical velocity. the coloured band is broken up by eddies and mixes with the water, as in Fig. 76.

(*b*) CHANGE OF TEMPERATURE. Barnes and Coker* determined the critical velocity by measuring the temperature of the stream for various velocities. As the frictional resistance below the critical velocity is proportional to v and, above the critical velocity, to v^n, it follows that more heat will be generated above the critical velocity. If the temperature of the water is plotted on a base representing the velocity, the curve will become much steeper beyond the critical velocity, as shown in Fig. 77. The critical velocity will be represented by the kink in the curve.

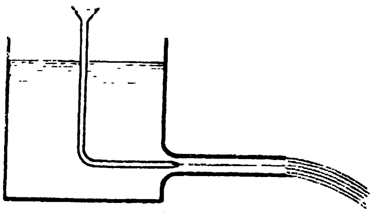


FIG. 75

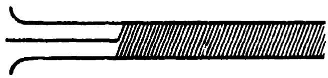


FIG. 76

70. Distribution of Velocity in a Pipe. The velocity of water flowing along a pipe will vary at different points of the cross section, its magnitude depending on the radius. The velocity of flow at any radius may be measured with a Pitot tube. It is found that the velocity is a maximum at the centre and a minimum at the circumference. The variation is shown in the curve of Fig. 78, the velocity being plotted horizontally on the diameter of the pipe as a base.

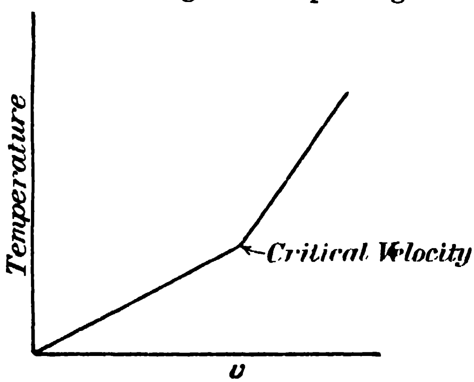


FIG. 77

It is found that the maximum velocity is about 1·2 times the mean velocity.

71. Flow through Long Pipes. The velocity of water flowing through a pipe may be found by applying Bernoulli's

* *Proceedings of the Royal Society*, vol. 74.

equation to the two ends of the pipe and allowing for any loss of head in the pipe. In all such problems the most convenient formula for the frictional head lost is

$$h_f = \frac{4 f l v^2}{d\, 2g}$$

as it is necessary to express all unknown terms as a function of the velocity head.

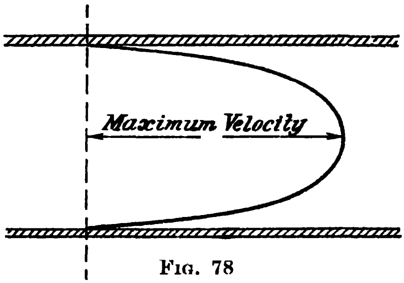


FIG. 78

Suppose water flows from a reservoir A (Fig. 79) under a constant head H_A into a reservoir B in which there is a constant head of H_B. Let the height of centre of pipe at A be Z_A, and at B be Z_B. Let v be the velocity of flow through the pipe, l be the length, and d the diameter.

Then, $$h_f = \frac{4 f l}{d} \frac{v^2}{2g},$$

and, head lost at entrance of pipe $\dfrac{\cdot 5\, v^2}{2g}$

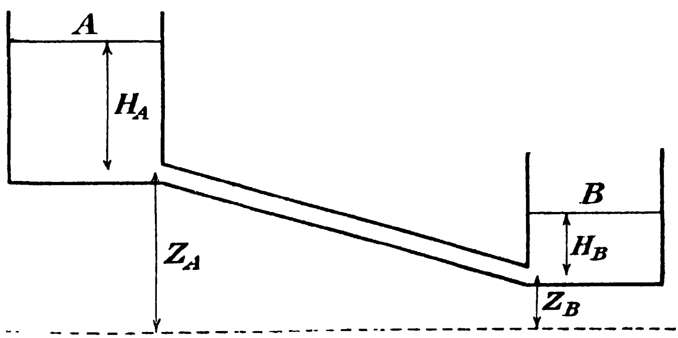


FIG. 79

Applying Bernoulli's equation to points just beyond each end of the pipe,

$$H_A + Z_A = H_B + Z_B + \frac{\cdot 5\, v^2}{2g} + \frac{4 f l}{d} \frac{v^2}{2g} + \frac{v^2}{2g}$$

The term $\dfrac{v^2}{2g}$ will be lost on entering B.

It will be noticed that $(H_A + Z_A) - (H_B + Z_B)$ is the difference in level of the water surfaces in A and B; hence,

$$\left.\begin{array}{l}\text{difference in level}\\ \text{of water surfaces}\end{array}\right\} = \frac{v^2}{2g}\left(1{\cdot}5 + \frac{4fl}{d}\right)$$

From this equation the unknown velocity may be obtained. If the pipe is long, the head lost in friction will be very large compared with the head lost at the two ends of the pipe; in which case the latter may be neglected.

If the pipe in Fig. 79, instead of discharging into the

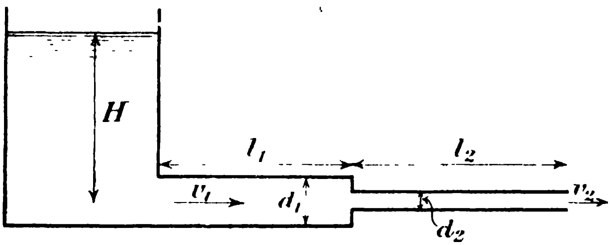


FIG. 80

reservoir B, discharged into the atmosphere, the equation would then be

$$H_A + Z_A = Z_B + \frac{\cdot 5\, v^2}{2g} + \frac{4fl}{d}\frac{v^2}{2g} + \frac{v^2}{2g}$$

the last term being the velocity head of the discharging water.

This may be written

$$H = \frac{v^2}{2g}\left(1{\cdot}5 + \frac{4fl}{d}\right)$$

where H is the height of water level in A above outlet of pipe.

Suppose water flows from a tank through a pipe of which the diameter is varied as in Fig. 80.

As quantity of water flowing per second is constant,

$$v_1 \frac{\pi}{4} d_1^2 = v_2 \frac{\pi}{4} d_2^2$$

then, $\quad v_1 = v_2 \left(\frac{d_2}{d_1}\right)^2$

Head lost in friction in large pipe $= \dfrac{4\,f\,l_1\,v_1^2}{d_1\ \ 2g}$

$$= \frac{4\,f\,l_1\,v_2^2\left(\dfrac{d_2}{d_1}\right)^4}{d_1\ \ 2g}$$

Head lost in friction in small pipe $= \dfrac{4\,f\,l_2\,v_2^2}{d_2\,2g}$

Total head lost in friction $= 4f\left\{\dfrac{l_1\left(\dfrac{d_2}{d_1}\right)^4}{d_1} + \dfrac{l_2}{d_2}\right\}\dfrac{v_2^2}{2g}$

Applying Bernoulli's equation to points just outside each end of pipe,

$H = \dfrac{\cdot 5\,v_1^2}{2g} + \dfrac{v_2^2}{2g} +$ head lost in friction $+$ head lost at contraction

$$= \frac{\cdot 5\left(\dfrac{d_2}{d_1}\right)^2}{2g}\,v_2^2 + \frac{1\cdot 5v_2^2}{2g} + 4f\left\{\frac{l_1\left(\dfrac{d_2}{d_1}\right)^4}{d_1} + \frac{l_2}{d_2}\right\}\frac{v_2^2}{2g}$$

From this equation, the velocity v_2 may be found. If the pipe is long, the velocity head, the head lost at entrance, and the head lost at the sudden contraction may be neglected as small.

EXAMPLE 1.

A cast-iron pipe, 6 in. diameter and 1,500 ft. long, connects two reservoirs. If the difference of water level in the two reservoirs is 96 ft., find the discharge through the pipe ; $f = \cdot 01$. Ignore all losses other than friction.

Total head $=$ velocity head $+$ head lost in friction

$$96 = \frac{v^2}{2g} + \frac{4\,f\,l\,v^2}{d\ 2g}$$

$$96 = \frac{v^2}{2g}\left(1 + \frac{4\,f\,l}{d}\right)$$

$$= \frac{v^2}{64\cdot 4}\left(1 + \frac{4 \times \cdot 01 \times 1500}{\cdot 5}\right)$$

$$= \frac{v^2}{64\cdot 4}(1 + 120)$$

Then, $v^2 = \dfrac{64{\cdot}4 \times 96}{121} = 51{\cdot}1$

$v = 7{\cdot}15$ ft. per sec.

Discharge $= \dfrac{\pi}{4}({\cdot}5)^2 \times 7{\cdot}15 = 1{\cdot}4$ cu. ft. per sec.

EXAMPLE 2.

Two reservoirs are connected by a straight pipe 1 mile long. For the first half of its length the pipe is 6 in. diameter; its diameter is then suddenly reduced to 3 in. The surface of the water in the upper reservoir is 100 ft. above that in the lower. Tabulate the losses of head which occur, including that at the sharp-edged entry, and determine the flow in gallons per minute. Assume $f = {\cdot}01$. (London Univ.)

Let v_1 = velocity in 6 in. pipe
And v_2 = velocity in 3 in. pipe
As quantity of flow is the same in both pipes

$$\frac{\pi}{4}({\cdot}5)^2 v_1 = \frac{\pi}{4}({\cdot}25)^2 v_2$$

Then, $v_1 = \dfrac{v_2}{4}$

Head lost at entrance $= \dfrac{{\cdot}5\, v_1^2}{2g} = {\cdot}03125\dfrac{v_2^2}{2g}$

Head lost in 6 in. pipe due to friction $= \dfrac{4 fl\, v_1^2}{d\, 2g}$

$$= \frac{4 \times {\cdot}01 \times 2640 \times v_2^2}{{\cdot}5 \times 16 \times 2g}$$

$$= 13{\cdot}2\frac{v_2^2}{2g}$$

Head lost at sudden contraction $= \dfrac{{\cdot}5\, v_2^2}{2g}$

Head lost in 3 in. pipe due to friction $= \dfrac{4 fl\, v_2^2}{d_2\, 2g}$

$$= \frac{4 \times {\cdot}01 \times 2640\, v_2^2}{{\cdot}25 \times 2g}$$

$$= 422\frac{v_2^2}{2g}$$

Head lost at exit $= \dfrac{v_2^2}{2g}$

$$\text{Total head lost} = \frac{v_2^2}{2g}(\cdot 03125 + 13\cdot 2 + \cdot 5 + 422 + 1)$$

$$= 436\cdot 73125\frac{v^2}{2g}$$

$$= 100 \text{ ft}$$

Then, $$v_2^2 = \frac{64\cdot 4 \times 100}{436\cdot 73125} = 14\cdot 73$$

$$v_2 = 3\cdot 835 \text{ ft. per sec.}$$

Discharge $$= \frac{\pi}{4}(\cdot 25)^2 \times 3\cdot 835 \times 60 \times 6\cdot 24$$

$$= 705 \text{ gallons per minute}$$

EXAMPLE 3.

The difference of surface level in two reservoirs connected by a syphon is 25 ft. The length of the syphon is 2,000 ft.; its diameter is 12 in.; and $f = \cdot 01$. If the barometric height is 34 ft. and if air is liberated from solution when the absolute pressure is less than 4 ft. of water, what will be the maximum length of inlet leg of the syphon to run full, if the vertex is 18 ft. above the surface level in the upper reservoir? What will then be the discharge? (London Univ.)

The problem is represented by Fig. 69; E being 18 ft. above the water level in A.

Let l = length of pipe between A and E.

First find the velocity of water in the pipe by applying Bernoulli's equation to points A and B, taking the water level in B as datum.

Then, $$25 = \frac{v^2}{2g} + \frac{4f\,2000\,v^2}{2g\,d}$$

$$= \frac{v^2}{2g}\left(1 + \frac{4 \times \cdot 01 \times 2000}{1}\right)$$

From which, $v = 4\cdot 45$ ft. per sec.

Next apply Bernoulli's equation to points A and E, taking the water level in A as datum. The limiting condition for the pipe to run full is when the absolute pressure at E is 4 ft. of water.

Take into account the atmospheric pressure at A.

Total energy at A = total energy at E.

Hence, $$34 = 18 + \frac{v^2}{2g} \times \frac{4\,f\,l\,v^2}{2g\,d} + 4$$

Or $$12 = \frac{v^2}{2g}\left(1 + \frac{4\,f\,l}{d}\right)$$

$$= \frac{(4{\cdot}45)^2}{2g}\left(1 + \frac{4 \times {\cdot}01\,l}{1}\right)$$

From which, $l = 947$ ft.

Discharge = area of pipe × velocity

$$= \frac{\pi}{4} \times 1 \times 4{\cdot}45$$

$$= 3{\cdot}49 \text{ cu. ft. per sec.}$$

72. **Parallel Flow through Pipes.** Suppose water to be flowing along a pipe which, at a certain point, divides into two branches as in Fig. 81. Then, any particle of water will flow along the route ABD or the route ABC.

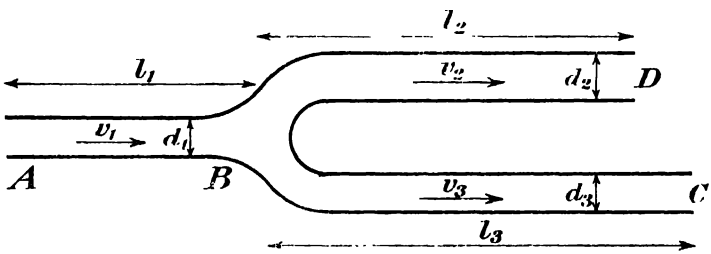


FIG. 81

Let H_A, H_D, and H_C be the total head at A, D, and C respectively.

Then, head causing flow along route $ABD = H_A - H_D$ − head lost in friction.

And, head causing flow along route $ABC = H_A - H_C$ − head lost in friction.

Let l_1, d_1 and v_1 refer to pipe AB

l_2, d_2 and v_2 refer to pipe BD

and l_3, d_3 and v_3 refer to pipe BC .

Then, for route ABD,

$$H_A - H_D - \frac{4fl_1 v_1^2}{d_1 2g} - \frac{4fl_2 v_2^2}{d_2 2g} = \frac{v_2^2}{2g} \quad . \quad . \quad . \quad (1)$$

And, for route ABC,

$$H_A - H_C - \frac{4fl_1 v_1^2}{d_1 2g} - \frac{4fl_3 v_3^2}{d_3 2g} = \frac{v_3^2}{2g} \quad . \quad . \quad . \quad (2)$$

Usually, the velocity heads given on the right of these equations are very small and may be written as zero.

Also, quantity flowing per second through AB equals sum of quantities through BD and BC.

Then, $\quad v_1 d_1^2 = v_2 d_2^2 + v_3 d_3^2 \quad . \quad . \quad . \quad . \quad (3)$

From Equations (1), (2), and (3) the three unknowns v_1, v_2, and v_3 may be obtained.

EXAMPLE.

Two pipes A and B, each 6 in. diameter, branch from a point C to a point D, which is 20 ft. below C. Pipe A is 300 yds. long and pipe B is 500 yds. long. Water is supplied at C under a head of 100 ft. A short pipe 3 in. diameter is fitted at D. Find the delivery when this pipe is fully open to the atmosphere. Take $v = 80\sqrt{mi}$ for pipes A and B. (Lond. Univ.)

Let v_A, v_B, and v be velocities in pipes A, B, and 3 in. pipe respectively.

Total head $= 100 + 20 = 120$ ft.

Consider pipe A.

$$m = \frac{d}{4} = \frac{\cdot 5}{4} = \frac{1}{8}$$

$$v_A = 80\sqrt{\frac{1}{8} \times \frac{h_f}{900}}$$

Then, $\quad h_f = 1{\cdot}125\, v_A^2$

Consider pipe B.

$$v_B = 80\sqrt{\frac{1}{8} \times \frac{h_f}{1500}}$$

Then, $h_f = 1{\cdot}875\, v_B^2$

As pressure at C and D is the same in both pipes,

$$\frac{v_A^2}{2g} + 1{\cdot}125\, v_A^2 = \frac{v_B^2}{2g} + 1{\cdot}875\, v_B^2$$

from which, $v_A = 1{\cdot}285\, v_B$ . . . . (1)

Consider route B,

$$\text{Total head} = \frac{v^2}{2g} + \text{frictional head lost in } B$$

Or, $120 = \dfrac{v^2}{2g} + 1{\cdot}875\, v_B^2$ . . . . . (2)

Also, quantity flowing through 3 in. pipe equals sum of quantities through A and B.

That is, $\dfrac{\pi}{4}(\cdot 25)^2\, v = \dfrac{\pi}{4}(\cdot 5)^2\,(v_A + v_B)$

Substituting from Equation (1),

$$v = 4(1{\cdot}285\, v_B + v_B) = 9{\cdot}14\, v_B$$

Substituting in Equation 2,

$$120 = \frac{v^2}{64{\cdot}4} + 1{\cdot}875\left(\frac{v}{9{\cdot}14}\right)^2$$

from which, $v = 56{\cdot}2$ ft. per sec.

Discharge $= \dfrac{\pi}{4}(\cdot 25)^2 \times 56{\cdot}2 = 2{\cdot}76$ cu. ft. per sec.

73. **Time of Emptying Tank through Pipe.** Let a reservoir or tank be emptied by means of a long pipe of length l and diameter d. Let the area of water surface in the reservoir be A, and the height of the water level above the outlet of

pipe be H_1 ft. Let v be the velocity of flow in the pipe. It is required to find the time taken to lower the water level in the reservoir from H_1 ft. to H_2 ft. above the outlet of pipe. Ignore all losses but friction.

$$\text{Head lost in friction in pipe} = \frac{4 f l v^2}{d\, 2g}$$

Consider the instant when the water level is h ft. above the outlet of pipe and let the water level fall by a small amount dh in the time dt, the term dh being negative.

Then, quantity flowing from reservoir equals quantity passing along pipe.

Or, $$-A\, dh = \frac{\pi}{4} d^2 v\, dt \quad . \quad .$$

But, $$h = \frac{v^2}{2g} + \frac{4 f l v^2}{d\, 2g}$$

$$= \frac{v^2}{2g}\left(1 + \frac{4 f l}{d}\right)$$

From which, $$v = \sqrt{\frac{2gh}{\left(1 + \frac{4 f l}{d}\right)}}$$

Substituting this value of v in Equation (1),

$$-A\, dh = \frac{\pi}{4} d^2 \frac{\sqrt{2g\, h}}{\sqrt{1 + \frac{4 f l}{d}}}\, dt$$

Therefore, $$dt = -\frac{4 A \sqrt{1 + \frac{4 f l}{d}}\, h^{-\frac{1}{2}} dh}{\pi d^2 \sqrt{2g}}$$

$$\text{Total time} = T = \int_0^T dt = -\frac{4 A \sqrt{1 + \frac{4 f l}{d}}}{\pi d^2 \sqrt{2g}} \int_{H_1}^{H_2} h^{-\frac{1}{2}} dh$$

$$= \frac{8 A \sqrt{1 + \frac{4 f l}{d}}}{\pi d^2 \sqrt{2g}} \left(H_1^{\frac{1}{2}} - H_2^{\frac{1}{2}}\right) \qquad (2)$$

EXAMPLE.

Two tanks, the bottom of which are on the same level, are connected with one another by a horizontal pipe 3 in. diameter, 1,000 ft. long, and bell mouthed at each end. One tank is of size 20 by 20 ft. and contains water to a depth of 20 ft., the other tank is of size 15 by 15 ft. and holds water to a depth of 10 ft.

If the tanks are put in communication with one another by means of the pipe (which is full of water), how long will it be before the water level in the larger tank falls from a height of 19 ft. to 17 ft. ? Assume $f = \cdot 01$. (London Univ.)

This question is shown diagrammatically in Fig. 82.

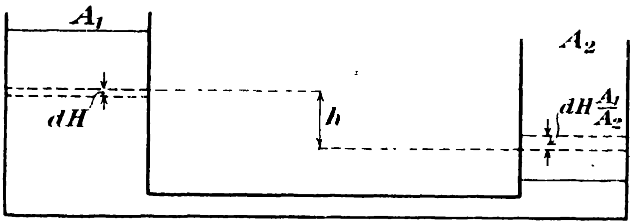


FIG. 82

Let A_1 and A_2 be the areas of the large and small tanks respectively.

Let h = difference of water level in tanks at any instant.

Let water level in A_1 fall by amount dH in time dt.

Then, level in small tank rises by $dH\frac{A_1}{A_2}$. Let dh be difference in head causing flow due to this change.

Then,
$$dh = dH + dH\frac{A_1}{A_2}$$
$$= dH\left(1 + \frac{A_1}{A_2}\right) \quad . \qquad (1)$$

Let a, v, d, and l be the area, velocity, diameter, and length of pipe respectively.

As quantity flowing from large tank equals quantity flowing along pipe, and as dH is negative,

$$-A_1\, dH = a\, v\, dt \quad . \quad . \quad . \quad . \qquad (2)$$

But,
$$h = \frac{v^2}{2g} + \frac{4 f l v^2}{d\, 2g}$$
$$= \frac{v^2}{2g}\left(1 + \frac{4 f l}{d}\right) \quad . \quad . \qquad (3)$$

Substituting Equations (1) and (3) in (2),

$$-A_1 \frac{dh}{\left(1+\frac{A_1}{A_2}\right)} = a\sqrt{\frac{2g\,h}{\left(1+\frac{4fl}{d}\right)}}\,dt$$

Therefore, $$dt = -\frac{A_1\sqrt{1+\frac{4fl}{d}}\,h^{-\frac{1}{2}}\,dh}{\left(1+\frac{A_1}{A_2}\right)a\sqrt{2g}}$$

Integrating between the limits of H_2 and H_1,

$$T = \int_0^T dt = \frac{2A_1\sqrt{1+\frac{4fl}{d}}\,\left(H_1^{\frac{1}{2}} - H_2^{\frac{1}{2}}\right)}{\left(1+\frac{A_1}{A_2}\right)a\sqrt{2g}} \qquad (4)$$

In this question, $A_1 = 20 \times 20 = 400$ sq. ft.

$$A_2 = 15 \times 15 = 225 \text{ sq. ft.}$$

$$a = \frac{\pi}{4}\left(\frac{1}{4}\right)^2 = \cdot 0491 \text{ sq. ft.}$$

$$H_1 = 19 - \left(10 + \frac{400}{225}\right) = 7 \cdot 22 \text{ ft.}$$

$$H_2 = 17 - \left(10 + \frac{3 \times 400}{225}\right) = 1 \cdot 66 \text{ ft.}$$

Substituting these values in Equation (4)

$$T = \frac{2 \times 400\sqrt{1+\frac{4 \times \cdot 01 \times 1000}{\cdot 25}}\,\left(7 \cdot 22^{\frac{1}{2}} - 1 \cdot 66^{\frac{1}{2}}\right)}{\left(1+\frac{400}{225}\right)\cdot 0491\sqrt{64 \cdot 4}}$$

$= 13{,}150$ sec.

$= 219$ minutes.

74. **Flow of Gases through Pipes.** The frictional resistance of gas flowing along a pipe may be found from the same

frictional formula as liquids. The head causing flow must be in feet of gas and if the pipe is sloping the slight difference of atmospheric pressure due to change of altitude must be taken into account.

Let gas be flowing up a sloping uniform pipe length of l and diameter d.

Then, head lost in friction $= \dfrac{4 f l v^2}{d\,2g}$ ft. of gas.

Let w = weight of 1 cu. ft. of water

w_1 = weight of 1 cu. ft. of gas

and w_2 = weight of 1 cu. ft. of air

Let h_1 be height of upper end of pipe above lower end.

Then, atmospheric pressure at lower end is h_1 ft. of air greater than at upper end. But pressure of gas at lower end is greater than pressure at higher end by h_1 ft. of gas.

Then, head causing flow due to change of altitude

$$= h_1 \text{ ft. of air} - h_1 \text{ ft. of gas}$$

$$= \left(h_1 \frac{w_2}{w_1} - h_1\right) \text{ ft. of gas.}$$

Suppose the pressure of gas above atmosphere be measured with a U-tube containing water.

Let y_1 = pressure of gas at lower end in feet of water

and y_2 = pressure of gas at higher end in feet of water.

Then, head causing flow due to difference of pressure

$$= (y_1 - y_2) \text{ feet of water}$$

$$= (y_1 - y_2)\frac{w}{w_1} \text{ ft. of gas}$$

Total head causing flow $= h_1 \dfrac{w_2}{w_1} - h_1 + (y_1 - y_2)\dfrac{w}{w_1}$ ft. of gas

$$\text{Then, } h_1 \frac{w_2}{w_1} - h_1 + (y_1 - y_2)\frac{w}{w_1} = \frac{4 f l v^2}{d\,2g} + \frac{v^2}{2g} \qquad (1)$$

If the gas is flowing down the pipe h_1 will be negative.

If the pipe is horizontal $h_1 = 0$,

Equation (1) then becomes

$$(y_1 - y_2)\frac{w}{w_1} = \frac{4\,f\,l\,v^2}{d\;2g} + \frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad (2)$$

Unwin found the coefficient of friction f to be equal to $\cdot 0044\left(1 + \frac{1}{7d}\right)$ for coal gas.

EXAMPLE.

Gas is supplied from a holder at a gauge pressure of 4 in. of water to a pipe 6 in. diameter, 300 ft. long, which rises to and discharges at a height of 50 ft. above the level of the outlet of the holder. The pressure at the pipe outlet must be not less than 1 in. of water by gauge. Find the delivery in cubic feet per hour. Take the weights of the gas and air as ·045 and ·08 lb. per cu. ft. respectively and the coefficient of friction as ·008. (London Univ.)

Applying Equation (1),

$$\left(50 \times \frac{\cdot 08}{\cdot 045}\right) - 50 + (\cdot 333 - \cdot 0834)\frac{62\cdot 4}{\cdot 045} = \frac{v^2}{2g}\left(1 + \frac{4 \times \cdot 008 \times 300}{\cdot 5}\right)$$

$$88\cdot 9 - 50 + 346\cdot 5 = \frac{v^2}{2g}(1 + 19\cdot 2)$$

From which, $v = 35$ ft. per sec.

$$\text{Delivery} = \frac{\pi}{4}(\cdot 5)^2 \times 35 \times 3600 = 24{,}750 \text{ cu. ft. per hour.}$$

75. **Transmission of Power through Pipes.** If power is transmitted through a considerable distance by means of water under pressure, the power supplied will be in proportion to the quantity of water per second passing through the pipe, and to the total head of the water. As the water flows along the pipe it will be subjected to a loss of head due to friction. It can be shown that the maximum power is transmitted by a pipe when the frictional loss of head is one-third of the total head supplied.

Let H = total head supplied at entrance to pipe

h_f = head lost due to friction

and let v, d, and l be the velocity of flow through pipe, the diameter of pipe, and length of pipe respectively.

Then, $$h_f = \frac{4\,f\,l\,v^2}{d\,2g}$$

Total head available at outlet of pipe $= H - h_f$

$$= H - \frac{4\,f\,l\,v^2}{d\,2g}$$

Available horse-power $= \dfrac{w\,\pi\,d^2\,v}{4 \times 550}\left(H - \dfrac{4\,f\,l\,v^2}{d\,2g}\right)$

as $w\,\dfrac{\pi}{4}\,d^2\,v =$ weight of water flowing per second.

From which, $$H.P. = \frac{w\,\pi\,d^2}{4 \times 550}\left(Hv - \frac{4\,f\,l\,v^3}{d\,2g}\right)$$

This will be a maximum when the amount inside the bracket is a maximum. Differentiating this with respect to v and equating to zero for a maximum,

$$\frac{d.(H.P.)}{dv} = H - 3\left(\frac{4\,f\,l\,v^2}{d\,2g}\right) = 0$$

Or $$H - 3\,h_f = 0$$

Therefore, $$H = 3\,h_f$$

That is, the horse-power transmitted is a maximum when the head lost in friction is one-third of total head supplied.

For any pipe line transmitting power,

$$\text{efficiency of transmission} = \frac{H - h_f}{H}$$

Power is transmitted through water pipes for working hydraulic machines. The supply of water under pressure for power purposes was being developed in large cities during the latter half of the nineteenth century; but the commercializing of electricity has mainly displaced this method of power transmission.

Example.

A hydraulic machine is supplied with water through a horizontal pipe 3,000 ft. long. The brake horse-power of the hydraulic machine is 50, and its mechanical efficiency is 80 per cent. Gauges fitted to the supply pipe show that the pressure at the power station end is 750 lb. per sq. in.; and at the machine 680 lb. per sq. in. If the coefficient of resistance, f, for the pipe is ·008, determine (1) the diameter of the supply pipe, (2) the velocity of flow. (London Univ.)

Let a, d, and v be area, diameter, and velocity of pipe respectively.

$$\text{Horse-power supplied by machine} = 50 \times \frac{100}{80} = 62 \cdot 5$$

$$= \frac{W\,H}{550}$$

$$= \frac{62 \cdot 4\, a\, v}{550} \times \frac{680 \times 144}{62 \cdot 4}$$

From which $\qquad av = \cdot 351 = \frac{\pi}{4} d^2 v$

Then, $\qquad d^2 v = \cdot 447 \qquad . \quad . \quad . \qquad (1)$

$$\text{Head lost in friction in pipe} = (750 - 680)\frac{144}{62 \cdot 4}$$

$$= 161 \cdot 8 \text{ ft. of water}$$

$$= \frac{4\,f\,l\,v^2}{d\;2g}$$

Therefore, $\qquad 161 \cdot 8 = \frac{4 \times \cdot 008 \times 3000\, v^2}{d \times 64 \cdot 4}$

From which, $\qquad \frac{v^2}{d} = 108 \cdot 6 \quad . \quad . \quad . \qquad (2)$

Substituting for d from Equation (1),

$$v^5 = 5280$$

Then, $\qquad v = 5 \cdot 55$ ft. per sec.

Substituting this value of v in Equation (2),

$$d = \frac{5 \cdot 55^2}{108 \cdot 6} = \cdot 284 \text{ ft.}$$

76. **Flow through Nozzles.** A nozzle is a tapering mouthpiece which is fitted to the outlet end of a pipe for the purpose of converting the total head of the water into velocity head. They are used on the end of hose pipes and in some forms of turbines. As the pressure of the jet issuing from the nozzle is atmospheric, the whole of the energy will be kinetic. The loss of energy in the nozzle itself will be small compared with the frictional loss in the pipe to which the nozzle is fixed, and may, therefore, be neglected.

In Art. 75 it was proved that for the maximum power to be transmitted along the pipe, the loss of head in the pipe due to friction must be one-third of the total head supplied. In which case, the loss of head due to friction will be one-half of the total head in the nozzle. By making use of this fact it is possible to obtain the ratio of the area of nozzle to area of supply pipe for maximum transmission of power.

Consider the pipe and nozzle of Fig. 83.

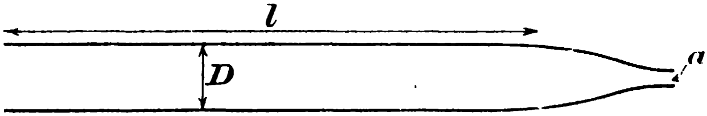


FIG. 83

Let l = length of supply pipe
D = diameter of supply pipe
V = velocity in supply pipe
f = coefficient of frictional resistance of supply pipe
a = area of outlet end of nozzle
and v = velocity of jet issuing from nozzle

For maximum transmission of power,

head lost in friction in supply pipe $= \frac{1}{3}$ total head supplied
$= \frac{1}{2}$ velocity head at nozzle

Or,
$$\frac{4 f l V^2}{D\, 2g} = \frac{1}{2} \times \frac{v^2}{2g} \qquad . \quad . \quad (1)$$

But, as quantity flowing through pipe equals quantity passing through nozzle,

$$VA = va$$

Therefore,
$$\frac{v}{V} = \frac{A}{a} \qquad . \quad . \quad . \quad (2)$$

From Equation (1),
$$\frac{8 f l}{D} = \frac{v^2}{V^2}$$

Substituting from Equation (2),
$$\frac{8 f l}{D} = \left(\frac{A}{a}\right)^2$$

Therefore,
$$\frac{A}{a} = \sqrt{\frac{8 f l}{D}} \qquad . \quad . \quad (3)$$

This gives the ratio between the areas of nozzle and supply pipe for maximum transmission of power.

$$\text{Horse-power of jet} = \frac{\text{Kinetic energy per second}}{550}$$

$$= \frac{wa\,v \times v^2}{2g\,550} = \frac{wa\,v^3}{2g\,550}$$

If the jet were projected vertically upwards the height the water would reach $= \dfrac{v^2}{2g}$

If the area of the supply pipe and jet do not conform with Equation (3), the pipe will not be transmitting the maximum horse-power possible. If such be the case,

$$\text{head transmitted by pipe} = H - \frac{4fL\,V^2}{2g\,D}$$

where H is the head supplied at the source.

Substituting from Equation (2) for V,

$$\text{head transmitted by pipe} = H - \frac{4\,f\,L}{2g\,D}\left(\frac{v\,a}{A}\right)^2$$

But, head transmitted = Kinetic energy at nozzle

$$= \frac{v^2}{2g} \text{ per lb. of water.}$$

Hence,
$$H - \frac{4\,f\,L}{2g\,D}\,\frac{a^2}{A^2}\,v^2 = \frac{v^2}{2g}$$

From which,
$$v = \sqrt{\frac{2g\,H}{1 + \dfrac{4\,f\,L\,a^2}{D\,A^2}}}$$

Horse-power transmitted = horse-power of jet

$$= \frac{(wa\,v)\,v^2}{550 \times 2g}$$

$$\text{Efficiency of transmission} = \frac{\text{Head transmitted}}{\text{Head supplied}}$$

$$= \frac{v^2}{2g\,H}$$

EXAMPLE 1.

The head of water at one end of a pipe, 200 yds. long, 3 in. in diameter, is 100 ft., and f for the main is ·01. What diameter of nozzle fitted to the end will give the maximum power, and what will the power then be? If a formula is used it must be proved. (London Univ.)

Using Equation (3) let d be the diameter of nozzle,

$$\frac{A}{a} = \sqrt{\frac{8fl}{D}}$$

Then, $$\frac{D^2}{d^2} = \sqrt{\frac{8 \times \cdot 01 \times 600}{\cdot 25}} = 13{\cdot}85$$

And, $$d = \sqrt{\frac{(\cdot 25)^2}{13{\cdot}85}} = \cdot 067 \text{ ft.} = \cdot 806 \text{ in}$$

$$h_f = \frac{H}{3} = \frac{4fLV^2}{2g\,D}$$

That is. $$\frac{100}{3} = \frac{4 \times \cdot 01 \times 600\ V^2}{64{\cdot}4 \times \frac{1}{4}}$$

From which, $V = 4{\cdot}72$ ft. per sec.

Then, $v = 4{\cdot}72 \times 13{\cdot}85 = 65{\cdot}4$ ft. per sec.

and, $$H.P = \frac{w\,a\,v^3}{550 \times 2g} = \frac{62{\cdot}4 \times \left(\frac{\pi}{4} \times \cdot 067^2\right) \times 65{\cdot}4^3}{550 \times 64{\cdot}4} = \cdot 171$$

EXAMPLE 2.

A horizontal pipe, 6 in. internal diameter and 540 ft. long, conducts water from a reservoir. When the water level in the reservoir is 4 ft. above the axis of the pipe the discharge through the pipe is 29·7 cu. ft. per min. If a nozzle tapering from 6 to 1½ in. internal diameter were fitted to the free end of the pipe, what would be the horse-power of the jet if the level of water in the reservoir were increased to 40 ft. above the axis of the pipe? (London Univ.)

In the first case, $$V = \frac{29{\cdot}7}{60 \times \frac{\pi}{4} \times \frac{1}{4}} = 2{\cdot}52 \text{ ft. per sec.}$$

Also, $$H = \frac{V^2}{2g}\left(1 + \frac{4fl}{D}\right)$$

That is, $$4 = \frac{2{\cdot}52^2}{64{\cdot}4}\left(1 + \frac{4 \times 540l}{\cdot 5}\right)$$

From which, $f = \cdot 00916$

In the second case, $H = \dfrac{V^2}{2g} + \dfrac{4flV^2}{2gD}$

and as $v = \dfrac{VA}{a}$

$$H = \frac{V^2}{2g}\left\{\left(\frac{A}{a}\right)^2 + \frac{4fl}{D}\right\}$$

That is, $40 = \dfrac{V^2}{64 \cdot 4}\left\{\left(\dfrac{36}{2 \cdot 25}\right)^2 + \dfrac{4 \times \cdot 00916 \times 540}{\cdot 5}\right\}$

From which, $V = 2 \cdot 95$ ft. per sec.

and, $v = 47 \cdot 2$ ft. per sec.

Horse-power of jet $= \dfrac{wav^3}{2g \times 550}$

$$= \frac{62 \cdot 4 \times \cdot 785 \times (\frac{1}{8})^2 \times 47 \cdot 2^3}{64 \cdot 4 \times 550}$$

$$= 2 \cdot 27$$

77. **Hammerblow in Pipes.** If water is flowing along a long pipe and is suddenly brought to rest by the closing of a valve. or by any similar cause, there will be a sudden rise in pressure due to the momentum of the moving water being destroyed. This will cause a pressure wave to be transmitted along the pipe which may set up noises known as knocking. The magnitude of this pressure will depend on the speed at which the valve is closed and on the length of the pipe. Knocking may often be heard in the water pipes of ordinary dwelling-houses if the tap be turned off quickly.

This sudden rise in pressure in a pipe due to the stoppage of the flow is known as the hammerblow.

Consider a long pipe of length "l" ft. and of cross-sectional area "a" sq. ft.; let water be flowing along the pipe with a velocity of v ft. per sec., and, due to the closing of a valve, let the water be brought to rest in t secs.

Then, retardation of water $= f = \dfrac{v}{t}$ ft. per sec.^{2}

Mass of moving column of water $= \dfrac{wal}{g}$

As force = Mass $\times$ retardation,

force on valve $= \dfrac{wal}{g} \times f$ lbs.

$$\text{Intensity of pressure on valve} = p = \frac{\text{force}}{\text{area}}$$

$$= \frac{w\,l\,f}{g} \text{ lb. per sq. ft.}$$

Or,
$$p = \frac{w\,l\,v}{g\,t} \text{ lb. per sq. ft.}$$

From this equation the magnitude of the pressure wave can be found.

This is the simple theory only; actually the water would compress on account of its bulk elastic modulus, and the pipe would expand laterally on account of its modulus of elasticity; these would both affect the problem.* The solution of this problem is dealt with in Arts. 221 to 224.

The Dorman Wave Transmission† for working hydraulic drills in mines is based on this principle. A blow from a piston is given to the water in a long pipe, which causes a pressure wave to travel the full length. At the other end of the pipe the pressure wave is made to work the drill. By this means, power is transmitted through a pipe containing water although the water itself does not flow.

EXAMPLE.

A hydraulic pipe line is 2 miles long. The velocity of flow is 4 f.s. A valve at the lower end is closed at such a rate as to produce uniform retardation in the water column. Calculate the rise in pressure behind the valve if the latter is closed: (*a*) in 20 secs.; (*b*) in 1 sec. (A.M.Inst. C.E.)

(*a*) Using the equation

$$p = \frac{w\,l\,v}{g\,t}$$

$$= \frac{62{\cdot}4 \times 2 \times 5280 \times 4}{32{\cdot}2 \times 20}$$

$$= 4{,}100 \text{ lb. per sq. ft.}$$

(*b*)
$$p = \frac{62{\cdot}4 \times 2 \times 5280 \times 4}{32{\cdot}2 \times 1}$$

$$= 82{,}000 \text{ lb. per sq. ft.}$$

* See *Water Hammer in Hydraulic Pipe Lines* (Gibson).

† See *Theory of Wave Transmission* (Constantinesco) for mathematical treatment; and article entitled "Dorman Wave Transmission" in *Conquest*, December, 1920, for description.

EXAMPLES 6.

(1) Find the loss of head due to friction in a pipe, 1,000 ft. long and 6 in. diameter, when the quantity of water flowing is 600 gallons per min. $f = \cdot 01$.

Ans.—83 ft. of water.

(2) Using the formula $v = C\sqrt{mi}$, find the loss of head due to friction in a circular pipe, 100 ft. long and 3 in. diameter, when the velocity of flow is 6 ft. per sec. $C = 100$.

Ans.— 5·76 ft.

(3) Draw curves showing the nature of the results obtained by Froude in regard to the surface friction of planes, of varying length and of different materials, moving through water. If $f = \cdot 0035$ and $n = 1\cdot 83$, find the horse-power required to overcome the skin resistance of a ship, wetted surface, 24,000 sq. ft., when going at 18 knots. One knot = 1·69 ft. per sec. (London Univ.)

Ans.—2,420.

(4) Assuming that $R = fAV^2$ to be the law of friction between a flow and a surface, find an expression for the work lost per second when a disc of radius r is rotated in water with a circumferential velocity v.

If the disc is surrounded by a forced vortex of double its diameter, compare the loss due to the friction of the vortex on the flat sides of the vortex chamber with the loss due to the friction on the above-mentioned disc. (London Univ.)

Ans.—$\frac{4}{5}\pi f v^3 r^2$; 32 - 1.

(5) The friction of a thin flat brass plate when towed edgewise through water at a velocity of 10 ft. per sec. is equal to ·21 lb. per sq. ft. of wetted surface, and the friction is found to vary as $V^{1\cdot 9}$.

Find how many foot-pounds of energy per minute are absorbed by the skin friction of the two surfaces of a flat circular disc, the external diameter of which is 24 in., and the internal diameter 12 in., if the disc makes 450 revolutions per minute. (London Univ.)

Ans.—27,800 ft. lb. per min.

(6) What do you understand by the expression "critical" velocity of flow in a pipe?

Describe experiments on the loss of head when water flows at known velocities through a horizontal pipe of constant cross section.

What is the law when (*a*) the velocity is less than the "critical"; (*b*) the velocity exceeds the "critical"? (London Univ.)

Ans.—(*a*) kv; (*b*) kv^n.

(7) Obtain an expression for the head lost in a pipe due to a sudden enlargement of area.

Comment on any assumption made.

A pipe 2 in. diameter is 20 ft. long and the velocity of the water in the pipe is 8 ft. per sec. What loss of head would be saved if the central 6 ft. length of pipe were replaced by 3 in. diameter pipe, the changes of section being sudden.

Take the frictional coefficient $f = \cdot 01$, and the coefficient of contraction ·62. (London Univ.)

Ans.—·52.

(8) Water is discharged from a reservoir through a pipe 1 ft. diameter for 1 mile of its length, the pipe then suddenly enlarging to 2 ft. diameter for the second mile.

There are two right-angled easy bends in each mile, and the difference of head between entrance and discharge ends of the pipe is 100 ft. Calculate the discharge in gallons per minute and all losses in the pipe if the coefficient of friction is ·008.

Draw the hydraulic gradient. (London Univ.)

Ans.—1,780 gallons per minute.

(9) Two reservoirs *A* and *B* discharge through circular pipes each 2 ft. in diameter and 1 mile long to a junction at *D*. From *D* the joint discharge is carried in a straight line with the discharge pipe from *A* to a third reservoir *C* by a 3 ft. diameter pipe of negligible length. The surface level at *A* is 50 ft., and that of *B* 30 ft. above that of *C*. Neglecting all losses other than pipe friction, find the discharge in gallons per minute from each reservoir.

Coefficient of friction = ·0075. (London Univ.)

Ans.—7,500 and 5,800 gallons per minute.

(10) A high level reservoir feeds two low service reservoirs by means of a single main 5 miles long, 30 in. diameter, laid at a slope of 10 ft. per mile. The main is then forked, and one branch, 2 miles long with a fall of 15 ft. per mile, serves one reservoir, whilst the other is served by a pipe 3 miles long with a fall of 12 ft. per mile. Calculate the diameters of these branch pipes so that each may deliver 4,000,000 gallons per day of 24 hours. Take f = ·006. (London Univ.)

Ans.—1·52 ft. and 1·6 ft.

(11) A cylindrical tank 16 ft. diameter discharges through a pipe 300 ft. long and 9 in. diameter. Find the time taken to lower water level in tank from 9 ft. above centre of pipe to 4 ft. above centre. f = ·01.

Ans.—7·82 minutes.

(12) Air initially at a pressure of 60 lb. per sq. in. absolute and a temperature of 16° C. flows through a 10 in. main which is 1 mile in length. Assuming that the coefficient of resistance to flow is ·0035, calculate the discharge in cubic feet per second, assuming that the pressure at the delivery end is to be maintained at 55 lb. per sq. in. absolute. (London Univ.)

Ans.—23·3 cu. ft. per sec.

(13) Air, initially at atmospheric pressure and 60° F., flows under a pressure difference of 10 in. of water through a 12 in. main 1,000 yds. long. Assuming that the coefficient of resistance to flow f is ·004, determine the number of cubic feet of air delivered per second. (London Univ.)

Ans.—24·2 cu. ft. per sec.

(14) In a water-power scheme, the total head is 503 ft., and 1,750,000 gallons of water are available per hour for utilization in an impulse turbine of the Pelton type. The proposed pipe line is 2 miles long.

Determine the diameter of the pipe necessary in order that the efficiency of transmission should be 80 per cent, and also calculate the horse-power available.

If the power is supplied to the Pelton wheel through two nozzles, determine their diameter.

Neglect the losses at inlet to the pipe and at the nozzles. f = ·0075. (London Univ.)

Ans.—3·44 ft.; 3,550; ·555 ft.

(15) In hydraulic transmission of power, state the losses which occur, and explain how they may be minimized. 100 h.p. is to be transmitted, the pressure at the inlet of the pipe being 1,000 lb. per sq. in. If the pressure drop per mile is to be 10 lb. per sq. in., and if $f = \cdot 006$, find the diameter of the pipe and the efficiency of transmission for 10 miles. (London Univ.)

Ans.—Efficiency = 90 per cent; ·477 ft.

(16) What is meant by "critical velocity" in fluid motion? State what factors in general have an effect on the value of this. (A.M.I. Mech. E.)

(17) The resistance to the motion, in the direction of its plane, of a thin flat body through water is proportional to v^2, and, at 10 ft. per sec., is ·5 lb. per sq. ft. Determine the horse-power required to rotate at 1,200 r.p.m. a submerged disc 2 ft. in diameter. (A.M.I. Mech. E.)

Ans.—45·4.

(18) Determine the levels of the hydraulic gradient at the points B, C, and D of a pipe-line discharging 12 cu. ft. per sec. The initial level of the gradient at A is 400 ft. above datum. AB is 24 in. diameter and 5,000 ft. long; BC is 18 in. diameter and 4,000 ft. long, and CD is 20 in. diameter and 3,000 ft. long. Short taper pipes are introduced at B and C. $f = \cdot 01$. (A.M.I. Civil E.)

Ans.—377·4 ft.; 300·7 ft.; 266·5 ft.

(19) Reservoir A at an elevation of 900 ft. supplies water to reservoirs B and C at levels respectively of 600 ft. and 500 ft. From A to D both supplies pass through a common pipe 12 in. diameter and 10 miles long; the branch D to B is 9 in. diameter and 6 miles long, and that from D to C is 6 in. diameter and 5 miles long. How many cubic feet per second will be delivered to B and C? $f = \cdot 01$. (A.M.I. Civil E.)

Ans.—1·00 cu. ft. per sec; ·7 cu. ft. per sec.

(20) The reservoir from which a Pelton wheel is supplied has an elevation of 1,050 ft. The pipe line is 18 in. diameter and 9,660 ft. long; it terminates at a level of 50 ft. in a nozzle which gives a jet with an effective diameter of 3 in. Taking for the nozzle $C_v = \cdot 97$, and for the pipe $f = \cdot 006$, determine the horse-power of the jet. (A.M.I. Civil E.)

Ans.—1282.

(21) Two pipe lines of equal length (10,000 ft.) are laid in parallel between two reservoirs whose difference of level is 50 ft. If their diameters are respectively 12 in. and 24 in., and if the frictional resistance is given by $h = \dfrac{f\,l\,v^{1\cdot 8}}{d^{1\cdot 2}}$, what will be the total discharge? Take $f = \cdot 005$. (A.M. Inst. C.E.)

Ans.—5·8 cu. ft. per sec.

(22) A 6-in. pipe line 10,000 ft. long is supplied with water at 1,200 lb. per sq. in. pressure. The coefficient f in the formula $h = \dfrac{f\,l\,v^2}{2g\,m}$ is 0·01. What is the maximum rate, in horse-power, at which energy can be delivered at the outlet from the pipe line? (A.M. Inst. C.E.)

Ans.—355 h.p.

(23) A thin flat disc enclosed in a casing containing water is to be used as a hydraulic dynamometer for absorbing and measuring the output from a petrol engine running at 1,800 revs. per min. Experiments on a similar type of surface show that its frictional resistance per square foot is equal to $0{\cdot}005\,v^2$ lb., where v is the velocity in feet per second. What diameter of disc will be necessary if the engine develops 50 b.h.p.? (A.M.I. Mech. E.)

Ans.—1·6 ft.

(24) The loss of head in a given pipe line is proportional to v. The following are corresponding experimental values of h and of v—

h	1·5	4·5	8·0	12·0
v	2·0	3·5	4·8	6·0

What is the value of n? (A.M.I. Mech. E.)

Ans.—n = 1·97.

(25) What is meant by "critical velocity" in pipe flow? Describe how you could determine, experimentally, the value of the lower critical velocity. (A.M. Inst. C.E.)

(26) Two reservoirs whose surface levels differ by 100 ft. are connected by a pipe 2 ft. diameter and 10,000 ft. long. The pipe line crosses a ridge whose summit is 30 ft. above the level of, and 1,000 ft. distant from, the higher reservoir. Find the minimum depth below the ridge at which the pipe must be laid if the absolute pressure in the pipe is not to fall below 10 ft. of water, and calculate the discharge in cubic feet per second. (f = ·0075.) (London Univ.)

Ans.—16·66 ft.; 20·5 cu. ft. per sec.

(27) Calculate the diameter of a pipe to convey gas from a holder to a power station, having given the following particulars: Gas consumption, 20,000 $ft.^3/hr.$; length of pipe, 0·5 mile; delivery, 50 ft. above the entrance to the pipe; pressure at holder, 4 inches of water, and at the power station 2 in.; density of gas 0·045 and of air 0·08 $lb./ft.^3$ Take f = 0·005. (Lond. Univ.)

Ans.—8·25 in.

(28) A chimney is 100 ft. high and 5 ft. in diameter. The horizontal flue is 60 ft. long and of the same section. The flue gas temperature is 250° C. and its speed 20 ft. per sec. Estimate the draught available at the boiler in inches of water. Take f = 0·01 and weight of gas per cu. ft. = ·078 lb. at N.T.P. (I. Mech. E.)

Ans.—·0518 in. of water.

(29) Compressed air is transmitted through 300 ft. of 2-in. pipe. The supply pressure is 100 lb. per sq. in., and the flow is 80 cu. ft. per min. at the supply end. Calculate the delivery pressure assuming the temperature remains at 15° C. throughout, and that $PV = 96T$ for 1 lb. of air. Prove any formula used. Take f = 0·005. (London Univ.)

Ans.—97·2 lb. per sq. in.

(30) Compressed air is transmitted through 5,000 ft. of 2-in. diameter pipe. The inlet pressure is maintained at 100 lb. per sq. in. (gauge) and the temperature throughout the pipe is 15° C. Calculate the maximum flow expressed in cu. ft. of free air per min. at 15° C. and 14·7 lb. per sq. in. (absolute) if the exit pressure is not to fall below 80 lb. per sq. in. (gauge). Assume a friction coefficient f = 0·004 and for air $\frac{PV}{T} = 96$.

Prove any formula used to allow for changing density. (London Univ.)

Ans.—231 cu. ft. per min.

CHAPTER VII

FLOW THROUGH OPEN CHANNELS

78. **Open Channels.** The term "open channel" applies to any passage through which water is flowing when the free surface of the water is in contact with the atmosphere. The water is then under atmospheric pressure throughout. The channel may be covered in at the top or open; if covered in, it must not be running full, otherwise the pressure might rise above or fall below atmospheric. A pipe which is not running full may be classed as an open channel. An open channel may be of uniform cross section, as a canal, sewer, and aqueduct or it may be of an irregular cross section, such as a river.

Water flowing through an open channel is subjected to a frictional resistance at the wetted surface of the channel which obeys the same laws as stated in the previous chapter. As the pressure throughout is atmospheric, the head causing flow will be due entirely to the slope of the channel. In channels of regular cross section the velocity of flow is constant; therefore, the head due to the slope of channel may be assumed to be lost in overcoming the frictional resistance of the sides.

The velocity of flow will vary at different points of the cross-section of the channel, being smaller towards the sides. All calculations on the flow through channels are based on the mean velocity of flow at any cross section.

79. **Formula for Flow in Open Channels.** An equation for the flow of water through an open channel may be deduced in a similar manner as for the flow in pipes. In an open channel the pressure is atmospheric and may, therefore, be neglected; the head due to the slope of the pipe is assumed to be lost in friction. Hence the hydraulic gradient is equal to the slope of the channel if the latter is uniform.

Let i = slope of channel
A = area of cross section of channel
P = wetted perimeter
v = mean velocity of flow
h_f = head lost in friction
f' = coefficient of friction between water and sides of channel for unit velocity.

Consider a section of the water of length l moving along the channel (Fig. 84). Assume slope of channel is uniform; it will therefore equal i.

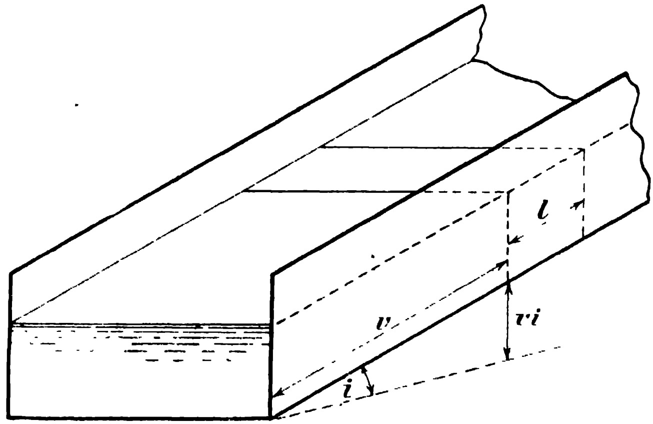


FIG. 84

Frictional resistance of section $= f' \times$ wetted area $\times$ (velocity)^n
$= f'\ Plv^n$

Work done per second in overcoming friction $= f'\ Plv^n \times v$

Loss of potential energy per sec. $=$ Weight $\times$ change of altitude
$= wAl \times vi$

But, potential energy lost $=$ work done against friction

Therefore, $wAl\ vi = f'\ Plv^n\ v$

$$wi = f'\ v^n \frac{P}{A}$$

Assuming $n = 2$ and substituting the hydraulic mean depth m for $\frac{A}{P}$,

$$i = \frac{f'\ v^2}{w\ m}$$

Or,

$$v = \sqrt{\frac{w}{f'}\ m\ i}$$
$$= C\sqrt{m\ i} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $C = \sqrt{\frac{w}{f'}}$ and is a constant depending on the shape and surface of the channel, and is determined experimentally.*

* Actually C will vary with the temperature, the velocity, and the size of the channel, as shown in Chapter XII on Viscous Flow.

Equation (1) is known as the Chezy formula. It was deduced by him empirically and is the same form as used for the flow through pipes.

From the results of experiments on the flow of water through channels, Bazin deduced the following formula for the value of C—

$$C = \frac{157 \cdot 5}{1 + \frac{k}{\sqrt{m}}} \text{ foot units,}$$

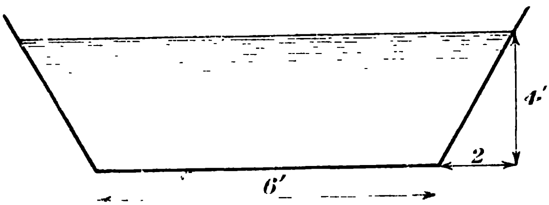


FIG. 85

where k is a constant depending on the surface of the channel and has the following values—

Clean smooth sides of wood, brick, stone, etc.	$k = \cdot 29$
Dirty sides of wood, brick, stone, etc.	$k = \cdot 5$
Sides of natural earth	$k = 2 \cdot 35$

EXAMPLE 1.

A trapezoidal channel, having sides of smooth stone, has a base of 6 ft. and side slopes of 2 vertical to 1 horizontal. The depth of water in the channel is 4 ft. Find the quantity of water flowing if the slope of the channel is 10 ft. per mile.

The section of channel is shown in Fig. 85.

Area of section $= 32$ sq. ft.

Wetted perimeter $= 6 + 2\sqrt{4^2 + 2^2}$

$= 14 \cdot 94$

$$m = \frac{A}{P} = \frac{32}{14 \cdot 94} = 2 \cdot 14$$

Using Bazin's formula for C,

$$C = \frac{157 \cdot 5}{1 + \frac{k}{\sqrt{m}}}$$

$$= \frac{157 \cdot 5}{1 + \frac{\cdot 29}{\sqrt{2 \cdot 14}}}$$

$$= 131 \cdot 5$$

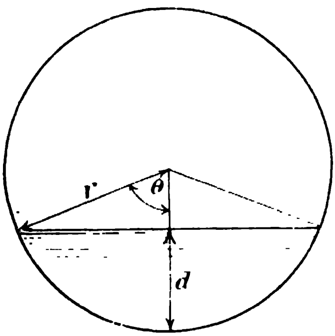


FIG. 86

Using the Chezy formula,

$$v = C\sqrt{m\,i}$$

$$= 131 \cdot 5 \sqrt{2 \cdot 14 \times \frac{10}{5280}}$$

$$= 8 \cdot 35 \text{ ft. per sec.}$$

$$\text{Quantity} = A \times v$$

$$= 32 \times 8 \cdot 35$$

$$= 268 \text{ cu. ft. per sec.}$$

EXAMPLE 2.

Find the depth of flow in a circular sewer 3 ft. diameter, having a fall of 1 in 200, when the discharge is 3,500 gallons per minute. Take $v = 100\sqrt{mi}$, and solve by plotting. (London Univ.)

Assume the water level to be at a height d (Fig. 86). Let r be the radius of the sewer, and θ be half the angle subtended at the centre by the water level.

From Fig. 86,

$$\cos \theta = \frac{r - d}{r}$$

from which the angle θ may be obtained in radians.

$$\text{Area of wetted cross-section} = A = \frac{r^2\, 2\theta}{2} - r^2 \sin \theta \cos \theta$$

$$= r^2\left(\theta - \frac{\sin 2\theta}{2}\right)$$

Wetted perimeter $= P = r2\theta$

$$m = \frac{A}{}$$

$$v = 100\sqrt{m \times \frac{1}{200}}\ \text{ft. per sec.}$$

Quantity $= Av \times 6{\cdot}24 \times 60$ gallons per min.

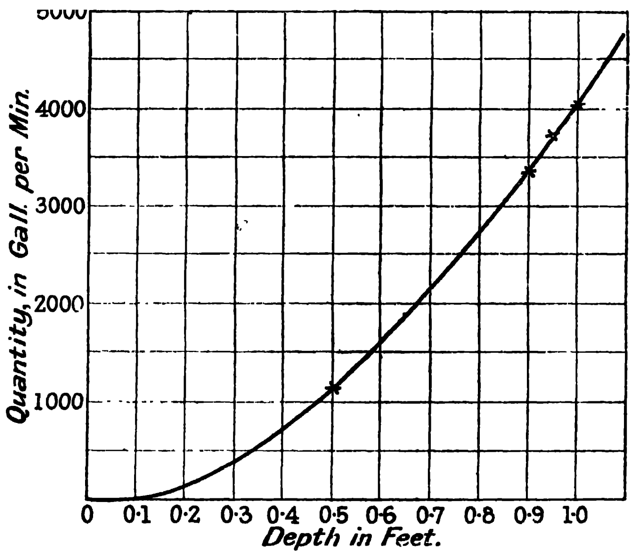


FIG. 87

These quantities are shown tabulated in the following table for assumed values of *d*—

d	cos *θ*	*θ* in rads	*A*	*P*	*m*	*v*	*Q*
·5	·666	·841	·775	2·523	·307	3·92	1137
·9	.4	1·157	1·778	3·471	·512	5·06	3360
·0	·333	1·225	2·04	3·675	·555	5·26	4020
·95	·3665	1·195	1·92	3·585	·535	5·17	3718

d and *Q* are shown plotted in Fig. 87, and a curve is drawn through the points. From this curve the depth to give a discharge of 3,500 gallons per minute may be obtained.

Required depth $=$ ·925 ft.

80. **Rectangular Channel: Depth for Maximum Discharge.** The channel with the most economical section is the one which gives the maximum discharge for a given amount of excavation. The discharge is proportional to the velocity and the area, whilst the excavation is proportional to the area. The proportions of the most economical section may be found by assuming the area to be constant, and finding the depth which gives the maximum velocity.

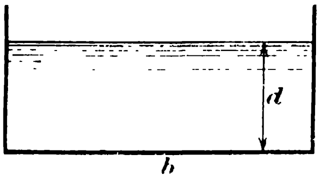


Fig. 88

Consider the rectangular-sectioned channel of Fig. 88; let b be the breadth and d the depth. Assume the area of cross-section to be constant.

As $$A = bd$$

$$b = \frac{A}{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{Discharge} = A \times v = A \times C\sqrt{\frac{A}{P}i}$$

As A, C, and i are constants for channel considered, the discharge will be a maximum when P is a minimum.

$$P = b + 2d$$

Substituting for b from Equation (1),

$$P = \frac{A}{d} + 2d$$

$$= Ad^{-1} + 2d$$

Differentiating P and equating to zero for a minimum,

$$\frac{dP}{dd} = -A\,d^{-2} + 2 = 0$$

Therefore, $$A = 2d^2$$

But, as $$A = bd,$$

$$bd = 2d^2$$

Or, $$d = \frac{b}{2}$$

That is, the maximum discharge for a rectangular channel occurs when the depth of water is one-half of the breadth.

81. **Trapezoidal Channel: Condition for most Economical Section.** The most economical section for a trapezoidal channel will be when the discharge is a maximum for a given excavation. The condition for this may be found, as in the previous case, by assuming the area to be a constant.

Consider the trapezoidal channel of Fig. 89. Let b be the breadth of the base, d be the depth of water, and let the slope of the sides be $\frac{1}{n}$; then the horizontal projection of the wetted side is nd.

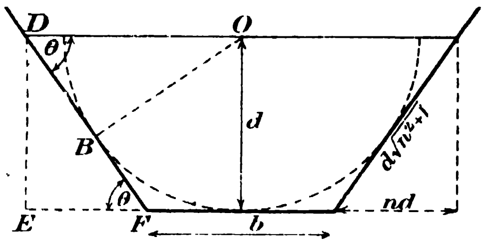


FIG. 89

$$\text{Discharge} = A \times v$$

$$= A \times C\sqrt{\frac{A}{P}i},$$

and will be a maximum when P is a minimum for the given channel.

$$A = (b + nd)d \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{Therefore,} \quad b = \frac{A}{d} - nd \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{Length of sloping side} = \sqrt{n^2d^2 + d^2}$$

$$= d\sqrt{n^2 + 1}$$

$$P = b + 2d\sqrt{n^2 + 1}$$

Substituting for b from Equation (2),

$$P = \frac{A}{d} - nd + 2d\sqrt{n^2 + 1}$$

Differentiating P and equating to zero for a minimum,

$$\frac{dP}{dd} = -\frac{A}{d^2} - n + 2\sqrt{n^2 + 1} = 0$$

$$\text{Therefore,} \quad \frac{A}{d^2} + n = 2\sqrt{n^2 + 1}$$

Substituting for A from Equation 1,

$$\frac{b + nd}{d} + n = 2\sqrt{n^2 + 1}$$

$$\text{Or,} \quad \frac{b + 2nd}{2} = d\sqrt{n^2 + 1} \quad . \quad . \quad . \quad . \quad (3)$$

Let θ be angle of slope of sides to horizontal. Let O be the centre of water surface. From O draw OB to meet a sloping side at B and perpendicular to it.

Consider triangle OBD,

$$\text{angle } ODB = \theta$$

then,
$$\sin\theta = \frac{OB}{OD} = \frac{OB}{\frac{b}{2} + nd}$$

Consider triangle DEF

$$\text{angle } DFE = \theta$$

then,
$$\sin\theta = \frac{ED}{DF} = \frac{d}{d\sqrt{n^2+1}}$$

Equating these two values of $\sin\theta$,

$$\frac{OB}{\frac{b}{2} + nd} = \frac{d}{d\sqrt{n^2+1}}$$

It will be seen from Equation (3) that these two denominators are equal.

Therefore, $OB = d$

That is, if a semicircle is drawn with centre at O and of radius d, the three sides of the section will be tangential to it.

Therefore, the most economical trapezoidal section is when the three sides are tangential to a semicircle described on the water line.

As triangle ODB is similar to triangle DFE, it follows that

$$OD = DF$$

Now,
$$m = \frac{A}{P} = \frac{\left(OD + \frac{b}{2}\right)d}{2\,DF + b}$$
$$= \frac{\left(OD + \frac{b}{2}\right)d}{(2\,OD + b)}$$
$$= \frac{d}{2}$$

This is another condition for maximum discharge which will be found useful for the solution of problems of this nature.

EXAMPLE.

A trapezoidal channel is to be designed for conveying 10,000 cu. ft. of water per minute. Determine the cross-sectional dimensions of the channel from the following data—

Slope 1 in 1,600; sides inclined 45°; cross section to be a minimum; $v = 90\sqrt{mi}$. (London Univ.)

Using Equation (3) and putting $n = 1$,

$$\frac{b + 2d}{2} = d\sqrt{1 + 1}$$

From which, $b = \cdot 828\,d$

$$\text{Area of section} = A = (b + d)d$$
$$= (\cdot 828d + d)d$$
$$= 1\cdot 828d^2$$

$$\text{Wetted perimeter} = P = b + 2\sqrt{2}\,d$$
$$= \cdot 828d + 2\cdot 828d$$
$$= 3\cdot 656d$$

$$m = \frac{A}{P} = \frac{1\cdot 828d^2}{3\cdot 656d}$$
$$= \cdot 5d$$

$$\text{Quantity per second} = A \times v$$
$$= A \times 90\sqrt{mi}$$

Therefore, $$\frac{10{,}000}{60} = 1\cdot 828d^2 \times 90\sqrt{\cdot 5d \times \frac{1}{1600}}$$

Squaring both sides and simplifying,

$$1\cdot 026 = d^4 \times \cdot 0003125d$$

Hence, $d^5 = 3280$

And, $d = 5\cdot 04$ ft.

$$b = \cdot 828d$$
$$= 4\cdot 17 \text{ ft.}$$

82. **Circular Section: Depth for Maximum Velocity.** The velocity of flow in a given circular channel will depend upon the depth of the water. As the velocity is proportional to the hydraulic mean depth, its maximum value may be obtained by differentiating $\frac{A}{P}$ and equating to zero.

Consider the circular channel of Fig. 86.

Let d = depth of water for maximum velocity

$\theta = \frac{1}{2}$ angle subtended at centre by water line, in radians

and r = radius of channel section.

Then,

area of wetted section $= A$ = area of sector − area of triangle

$$= r^2\theta - r^2\frac{\sin 2\theta}{2}$$

$$= r^2\left(\theta - \frac{\sin 2\theta}{2}\right)$$

Wetted perimeter $= P = 2r\theta$

And, $m = \dfrac{A}{P}$

As $v = C\sqrt{mi}$,

v = maximum when m is a maximum. That is, when $\dfrac{A}{P}$ is a maximum.

Differentiating $\dfrac{A}{P}$ and equating to zero,

$$\frac{d\left(\frac{A}{P}\right)}{d\theta} = P\frac{dA}{d\theta} - A\frac{dP}{d\theta} = 0$$

From which, $Pr^2(1 - \cos 2\theta) - A\,2r = 0$

Substituting for P and A,

$$2r^3\theta(1 - \cos 2\theta) = 2r^3\left(\theta - \frac{\sin 2\theta}{2}\right)$$

Therefore, $2\theta = \tan 2\theta$

The solution of which is when $2\theta = 257\tfrac{1}{2}^\circ$

$$d = r - r\cos\theta$$

For maximum velocity, $d = r - r\cos\dfrac{257\frac{1}{2}}{2}$

$$= r(1 + \cdot 62)$$
$$= 1\cdot 62r$$

Or,

depth for maximum velocity $= \cdot 81 \times$ diameter of channel.

83. **Circular Section: Depth for Maximum Discharge.** Referring to Fig. 86, and using the same notation as in Art. 82,

$$\text{discharge} = A \times C\sqrt{mi}$$
$$= A \times C\sqrt{\frac{A}{P}i}$$
$$= C\sqrt{\frac{A^3}{P}i}$$

Therefore, discharge is a maximum when $\frac{A^3}{P}$ is a maximum.

$$\frac{d\left(\frac{A^3}{P}\right)}{d\theta} = \left(P\,3\,A^2\frac{dA}{d\theta} - A^3\frac{dP}{d\theta}\right) \times \frac{1}{P^2} = 0$$

From which, $$3\,P\frac{dA}{d\theta} - A\frac{dP}{d\theta} = 0$$

$$6r^3\theta(1 - \cos 2\theta) - 2r^3\left(\theta - \frac{\sin 2\theta}{2}\right) = 0$$

$$3\theta(1 - \cos 2\theta) = \theta - \frac{\sin 2\theta}{2}$$

Or, $$4\theta - 6\theta\cos 2\theta = -\sin 2\theta$$

The solution of this equation is $\theta = 154°$

Then, for maximum discharge, $d = r - r\cos\theta$

$$= r(1 + \cdot 9)$$
$$= 1{\cdot}9r$$

Or, depth for maximum discharge $= 95 \times$ diameter.

Example.

Find, either graphically or by calculation, the depth for the maximum discharge for a circular culvert.

Find the depth of water for maximum velocity along a 6 ft. diameter culvert. (London Univ.)

The depth for maximum discharge may be found by the above method; or, it may be found by plotting as was done in Example (2), Art. 79.

Depth for maximum velocity may be found from the equation of Art. 82—

$$\text{depth} = \cdot 81 \times \text{diameter}$$
$$= \cdot 81 \times 6 = 4{\cdot}86 \text{ ft.}$$

84. **Variation of Velocity over Cross-section of a Channel.** The velocity of flow varies at different points of the cross-

section of the channel. The frictional resistance of the sides causes the water to slow down towards the sides of the channel, and the frictional resistance between the water surface and the atmosphere causes a slight reduction of velocity at the free

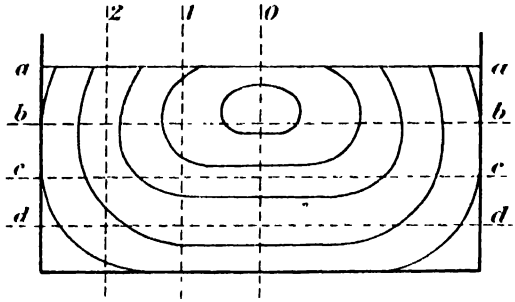


Fig. 90

surface. The maximum velocity will be on the vertical centre line of the channel at a point a little below the free surface.

The variation of velocity over the cross-section of a rectangular channel is shown in Fig. 90. The curves shown are lines of equal velocity; they have the greatest value at the centre, just below the water surface, and decrease towards the sides and base. In Fig. 91 are shown the variations of velocity on horizontal section lines taken at different depths. The velocities at different points of the section lines *a*, *b*, *c*, and *d* (Fig. 90) are plotted on a base representing the width of the channel.

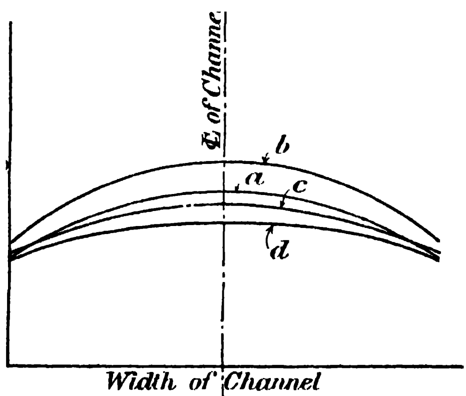


Fig. 91

Fig. 92 shows the variation of velocity on the vertical section lines 0, 1, and 2 (Fig. 90). The horizontal ordinate represents the velocity and the vertical ordinate the depth.

The mean velocity on any vertical section occurs at approximately ·6 of the depth; it varies with the type of channel

and with the nature of the sides. The discharge of the whole channel may be obtained by dividing the section into vertical rectangles and finding the mean velocity of each rectangle. Using this mean velocity, the discharge through each rectangle may be obtained. The sum of all these discharges will be the total discharge of the channel.

The mean velocity of each rectangular strip may be taken, approximately, as the velocity at a depth of ·6 of the total depth.

85. **Measurement of Flow of Irregular Channels.*** By the term "irregular channels" is included large rivers and small streams. The quantity of flow of a small stream may be obtained by fitting a notch or weir across the stream; the discharge may then be calculated by measuring the head over the notch. This method could not be used for a large river on account of the expense and of the obstruction to navigation it would cause. In this case it is necessary to measure the cross-section of the river, and to measure the velocity of flow at various points of this cross-section.

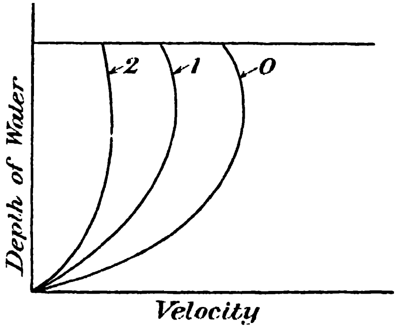


FIG. 92

Let Fig. 93 represent the cross-section of the river at the point chosen. This should be on a straight uniform portion of the river. The cross-section is then divided into vertical rectangles as shown, and the mean velocity of each rectangle

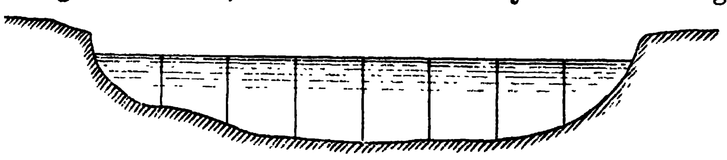


FIG. 93

found. This can be obtained approximately by assuming the mean velocity to occur at ·6 of the depth and measuring the velocity at that point, or it may be found more accurately by measuring the velocity at several depths and calculating the mean from these measurements. The discharge through each

* For the measurement of flow in a channel by a Venturi flume, see Art. 93.

rectangle may then be obtained by multiplying the area by the mean velocity. Then, by adding together the discharge of each rectangle the total flow of the river is obtained.

The velocity of flow may be measured by the following methods—

(*a*) Pitot Tube. The Pitot tube is held with the orifice facing up stream at the place at which the velocity is required. The velocity is then obtained by the method given in Art. 33.

(*b*) Current Meter.* A type of current meter is shown in Fig. 94. It consists of a wheel containing blades or cups,

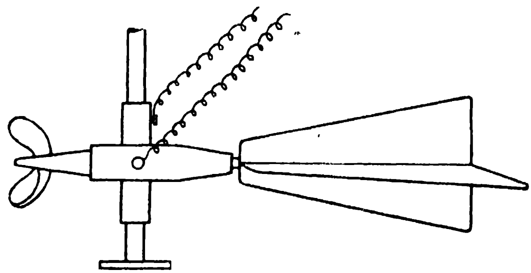


Fig 94

which are rotated by the flowing water; these are headed towards the current by means of a tail on which vanes or fins are fixed. An electric current is passed to the wheel from a battery above the water by means of wires, and a commutator is fixed to the shaft of the revolving blades which makes and breaks the electric circuit each revolution. A revolution counter above the water is worked by this electric current. The meter is lowered into the water at the required point, and the velocity obtained from the revolution counter.

The Amsler Current Meter† is shown in Figs. 95 and 96; this is a universal instrument suitable for both measurements in very slow running waters and yet strong enough for use in great velocities. The propeller is of a strictly helical shape; it is made of one single piece of hard and very resistant aluminium. There is no friction between the axis of the propeller and the electric contact. The shaft of the propeller runs on the one side in a ball bearing and on the other in a sapphire bearing, ensuring smooth running and consequently great accuracy of results, even when using the current meter for

* For a description of the Aerofoil flow recorder, see Art. 165.

† By courtesy of Messrs. Amsler & Co., Schaffhausen, Switzerland.

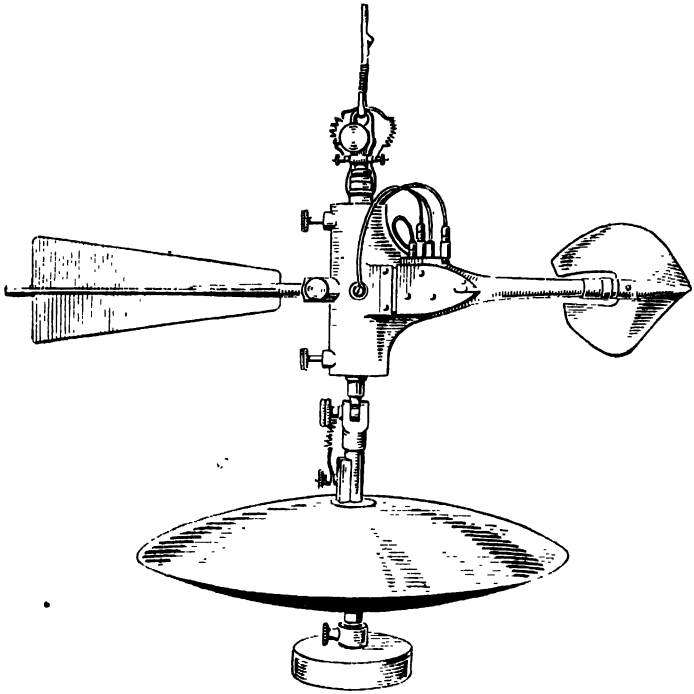


FIG. 95.—AMSLER CURRENT METER WITH UNIVERSAL JOINT, WEIGHT, AND GROUND SINKER

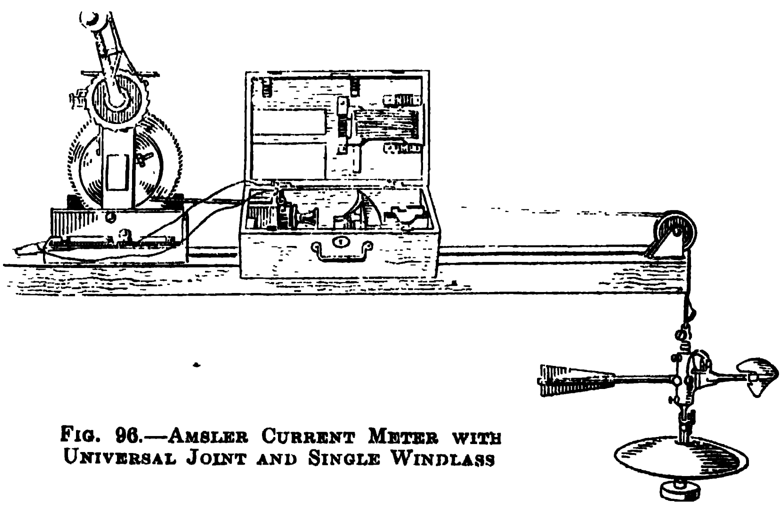


FIG. 96.—AMSLER CURRENT METER WITH UNIVERSAL JOINT AND SINGLE WINDLASS

slow speeds. The current meter commences rotating at a speed of 1 in. per sec.

The contact for transmitting the rotation of the propeller to the electric bell inside the casing of the current meter is arranged in a watertight chamber, thus preventing any corrosive and electrolytic influence of the water, especially sea water or acidulated water. The instrument can be taken to pieces quickly and conveniently without any tools. The ball bearings remain fast to their axis and cannot therefore be lost.

This current meter can also be provided with an additional contact for single revolutions of the propeller and with an observation telephone. At every revolution of the propeller a crack is then heard in the telephone. If the propeller turns backward, as may be the case in whirlpools or backwater, a double crack is heard at every revolution. By means of this telephone it is possible to ascertain whether the propeller turns regularly, forwards or backwards, and also to count directly the number of revolutions of propeller for a certain interval of time if the water flows very slowly. The instrument makes contact at every fifty revolutions of the propeller. The two terminals on the instrument casing are connected, by means of a double wire, with the electric bell and the battery, which are located in the instrument case.

(*c*) FLOATS. A simple way of measuring the velocity of flow of a river is by means of floats. The surface velocity at any section may be obtained by a single float. The time taken for the float to traverse a known distance is measured and the velocity calculated. A single float gives the surface velocity only, and is affected by the wind and air resistance.

A better method is to use double floats. A double float consists of a surface float on to which is attached a hollow metal sphere, heavier than water, and suspended from it by a cord of known length. (Fig. 97.) The depth of the lower float may be regulated by the length of the cord. The velocity is then obtained by timing the top float over a known distance. This gives the mean between the velocity of the surface and the velocity of the layer traversed by the lower float.

The best type is the rod float. This consists of a vertical wooden rod which is weighted at the bottom to keep it vertical. The rod will travel with a velocity equal to the mean velocity of the section. It should be as long as the depth of the river will permit, and the top should be made conspicuous by

painting it white. Some types of rods are made telescopic, so that the length may be adjusted to suit any depth.

Weeds at the bottom of a river will interfere with the use of a rod or double float. If possible, a section of the river which is free from weeds should be chosen.

(*d*) CHEMICAL METHOD. Another method for finding the discharge of an irregular channel is by inserting a chemical solution of known weight and strength uniformly at a certain section. Then, by finding the strength of the solution at another section lower down the stream, the discharge may be calculated. A solution must be chosen which readily mixes

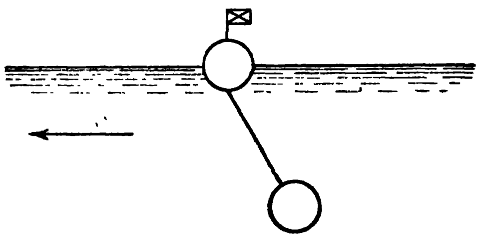


FIG. 97

with water; for this reason, and on account of cheapness, common salt is generally used. Great care must be taken with this method, a uniform stretch of channel should be used, and the solution should be inserted at several places over the cross-section.

At a section of the stream, sufficiently below the inserting section for the solution to have mixed evenly with the stream, samples of the stream are taken at various points, from which the weight of salt per cubic foot of water is measured.

Let Q = discharge of stream in cu. ft. per sec.

q = quantity of solution injected in cu. ft. per sec.

W = weight of salt per cu. ft. of stream water at lower section.

w = weight of salt per cu. ft. of solution injected.

Then, as the weight of salt injected per second must equal the weight per second passing the lower section of stream,

$$q\,w = Q\,W$$

From which $$Q = \frac{q\,w}{W}$$

This method is very unreliable unless the water is well mixed before reaching the lower section at which the samples are taken. The average results from all the samples must be used.

86. **River Bends.** It is known from experience that a river flowing round a bend scours the bank on the outside of the bend, and material is deposited on the inside. This means that the bend is continually increasing, and eventually the river breaks through the narrow neck thus formed and makes an island of the land which previously formed the inside of the bend. After a time the main water course will be through the breach and the bent channel will be partly silted up, until finally it becomes a horse-shoe lake. These horse-shoe lakes are frequently found at the sides of rivers.

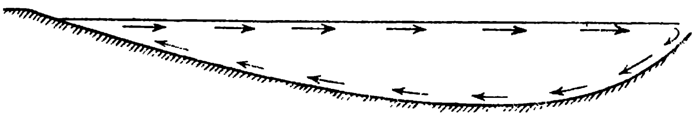


FIG. 98

The scouring of the outside of a river bend is mainly due to the impact of the water as it strikes the bank. Another explanation is given by Prof. James Thomson, who accounts for it by the action of a transverse current which flows along the bottom of the river from the outside of the bend to the inside, as shown in Fig. 98. Owing to the centrifugal force the pressure of the water on the outside of the bend will increase; but as the water near the surface has a greater velocity than that near the bottom, the pressure at the surface will be greater than that at the bottom. This will cause the water to flow downwards and form a cross current which will transport material from the outside of the bend to the inside. This explanation may be the cause of some of the silting, but the main quantity is probably due to impact on the outside of the bend and to still water at the inside.

87. **Water Supply and Rainfall.** The water supply for a district is usually obtained by building a dam across certain water-courses, such as mountain streams. As this stops the flow of the stream, the inhabitants of the land below the dam, who were formerly supplied with water by the stream, must be compensated by a daily supply of water from the dam.

Such water is known as compensation water, and the quantity is fixed by law to be one-third of the total amount collected by the dam. As the greater part of this compensation water must be supplied in the day time, it is usual to have a special reservoir for it, so that the claimants may use it as they wish.

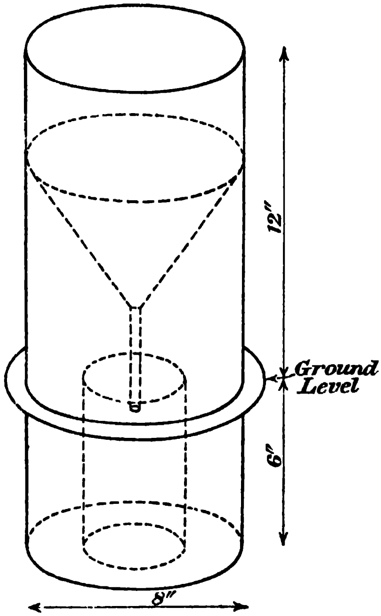


FIG. 99

In supplying the population of a district with water, the following considerations are necessary—

(1) Rainfall.

(2) Amount lost by evaporation, absorption, and percolation.

(3) Maximum period in which available supply falls short of demand.

The rainfall of a district is measured with a rain gauge, such as shown in Fig. 99, and is averaged over several years. The following list gives the average rainfall, of a few districts—

Seathwaite	136 in. per year
Keswick	60 ,,
Buxton	54 ,,
Midlands	
London	25
Manchester	37·4

The water supply is reckoned on the three consecutive driest years, and the following rules are used—

(1) The wettest year has a rainfall of $1\frac{1}{2}$ times the mean rainfall.

(2) The driest year has $\frac{2}{3}$ the mean rainfall.

(3) The driest two consecutive years have $\frac{3}{4}$ of mean rainfall per year.

(4) The driest three consecutive years have $\frac{4}{5}$ of mean rainfall per year.

It is usual to work on the last of these rules.

Some of the rainfall is lost by evaporation, absorption, and percolation, the amount depending on the nature of the ground on which the rain falls. The following table will give an idea of the amounts lost by these causes—

Rainfall.	Percolation.		Evaporation.		
	Soil.	Sand.	Soil.	Sand.	Water.
in. 25·7	in. 7·6	in. 21·4	in. 18·1	in. 4·3	in. 20·6

The average loss from these causes may be taken as from 10 to 20 in. of the annual rainfall.

Reservoirs are built in which the water is stored. The amount stored should be equal to about 150 days' supply. In towns, small reservoirs, known as service reservoirs, are built for district supply.

The total amount of water required to be stored for the whole water supply is given by the following empirical rule—

$$N = \frac{1000}{\sqrt{h}}$$

where N = number of days' supply to be stored

and h = inches of rainfall in three consecutive dry years.

The demand for the water is not regular, and it is found that one-half of the amount used daily will be drawn off in 6 hours. That is, the maximum rate of flow is 100 per cent more than the average rate. This must be taken into account in designing the supply pipes.

The amount of water consumed varies in different districts. The following figures are an average of the amounts supplied—

Domestic supply:

17 gallons per day per head in towns.

12 gallons per day per head in rural districts.

Trade supply:

5 to 20 gallons per day per head.

The following gives the quantities of water supplied per day for several large towns—

Philadelphia	215 gallons per head
Glasgow	52 " " "
Perth	50 " " "
Manchester . . .	27 " " "
Liverpool . .	25 " " "
Leicester. . .	16 " " "

These figures include the consumption for trade, domestic, municipal, and leakage. The large variation is probably due to trade consumption and leakage. For an average, a total consumption of 30 gallons per head per day may be used.

EXAMPLE.

The average rainfall over a catchment area of 1,680 acres, as determined for a period of 35 years, is 36·6 in. per annum. Assuming that there is a possibility of three consecutive dry years, during which the average rainfall is only 80 per cent of the above average, and assuming that the evaporation loss in such dry years is equivalent to 15 in. of rainfall per annum, determine in gallons the minimum annual yield from this catchment area.

If one-third of this yield has to be supplied for compensation water, what population could be supplied from this catchment area, if the daily supply is 48 gallons per head? (London Univ.)

$$\text{Minimum average rainfall} = 36{\cdot}6 \times \frac{80}{100} = 29{\cdot}25 \text{ in.}$$

Deducting loss due to evaporation,

$$\text{collectable rainfall} = 29{\cdot}25 - 15{\cdot}0 = 14{\cdot}25 \text{ in.}$$

Volume of rain collected per annum

$$\begin{aligned} &= \text{area} \times \text{depth} \\ &= 1680 \times 4840 \times 9 \times \frac{14{\cdot}25}{12} \\ &= 86{,}800{,}000 \text{ cu. ft.} \\ &= 542{,}000{,}000 \text{ gallons.} \end{aligned}$$

Deducting one-third for compensation water, actual amount available per annum

$$\begin{aligned} &= 542{,}000{,}000 \times \tfrac{2}{3} \\ &= 361{,}000{,}000 \text{ gallons} \end{aligned}$$

$$\text{Population supplied} = \frac{361{,}000{,}000}{48 \times 365}$$

20,600

88. **Hydraulic Jump.** Fig. 100 shows water issuing from a sluice, at a high velocity, into a channel of uniform width. Owing to the frictional resistance of the sides of the channel, there will be a loss of energy as the water flows along the channel; this will reduce the velocity. At the vertical section B the normal type of channel flow is proceeding; at this section the slope of the channel bed is just sufficient to overcome the frictional resistance of the sides, and the flow proceeds with a normal velocity V, in agreement with the Chezy formula.

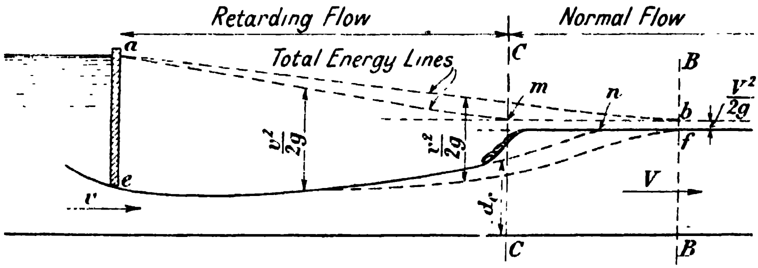


FIG. 100.—FORMATION OF STANDING WAVE

The total energy line for this portion of the channel is the dotted line mb, which is at a height of $\frac{V^2}{2g}$ above the water surface. If the dotted line ab represents the total energy line of the water as it flows from the sluice, the normal channel flow commences at b, where it meets the normal energy line mb. The distance between the total-energy line and the water surface represents the velocity head.

The water flowing from the sluice at e has its high velocity continuously reduced by the friction of the channel; this causes the depth to increase in proportion to the reduction of velocity. Let the line ef represent the water surface. Now f must be on the same vertical section as the point b for the flow to remain stable; in which case the rate of loss of total energy has been in proportion to the rate of increase in depth, in such a way that both reach their normal values on the same vertical section.

Now suppose the loss of total energy is greater than the previous supposition and follows the dotted line am; let this total energy line meet the normal energy line at m on the vertical section C. The increase of depth will now be at a

different rate; let the water surface now lie on the line *en*, meeting the normal water surface at *n*. This new condition will not be stable because the energies on the vertical section *C* do not balance. For a stable condition at *C* the velocity head should be $\frac{V^2}{2g}$, whereas it is actually $\frac{v^2}{2g}$. At this section there will now occur a sudden heaping up of the water surface until it reaches the normal channel level at *m*. This sudden increase of depth will reduce the velocity from v to V, and the energies of the water will now balance. The sudden heaping up of the water at section *C* is known as the *hydraulic jump* or *standing wave*. It is caused by the fact that whilst the energy loss is proportional to v^2, the increase in depth is only proportional to v. The phenomenon is liable to occur at the foot of waterfalls, in the exit channels of sluices, and in the vicinity of under-water obstructions. It will be noticed that the hydraulic jump occurs at the section where the total-energy line from the sluice meets the normal total-energy line of the channel.* Actually, there is a small loss of energy at the jump which causes the jump to advance upstream to the left of section *C*. This causes a sudden drop in the total energy line *am* at the commencement of the jump.

For any given problem the total energy line *am* and the water surface *en* can be plotted to scale if the following assumptions are made:

(1) The velocity of the water in the channel is constant over the cross-section. Actually, the velocity varies considerably as shown in Figs. 90 to 92.

(2) The value of the frictional coefficient f is constant at all velocities. This is not strictly true as f varies with the velocity (Art. 140 and 141).

89. **Specific Energy of a Channel's Cross-section.** Consider a constant quantity of water to be flowing through the cross-section of the rectangular channel shown in Fig. 101. As the quantity is assumed constant, the velocity of the stream will depend on the depth. If the depth is small, the velocity will be large, as,

$$v = \frac{Q}{bd}$$

* For further information on hydraulic jump see "The Standing wave or Hydraulic Jump," Central Board of Irrigation, Publication No. 7. (Simla, India.)

On the other hand, if the depth is large the velocity is correspondingly small.

The specific energy of the stream at this cross-section consists of the static energy due to its depth, plus the kinetic energy. Or, specific energy

$$= E = d + \frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad (1)$$

This is not the total energy of the water, as no reference has been made to any datum height; consequently, the slope of the channel is not included in the term specific energy.

In Fig. 102 the static energy, kinetic energy and specific energy for this cross-section have been plotted for a fixed quantity of flow and for various depths of the stream. The static energy is represented by the straight line ox; the corresponding kinetic energy by the curve yz. By adding the horizontal ordinates of these two curves the specific energy line abc is obtained. It will be noticed that the specific energy at first becomes less as the depth increases. At the point b the specific energy has its minimum value; beyond this point there is an increase of specific energy as the depth increases. The depth at the point b is the depth at minimum energy, and is called the *critical depth*. For each value of specific energy to the right of b, there are two depths for the given quantity of flow considered; both of these depths produce the same specific energy.

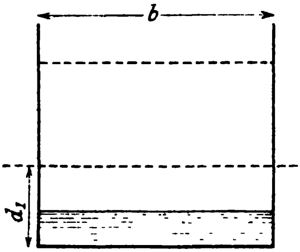


Fig. 101

If a horizontal line is drawn through b, the area above this line is known as the area of tranquil flow, the area below is known as the area of rapid or streaming flow.

Consider the condition represented by the line ABC. If the total energy conditions are favourable for a hydraulic jump (Art. 88) to occur at this section, before the jump occurs the flow is rapid, and the depth of the channel is CB; after the jump occurs the flow is tranquil and the depth is CA. Thus, the depth of the stream has increased from CB to CA whilst the energy remains constant, excepting for losses due to turbulence. It will be noticed that if a hydraulic jump occurs, it must do

so before the depth reaches the critical depth, because there are two depths for equilibrium only before the critical depth is reached.

The depth at minimum specific energy, or critical depth, can be obtained by differentiating Equation (1) and equating to zero; or,

$$\frac{dE}{d(d)} = 0$$

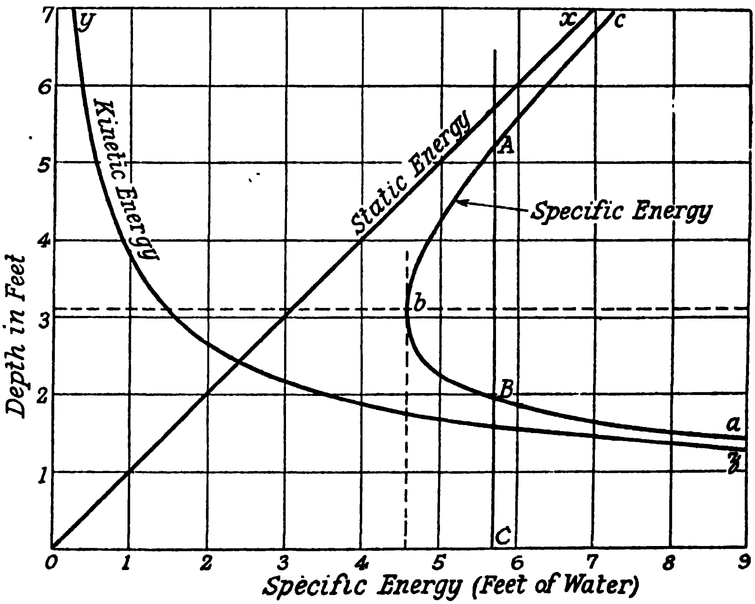


FIG. 102

Consider unit width of channel and let Q be the quantity of flow through unit width per sec. Then,

$$v = \frac{Q}{d}$$

It should be noticed that this is an approximation only as it assumes the velocity to be uniform throughout the section considered.

Substituting for v in Equation (1),

$$E = d + \frac{Q^2 d^{-2}}{2g}$$

then, $$\frac{dE}{d(d)} = 1 - \frac{2Q^2d^{-3}}{2g} = 0$$

from which, $$\left(\frac{Q}{d}\right)^2 = gd$$

or $$v^2 = gd \quad . \quad . \quad . \quad . \quad . \quad (2)$$

hence, depth for minimum specific energy $= \dfrac{v^2}{g}$

From Equation (2) it will be seen that $\dfrac{v}{\sqrt{gd}} = 1$

The term $\dfrac{v}{\sqrt{gd}}$ is called the *Froude number* (Art. 145); hence, minimum specific energy occurs when the Froude number is unity.

EXAMPLE

Water enters a horizontal channel of uniform width, with a velocity of 20 ft. per sec. The depth of the water at entrance is 2 ft. Calculate the critical depth of the water.

Using Equation (2), $d = \dfrac{\;}{32{\cdot}2}$

also, quantity per sec. per ft. width $= 20 \times 2 = v \times d$

hence, $$d = \frac{40}{v} = \frac{v^2}{32{\cdot}2}$$

from which, $$v^3 = 40 \times 32{\cdot}2$$

hence, $$v = 10{\cdot}9 \text{ ft. per sec.}$$

Then, $$d = \frac{40}{10{\cdot}9}$$
$$= 3{\cdot}66 \text{ ft.}$$

90. **Depth for Maximum Flow at Given Specific Energy.** In Art. 89 the critical depth of a channel was defined as the depth producing minimum specific energy for a given quantity of flow Q. The critical depth may also be defined as the depth producing a maximum flow for a given specific energy. The critical depth satisfies both of these conditions; hence, either definition may be used.

Using the notation of Art. 89, consider the channel of Fig.

101, but this time assume the specific energy E remains constant whilst the flow Q is varied. Then,

$$E = d + \frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

from which, $v = \sqrt{2g\,(E - d)}$

Now,
$$\begin{aligned} Q &= AV \\ &= bd\,\sqrt{2g\,(E - d)} \\ &= b\,\sqrt{2g\,(Ed^2 - d^3)} \end{aligned}$$

hence, Q is a maximum when the term $(Ed^2 - d^3)$ is a maximum. Differentiating this term and equating to zero for a maximum,

$$\frac{dQ}{d(d)} = 2Ed - 3d^2 = 0$$

from which, $E = \frac{3}{2}d$

Substituting this value of E in Equation (1),

$$\frac{3}{2}d = d + \frac{v^2}{2g}$$

hence,
$$d = \frac{v^2}{g}$$

or,
$$\frac{v}{\sqrt{gd}} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is the same result as that obtained in Art. 89. Hence, the depth of a channel for maximum flow, for a given specific energy, is the critical depth.

It will be proved in Art. 219 that the velocity given by Equation (2) is also the velocity of surface waves in shallow water; that is, when the Froude number equals unity. It is found from observation that the flow in a channel changes to surface wave formation as soon as the critical depth is reached; the flow of these shallow water surface waves is dealt with in Art. 219.

91. **Depth at Hydraulic Jump.** The depth of the water in the channel after the hydraulic jump has occurred can be calculated by equating the force on a section of the water to its rate of change of momentum. Let Fig. 103 represent the water in the vicinity of the hydraulic jump. Consider the equilibrium of the mass of water between two vertical sections

(1) and (2), one before the jump and the other after. Consider unit width of channel, dimensions being in feet.

Let p_1, v_1 and d_1 apply to water at section (1)
p_2, v_2 and d_2 apply to water at section (2)
q = quantity of water flowing per sec., per unit width $= v_1 d_1 = v_2 d_2$

Now, average $p_1 = \dfrac{wd_1}{2}$

and average $p_2 = \dfrac{wd_2}{2}$

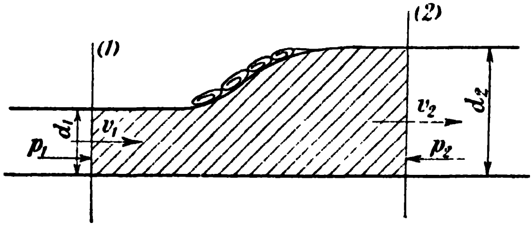


FIG. 103.—DEPTH AT HYDRAULIC JUMP

Horizontal force on section (2) $= p_2 \times$ cross-sectional area

$$= p_2 d_2 = \frac{wd_2^2}{2}$$

Horizontal force on section (1) $= \dfrac{wd_1^2}{2}$

Now, force = mass per sec. $\times$ change of velocity

that is,
$$\frac{wd_2^2}{2} - \frac{wd_1^2}{2} = \frac{wq}{g}(v_1 - v_2) \qquad (1)$$

but, $v_1 = \dfrac{q}{d_1}$; and $v_2 = \dfrac{q}{d_2}$

hence, substituting in Equation (1),

$$d_2^2 - d_1^2 = \frac{2q}{g}\left(\frac{q}{d_1} - \frac{q}{d_2}\right)$$

or, $$(d_2 - d_1)(d_2 + d_1) = \frac{2q^2}{g}\left(\frac{d_2 - d_1}{d_2 d_1}\right)$$

hence, $$d_2 + d_1 = \frac{2q^2}{g d_2 d_1}$$

that is, $$d_2^2 + d_2 d_1 = \frac{2q^2}{gd_1}$$

Solving this quadratic for d_2,

$$d_2 = -\frac{d_1}{2} + \sqrt{\frac{2q^2}{gd_1} + \frac{d_1^2}{4}} \quad . \quad . \quad . \quad . \quad (2)$$

Substituting for $q = v_1 d_1$,

$$d_2 = -\frac{d_1}{2} + \sqrt{\frac{2v_1^2 d_1}{g} + \frac{d_1^2}{4}} \quad . \quad . \quad (3)$$

From this equation the depth of the channel on the downstream side of the jump can be calculated.

Loss of energy due to jump

$$= \left(\frac{v_1^2}{2g} + d_1\right) - \left(\frac{v_2^2}{2g} + d_2\right) \quad . \quad . \quad (4)$$

EXAMPLE.

Water is flowing along a channel of uniform width, the quantity of flow being 40 cu. ft. per sec. per ft. width of channel, causing a standing wave to occur. If the depth of the water on the upstream side of the standing wave is 3 ft., find the height of the wave.

Calculating on 1 ft. width of channel,

$$q = 40 \text{ cu. ft. per sec.}$$
$$d_1 = 3 \text{ ft.}$$

Using Equation (2),

$$d_2 = -\frac{3}{2} + \sqrt{\frac{2 \times 40^2}{32{\cdot}2 \times 3} + \frac{3^2}{4}}$$
$$= -1{\cdot}5 + \sqrt{33{\cdot}2 + 2{\cdot}25}$$
$$= -1{\cdot}5 + 5{\cdot}94 = 4{\cdot}44 \text{ ft.}$$

Height of wave $= d_2 - d_1$

$$= 4{\cdot}44 - 3{\cdot}0$$
$$= 1{\cdot}44 \text{ ft.}$$

92. **Non-uniform Flow in Channels.** Fig. 104 represents a longitudinal section of a channel in which the velocity of the water is not constant. Owing to the frictional resistance of the sides and base of the channel the water is slowing down; this causes an increase in the depth, as the quantity of flow is constant.

Consider two vertical sections at a distance dl apart.

Let v = velocity of flow at left-hand section

d = depth at left-hand section

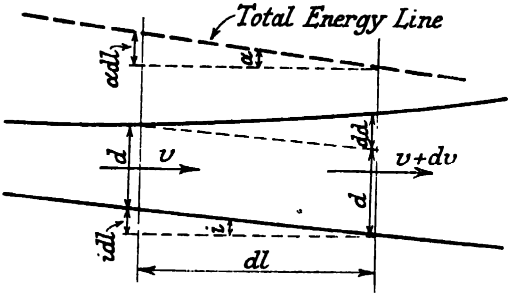


Fig. 104

(In the present article, the symbol d in roman type represents "depth," whereas the italic symbol d represents the differential coefficient.)

Let the velocity change by dv and the depth by d(d) over the length dl. Actually dv is negative for the case shown in Fig. 104.

Let a = slope of total energy line

i = slope of base of channel.

Applying Bernoulli's equation to the two vertical sections, and assuming the base of the channel at the right-hand section to be the datum line,

$$idl + \mathrm{d} + \frac{v^2}{2g} = \mathrm{d} + d(\mathrm{d}) + \frac{(v + dv)^2}{2g} + adl$$

From which, $$idl = d(\mathrm{d}) + \frac{vdv}{g} + adl$$

if small quantities of the second order are ignored.

Hence, $$\frac{d(\mathrm{d})}{dl} = i - \frac{vdv}{gdl} - a \quad . \quad . \quad . \qquad (1)$$

But, quantity of flow per ft. width of channel

$$= Q = v \times \mathrm{d} = \text{a constant}$$

hence, $$\frac{dQ}{dl} = 0$$

or $$\frac{d(v\mathrm{d})}{dl} = 0$$

differentiating, and treating both v and d as variables,

$$v\frac{d(\mathrm{d})}{dl} + \mathrm{d}\frac{dv}{dl} = 0$$

from which
$$\frac{dv}{dl} = -\frac{v}{\mathrm{d}} \times \frac{d(\mathrm{d})}{dl} \quad . \quad . \quad . \quad (2)$$

Substituting Equation (2) in Equation (1),

$$\frac{d(\mathrm{d})}{dl} = i + \frac{v^2}{\mathrm{d}g}\frac{d(\mathrm{d})}{dl} - a$$

that is,
$$\frac{d(\mathrm{d})}{dl}\left(1 - \frac{v^2}{g\mathrm{d}}\right) = i - a$$

or
$$\frac{d(\mathrm{d})}{dl} = \frac{i - a}{\left(1 - \frac{v^2}{g\mathrm{d}}\right)} \quad . \quad . \quad . \quad . \quad (3)$$

This equation represents non-uniform flow in a channel; it will be noticed that $\frac{d(\mathrm{d})}{dl}$ is the slope of the water surface.

From Equation (3) it will be seen that if $\alpha = i$, then

$$\frac{d(\mathrm{d})}{dl} = 0$$

and the flow is then at uniform depth.

It was shown in Art. 89 that when

$$\frac{v^2}{g\mathrm{d}} = 1$$

the critical depth for the channel is reached. If this condition is inserted in Equation (3), it will be noticed that $\frac{d(\mathrm{d})}{dl}$ becomes equal to ∞; the water surface would then be vertical. This condition is never reached in practice as Equation (3) would not hold for this extreme limit, as small changes only have been assumed. Either steady flow would have been reached, or a hydraulic jump would have occurred, before this condition of critical depth could be attained (Arts. 88 and 89).

It will also be noticed that $a = \frac{h_f}{dl}$ where h_f is the total head lost in friction and eddies, but $h_f \propto f$ where f is a frictional coefficient.

Actually f will vary with the velocity and with the surface of the channel sides; it will not be constant, as has been assumed in the above solution; therefore a is not constant throughout the length of channel. From this it appears that Equation (3) has no practical value.

93. **Venturi Flume in Channel.** The quantity of water flowing along a channel can be measured by restricting the width as shown in Fig. 105. This is known as a Venturi flume

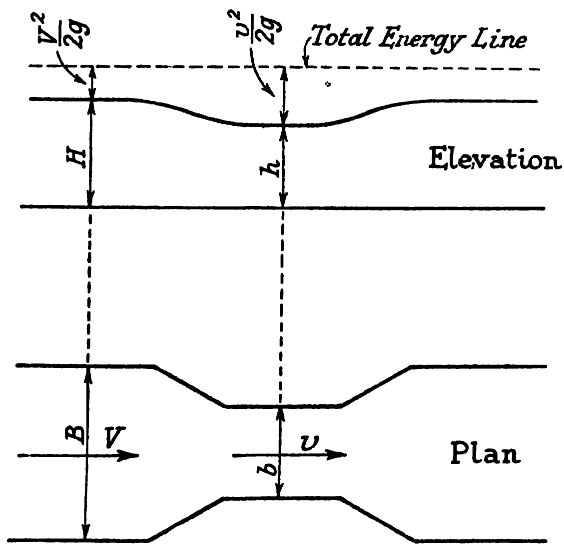


FIG. 105

and corresponds to the throat of a Venturi meter, used for the measurement of pipe flow.

Let B, V, and H represent the normal breadth, velocity, and depth of the channel at entrance of the flume. Let b, v, and h represent the breadth, velocity, and depth at the narrowest part of the flume.

As the quantity of flow past all sections is the same,

$$Q = VBH = vbh \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It will be noticed that v is larger than V on account of the restriction of the cross-section; hence, the water level may fall in the narrow portion of the flume, as the total energy remains practically constant. This is demonstrated in the sectional elevation of Fig. 105.

Let H_1 = total energy at any section.

Applying Bernoulli's equation to the inlet and throat, and neglecting all losses,

$$H_1 = H + \frac{V^2}{2g} = h + \frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

as the depth of water represents its static head plus datum head.

From Equation (2),

$$v = \sqrt{2g}\,(H_1 - h)^{\frac{1}{2}}$$

also,

$$Q = bhv$$

hence,

$$Q = b\sqrt{2g}\,h\,(H_1 - h)^{\frac{1}{2}} \quad . \quad . \quad . \quad (3)$$

$$= k\,(H_1h^2 - h^3)^{\frac{1}{2}}$$

where k is a constant for a given flume, which includes all losses. Differentiating this expression in terms of h, for a maximum flow,

$$\frac{dQ}{dh} = \tfrac{1}{2}(H_1h^2 - h^3)^{-\frac{1}{2}}(2H_1h - 3h^2) = 0$$

or

$$\frac{2H_1h - 3h^2}{(H_1h^2 - h^3)^{\frac{1}{2}}} = 0$$

hence,

$$h(2H_1 - 3h) = 0$$

then,

$$2H_1 - 3h = 0$$

or

$$h = \tfrac{2}{3}H_1 \quad . \quad . \quad . \quad (4)$$

This proves that the flow through the flume is a maximum when the depth at the throat is two-thirds of the total energy of flow.

Substituting this value of h in Equation (3),

$$\text{Maximum } Q = b\sqrt{2g}\,.\,\tfrac{2}{3}H_1(H_1 - \tfrac{2}{3}H_1)^{\frac{1}{2}}$$

$$= 3{\cdot}09\,bH_1^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For any flow through the channel, the quantity of flow can be calculated by measuring the depths at the entrance and throat of the flume. The equation for the quantity is the same as the equation for a pipe Venturi meter deduced in Art. 27. Or

$$Q = \frac{Aa\sqrt{2g}}{\sqrt{A^2 - a^2}}\sqrt{(H - h)} \text{ cu. ft. per sec.}$$

where A = area of channel at entrance in sq. ft.

and $\qquad a =$ area of flume at throat in sq. ft.

Then, $$Q = \frac{BH \times bh\sqrt{2g}}{\sqrt{B^2H^2 - b^2h^2}} \sqrt{(H - h)} \quad . \quad . \quad . \qquad (6)$$

This should be multiplied by a coefficient, found experimentally, in order to allow for any losses in the flume.*

It is possible for a hydraulic jump to occur in the downstream portion of the flume if the conditions for the formation of the jump are favourable (Art. 88).

EXAMPLE.

In a Venturi flume the floor is horizontal and the throat is rectangular.

The width of the throat is 1 ft., and the width at inlet is 1·5 ft. On the downstream side a "jump" occurs so that the conditions for maximum flow exist at the throat.

If the rate of flow is 1·5 cu. ft. per sec., find the depth of water at the throat and at inlet.

Assume that the coefficient of discharge is unity. (London Univ.)

Using Equation (5) for maximum flow,

$$Q = 3{\cdot}09\, b\, H_1^{\frac{3}{2}}$$

that is, $$1{\cdot}5 = 3{\cdot}09 \times 1 \times H_1^{\frac{3}{2}}$$

from which $$H_1 = \left(\frac{1{\cdot}5}{3{\cdot}09}\right)^{\frac{2}{3}}$$
$$= {\cdot}618 \text{ ft.}$$

From Equation (4),

$$h = \tfrac{2}{3}H_1$$
$$= \tfrac{2}{3} \times {\cdot}618$$
$$= {\cdot}413 \text{ ft.}$$

Using Equation (2),

$$\frac{v^2}{2g} = \frac{H_1}{3}$$

then, $$v = \sqrt{2g \times \frac{{\cdot}618}{3}}$$
$$= 3{\cdot}64 \text{ ft. per sec.}$$

Also, $$H + \frac{V^2}{2g} = H_1$$

and, from Equation (1), $H = \dfrac{Q}{BV}$

* For practical information on hydraulic flumes, see Central Board of Irrigation Publication No. 6 (Simla, India), by A. M. R. Montagu, entitled "Fluming."

hence, $$\frac{Q}{BV} + \frac{V^2}{2g} = H_1$$

that is, $$\frac{1{\cdot}5}{1{\cdot}5V} + \frac{V^2}{2g} = {\cdot}618$$

or $$64{\cdot}4 + V^3 = 39{\cdot}8V$$

Solving this equation by trial or by plotting,

$$V = 1{\cdot}76 \text{ ft. per sec.}$$

Then, $$H = H_1 - \frac{V^2}{2g}$$

$$= {\cdot}618 - \frac{(1{\cdot}76)^2}{2g}$$

$$= {\cdot}57 \text{ ft.}$$

EXAMPLES 7.

(1) Using Bazin's formula $C = \dfrac{157{\cdot}5}{1 + \dfrac{k}{\sqrt{m}}}$, find the value of C for a broad shallow river of 2 ft. deep. (Take wetted perimeter as breadth of river.)

Ans.—59·2.

(2) A channel 10 ft. wide at the bottom and with sides sloping 1 to 1, has a slope of 3 ft. per mile. What would be the discharge if the water is 4 ft. deep in the channel and $C = 95$ in the equation $v = C\sqrt{mi}$. (London Univ.)

Ans.—205·5 cu. ft. per sec.

(3) Find the maximum discharge for least excavation of a rectangular channel 10 ft. wide, when $C = 105$ and slope $= 1$ in 1,000.

Ans.—262·5 cu. ft. per sec.

(4) Explain what is meant by the "best cross-section" for a channel, and how it is determined.

A channel with side slopes at 45° is to have a cross-section of 120 sq. ft. Determine the dimensions for the best section. (London Univ.)

Ans.—Depth = 8·1 ft. Base = 6·7 ft.

(5) Deduce the formula for the depth of water in a circular conduit for maximum discharge.

Find the depth for maximum discharge in a circular brick sewer 4 ft. diameter. (London Univ.)

Ans.—3·8 ft.

(6) A district has a drainage area of 2,500 acres, with a population of 20 persons per acre. The daily water supply to the district is equal to 40 gallons per head. During dry weather it is found that 7 per cent of the daily dry weather flow passes along the sewer between the hours of 12 noon and 1 p.m.

Assuming a maximum rainfall of 1 in. in 24 hours over the whole area, determine the diameter of a circular sewer, having a slope of 1 in 3,000, which will take the maximum dry weather flow and the rainfall, without the sewer becoming more than half full. [Assume $C = 130$.] (London Univ.)

Ans.—8·93 ft.

(7) The bed of a stream has a slope of 1 in 1,000, and the depth of the water is 3 ft. A dam is to be built across the stream and provided with a sluice gate. Find the height of the dam so that the rise in level of the water, when the sluice gate is closed, may be limited to 7·5 ft. Take $C = 65$ in the formula $v = C\sqrt{mi}$, the coefficient of discharge of the dam, as a weir, ·56, and assume, in calculating m, that the breadth of the stream is large in comparison with the depth. (London Univ.)

Ans.—8·17 ft. above bed of stream.

(8) A rectangular channel is 5 ft. deep and 10 ft. wide. If the value of C in Chezy's formula is 100, determine the discharge if the gradient is 1 in 1,000. (A.M.I.Mech.E.) *Ans.*—250 cu. ft. per sec.

(9) Show how the basic formula for steady flow in channels of constant slope and section is derived.

The depth of water in a circular brick-lined conduit, 6 ft. in diameter, is to be 5 ft., and its capacity 50 million gallons a day. The water surface subtends an angle of 96° 20′ at the axis of the conduit. What must the gradient be? $C = 123$. (A.M.I. Civil E.) *Ans.*—$\frac{1}{2030}$.

(10) You are required to ascertain the discharge of a river by means of current meter observations. Describe with diagrams the procedure at the site, and explain carefully how you would arrive at the discharge from your meter readings. The meter may be assumed calibrated and ready for use. (A.M.I. Civil E.)

(11) A concrete-lined channel has a bottom width of 10 ft., side slopes of 1 horizontal to 3 vertical, and a gradient of 1 in 800. When flowing 3 ft. deep, it is found to have a capacity of 220 cusecs. What is the value of C? (A.M.I. Civil E.) *Ans.*—133.

(12) A stream is 40 ft. wide at water level. At horizontal intervals of 5 ft. the following results are obtained by current meter—

Distance from bank. Ft.	0	2·5	7·5	12·5	17·5	22·5	27·5	32·5	37·5	40
Depth of water. Ft.	0	1·0	2·2	3·2	4·4	5·0	3·4	2·2	1·2	0
Mean velocity on vertical Ft. per sec.	—	1·5	1·9	2·2	2·9	3·3	2·7	1·8	1·4	—

What is the discharge in cusecs? (A.M.I. Mech. E.) *Ans.*—283·8 cusecs.

(13) Deduce the Chezy formula for uniform flow in channels. An irrigation channel has a gradient of 1 in 2,000, a bottom width of 16 ft., and side slopes of 1 vertical to 2 horizontal. If the depth of water is 4 ft. and the value of C is 90, what is the mean velocity and the capacity in cusecs? (A.M.I. Mech. E.)

Ans.—$v = 3·38$ ft. per sec.; $Q = 324$ cusecs.

(14) A brick-lined sewer has a semicircular bottom and vertical side walls 2 ft. apart. If the slope is 1 in 1,000 determine the discharge when the maximum depth of water is 3 ft. Take C in Chezy's formula as 90. (A.M.Inst. C.E.) *Ans.*—14·05 cusecs.

(15) Find the downstream height of a hydraulic jump occurring on a level bed when the upstream depth is 3 ft. and velocity 35 ft. per sec.

What conditions are necessary for a jump to occur? (London Univ.)

Ans.—14·7 ft.

(16) Explain the meaning of the term *critical depth* used in connection with non-uniform flow along open channels. Show that in the case of non-uniform flow along a rectangular channel having a constant width of B ft. the critical depth is given by

$$D_c = \sqrt[3]{\frac{Q^2}{gB^2}}$$

where Q is the quantity of water flowing along the channel in cusecs. (London Univ.)

(17) A sluice spans a channel of rectangular section 60 ft. wide, has an opening $2\frac{1}{2}$ ft. deep, and discharges 1720 ft.^{3}/sec. of water. If a standing wave is formed on the downstream side of the sluice, determine the probable height of the crest above the upper edge of the sluice. (London Univ.)

Ans.—0·92 ft.

(18) A Venturi flume is now frequently employed for the purpose of measuring the quantity of water flowing along an open channel. Explain how the venturi flume operates and describe the method of calculating the quantity from the experimental observations. (Lond. Univ.)

(19) Show that the slope of the water surface at any point along a horizontal rectangular open channel is given by

$$\frac{dD}{dL} = \frac{fDQ^2}{2m(Q^2 - gB^2D^3)}$$

where Q is the quantity in cusecs; D is the depth, B is the width and m is the hydraulic mean depth; f is the coefficient of friction. (London Univ.)

(20) Derive a formula for the change in level produced at a hydraulic jump. Show that a jump is impossible if the depth exceeds the critical depth. Water flows in a horizontal conduit of rectangular section with a velocity of 10 ft. per sec. where the depth is 2 ft., at which section a jump occurs. Calculate the depth after the jump and the energy lost per lb. of water. (London Univ.)

Ans.—2·67 ft; ·014 ft. lb.

(21) Show that the discharge of a Venturi flume with a horizontal bed can be expressed as $Q = kbH^{\frac{3}{2}}$ where b is the breadth at the throat and H is the "still-water" head measured upstream. Assume that a standing wave will occur after the throat, and neglect friction. Calculate the discharge where the throat is 6 in. wide, the contraction of width is $1\frac{1}{2}$: 1, and the depth in the 9 in. section is 8 in., using a method of successive approximation. (London Univ.)

Ans.—2·03 cu. ft. per sec.

(22) A Venturi flume is to be installed in a channel conveying water with the object of raising the level of water upstream. The channel is rectangular in section and is 40 ft. wide, has a gradient of 1 in 6400 and a depth of water 5 ft. The width of the throat section of the flume is 20 ft. If the bed of the flume at the throat is a streamlined hump, find the necessary height of the hump in order that the depth of water on the upstream side shall be 6 ft. Take $v = 140\sqrt{mi}$ for the channel, ignore hydraulic losses in the flume, and assume that a standing wave is formed on the downstream side of the hump. (London Univ.)

Ans.—2 ft.

CHAPTER VIII

RECIPROCATING PUMPS

94. **Types of Reciprocating Pumps.** There are two main types of pumps, centrifugal and reciprocating; the latter type only will be dealt with in this chapter. A reciprocating pump is driven by power from an external source and consists of a cylinder in which a piston or plunger is moved backwards and forwards. This movement of the plunger creates alternately a vacuum pressure and a positive pressure in the cylinder by means of which the water is raised. If a plunger is used, or if the water acts on one side of the piston only, the pump is single acting. In this case it sucks the water into the cylinder on the outward stroke and forces it out during the inward stroke. If the water acts on both sides of the piston it will suck and deliver during one stroke; such a pump is said to be double acting.

Pumps which raise the water by suction only are known as suction pumps. Such pumps are only suitable for low lifts, as the maximum height through which water could be lifted by this type of pump is theoretically equal to the barometer reading and actually to about 25 ft.

Pumps which lift water by means of pressure are known as force pumps.

95. **Force Pump.** A diagrammatic view of a force pump is shown in Fig. 106. The rotation of the crank causes the plunger P to move backwards and forwards in the cylinder C. During the suction stroke the plunger moves to the right, which causes a vacuum in the cylinder. The atmospheric pressure on the water surface forces the water up the suction pipe S; this forces open the suction valve a, and the water enters the cylinder. On the return stroke of the plunger the water pressure closes the suction valve and opens the delivery valve b; the water is then forced up the delivery pipe D and so raised to the required height.

The theoretical volume of water raised per revolution is equal to the stroke volume of the cylinder if the pump is single acting, and to twice this volume if double acting.

Actually, the amount lifted is less than this volume, owing to losses.

96. **Work done by Pump.** Referring to Fig. 106, let r be the radius of crank and L be the length of stroke, in feet.

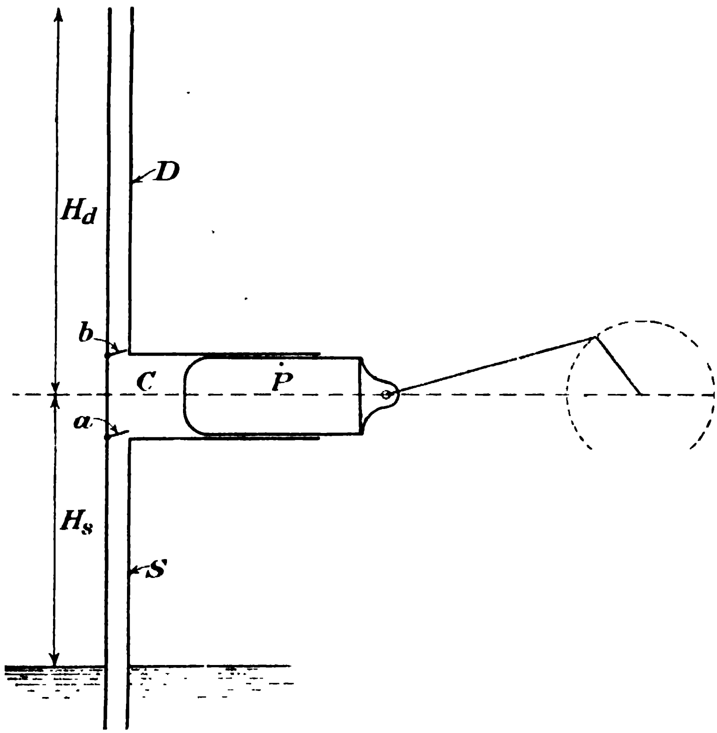


Fig. 106

Then, $$L = 2r$$

Let A be the cross-sectional area of piston in square feet. Then,

theoretical volume of water pumped per stroke $= AL$

and,

theoretical weight of water per stroke $= wAL$

Let $H_s =$ height of centre of cylinder above water surface

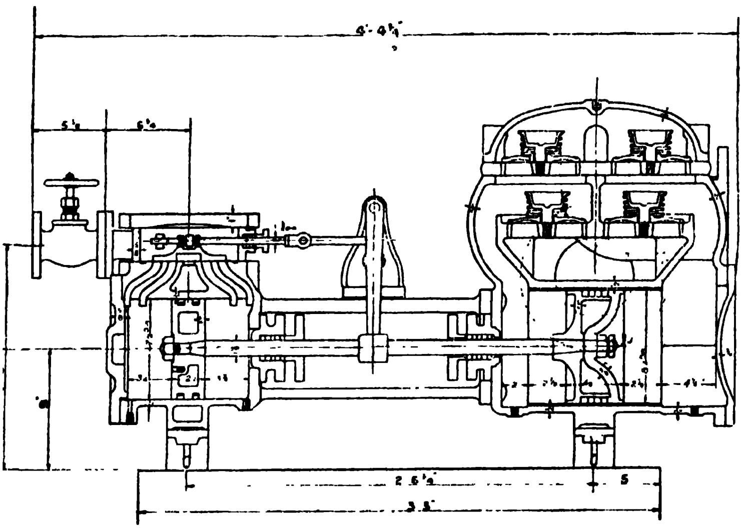
and $H_d =$ height to which water is raised above centre of cylinder.

Then, total height lifted $= H_s + H_d$

Let v_d = velocity of water in delivery pipe

Velocity head of water leaving delivery pipe $= \frac{v_d^2}{2g}$

As v_d is usually small and varies during the stroke, it may be neglected unless the total lift is very small.



(*Worthington Simpson, Ltd.*)

FIG. 107.—SECTION OF HORIZONTAL DUPLEX STEAM PUMP

Let W be the weight of water per second actually lifted.

Work done $= W(H_s + H_d)$ ft. lb. per sec.

Theoretical horse-power required $= \frac{W(H_s + H_d)}{550}$

The actual horse-power required would be greater than this on account of frictional resistance of water and mechanical parts, and of leakage.

The ratio of the actual volume of water discharged to the volume swept through by the plunger is called the coefficient of discharge.

$$\text{Or, coefficient of discharge} = \frac{W}{62{\cdot}4\, A L n}$$

where n = number of suction strokes per second.

The difference between the volume swept through by the plunger and the actual discharge is called the " slip."

In the case of pumps with a long suction pipe and a low delivery head, the pressure due to the inertia of the column of water in the suction pipe will be large compared with the

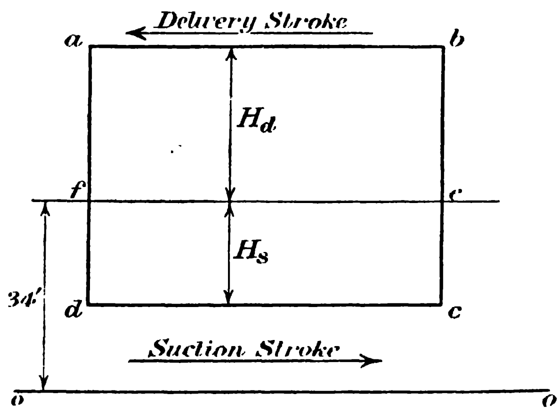


FIG. 108

pressure on the outside of the delivery valve, especially if the speed is great. This may cause the delivery valve to open before the end of the suction stroke, and a greater volume of water will be delivered than that swept through by the plunger. This makes the theoretical discharge less than the actual; the slip will then be negative and the coefficient of discharge will be greater than unity.

A diagram showing the work done by the pump during a complete cycle is shown in Fig. 108. The diagram shows the pressure on the plunger, or on one side of the piston if double acting, plotted as the vertical ordinate, whilst the length of the stroke is represented by the horizontal ordinate. The horizontal line *fe* represents atmospheric pressure. The line *dc* is the pressure in the cylinder during the suction stroke, it being below the atmospheric line by the amount H_s. The line *a b* represents the pressure in the cylinder during the

delivery stroke, and is above the atmospheric line by the amount H_d. The area $d\,c\,e\,f$ is the work done by the plunger on the suction stroke, and $a\,b\,e\,f$ is that done on the delivery stroke. Then, total work done per revolution is given by the area $a\,b\,c\,d$. If the pump is double acting, the work done is twice this amount.

Such a diagram may be obtained automatically by means of an indicator placed on the cylinder, and is consequently called an indicator diagram.

An actual diagram taken by an indicator would be similar to Fig. 108 if the pump were running at a low speed.

EXAMPLE.

A single acting reciprocating pump has a piston area of 1·5 sq. ft. and a stroke of 12 in. The cross-sectional area of the delivery pipe is ·3 sq. ft. and the water is lifted through a total height of 40 ft. If the speed of the pump is 60 revs. per min., and the actual quantity of water lifted 550 gallons per min., find the slip, the coefficient of discharge, and the theoretical horse-power required to drive the pump.

Volume swept through by piston $= 1{\cdot}5 \times 1{\cdot}0$
$= 1{\cdot}5$ cu. ft.

Theoretical volume pumped per sec. $= 1{\cdot}5 \times \dfrac{60}{60}$
$= 1{\cdot}5$ cu. ft.

Actual volume pumped per sec. $= \dfrac{550}{60 \times 6{\cdot}24}$
$= 1{\cdot}47$ cu. ft.

Slip $= 1{\cdot}5 - 1{\cdot}47$
$= {\cdot}03$ cu. ft. per sec.
$= \dfrac{{\cdot}03}{1{\cdot}5} \times 100$
$= 2$ per cent.

Coefficient of discharge $= \dfrac{1{\cdot}47}{1{\cdot}5} = 98$ per cent.

Total pressure head $= H_s + H_d$
$= 40$ ft. of water.

Theoretical horse-power $= \dfrac{550 \times 10}{60} \times \dfrac{40}{550}$
$= 6{\cdot}67.$

97. Variation of Pressure due to Acceleration of Piston. Owing to the reciprocating motion of the plunger or piston, it will have an acceleration at the beginning and a retardation at the end of each stroke. This will transmit a corresponding acceleration and retardation to the water in the suction and delivery pipes, the inertia of which will cause a variation of the pressure in the cylinder.

In order to simplify the problem it is usual to assume that the piston moves with simple harmonic motion. This would be the case if the connecting rod were very long compared with the length of the crank.

Consider the diagrammatic view of the crank and connecting rod of Fig. 109. Let the crank be rotating with an angular

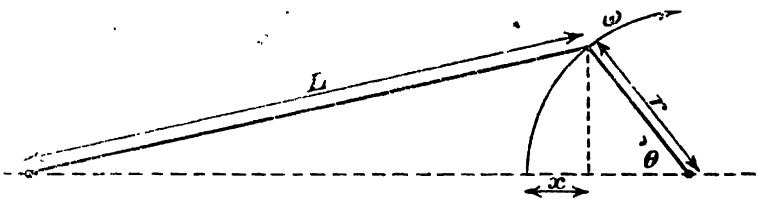


FIG. 109

velocity ω. Suppose it has turned through an angle θ in the time t sec., and assume simple harmonic motion.

Then, $\theta = \omega t$

Displacement of piston from end of stroke $= x = r - r\cos\omega t$

Velocity of piston $= v = \dfrac{dx}{dt} = \omega r \sin\omega t$ . . (1)

Acceleration of piston $= f = \dfrac{dv}{dt} = \omega^2 r \cos\omega t$ . . (2)

Let A be the area of piston and a be the area of pipe. Then, as volume of water flowing from pipe per second equals volume of water flowing into cylinder per second,

velocity of water in pipe $=$ velocity of piston $\times \dfrac{A}{a}$

$$= \frac{A}{a}\,\omega r \sin\omega t$$

$$= \frac{A}{a}\,\omega r \sin\theta \qquad (3)$$

acceleration of water in pipe $= f \times \dfrac{A}{a}$

$$= \frac{A}{a}\, \omega^2 r \cos \omega t$$

$$= \frac{A}{a}\, \omega^2 r \cos \theta \quad . \quad . \quad . \quad (4)$$

Let l = length of pipe through which water is flowing.

Then, weight of water in pipe $= w\,a\,l$.

Let p_a = intensity of pressure due to acceleration of water in pipe.

From Newton's laws of motion,

accelerating force = mass × acceleration

That is, $$p_a\, a = \frac{wal}{g} \times f\frac{A}{a}$$

Or, $$p_a = \frac{wl}{g} \times f\frac{A}{a}$$

Let H_a acceleration pressure in feet of water

$$\frac{p_a}{w}$$

Then, $$H_a = \frac{p_a}{w} = \frac{l}{g} \times f\frac{A}{a}$$

Substituting for f from Eq. 2.

$$H_a = \frac{l}{g} \times \frac{A}{a}\, \omega^2 r \cos \theta \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The pressure head due to acceleration acting on the piston will, therefore, vary with the angle θ.

At the beginning of the stroke when $\theta = 0$, $\cos \theta = 1$

then, $$H_a = \frac{l}{g}\frac{A}{a}\, \omega^2 r$$

At the middle of the stroke when $\theta = 90$, $\cos \theta = 0$

then, $H_a = 0$

At the end of the stroke when $\theta = 180$, $\cos \theta = -1$

then, $$H_a = -\frac{l}{g}\frac{A}{a}\, \omega^2 r$$

If simple harmonic motion is not assumed, the acceleration of piston at dead centres $= \omega^2 r\left(1 \mp \frac{r}{L}\right)$ where L is the length of the connecting rod.* Then, at beginning of stroke,

$$H_a = \frac{l}{g}\frac{A}{a}\omega^2 r\left(1 + \frac{r}{L}\right)$$

At end of stroke,

$$H_a = \frac{l}{g}\frac{A}{a}\omega^2 r\left(1 - \frac{r}{L}\right)$$

98. **The Effect of Acceleration in Suction Pipe.** Consider the suction stroke of the pump of Fig. 106. As the piston moves along the cylinder it must produce a vacuum sufficient to lift the water up the height H_s, and also to accelerate the water. The vacuum pressure in the cylinder must, therefore, equal $H_s + H_a$. If this vacuum pressure reaches 26 ft. of water, that is 8 ft. absolute, the water commences to vaporize and cavities of dissolved gases and vapour are formed. This will cause the water in the pipe to separate and flow in sections; the flow is then no longer continuous and vibrations and "knocking" will occur. This phenomenon is known as separation or cavitation and must be prevented.

The suction stroke of the indicator diagram of Fig. 108 must now be modified to take into account the acceleration head.

Let l_s = length of suction pipe
and a_s = cross-sectional area of suction pipe.

Then, $H_a = \frac{l_s}{g} \times \frac{A}{a_s}\omega^2 r \cos\theta$

At the beginning of the suction stroke this must be added to the suction head as the piston is accelerating the water. The equation gives a straight sloping line, the accelerating head being zero at the centre of the stroke. At the end of the stroke the water causes a positive pressure on the piston in retarding, which reduces the vacuum pressure in the cylinder.

The new indicator diagram for the suction stroke is shown in Fig. 110. The acceleration head H_a is added to the vacuum head at the beginning of the stroke and subtracted at the end. The work done is now represented by the area $f\,m\,n\,e$; but as this equals the area $f\,d\,c\,e$, the net work done remains as

* See textbooks on Mechanism.

before. Thus, the inertia of the water does not affect the net work done, but only causes a variation of the pressure in the cylinder. The piston does work on the water in accelerating it during the first half of the stroke, but receives it back in retarding it during the latter half.

If simple harmonic motion had not been assumed, the straight sloping line $m\,n$ would have been slightly curved.

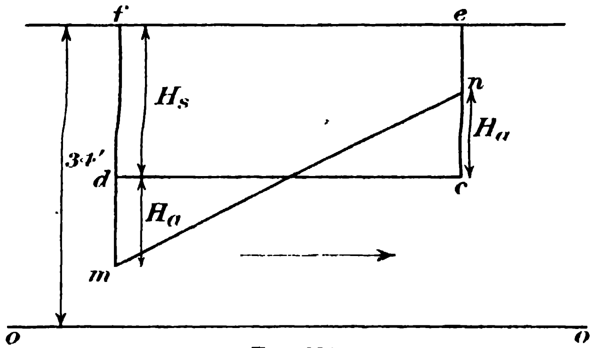


FIG. 110

In designing pumps, the point m (Fig. 110) must not fall below the separation pressure of the water. Or,

$H_s + H_a$ must not be greater than 26 ft. of water.

This may be arranged by varying H_s, l_s, the ratio $\frac{A}{a_s}$, or the speed of the pump.

EXAMPLE.

A single acting pump has a plunger diameter of 5 in. and a stroke of 1 ft. The length of the suction pipe is 30 ft. and the diameter 3 in. Find the acceleration head at the beginning of stroke when the pump is running at 30 revs. per min. If the height of pump's centre is 10 ft. above the water level in the sump, find the pressure head in the cylinder at beginning of stroke.

At beginning of stroke, $\cos\theta = 1$,

$$\text{Then,}\quad H_a = \frac{l_s}{g} \times \frac{A}{a}\,\omega^2 r$$

$$= \frac{30}{32 \cdot 2} \times \left(\frac{5}{3}\right)^2 \left(\frac{2\pi 30}{60}\right)^2 \cdot 5$$

$$= 12 \cdot 75 \text{ ft. of water.}$$

$$\text{Pressure head in cylinder} = 34 - H_s - H_a$$

$$= 34 - 10 - 12 \cdot 75$$

$$= 11 \cdot 25 \text{ ft. of water.}$$

99. The Effect of Acceleration in the Delivery Pipe. The column of water in the delivery pipe will be accelerated at the beginning of the delivery stroke and retarded at the end, in the same way as that in the suction pipe. But, as the delivery pipe may be much longer than the suction pipe, the lift of the latter being limited to 26 ft., the accelerating head in this case may be very large.

Let l_d = length of delivery pipe

and a_d = cross-sectional area of delivery pipe.

Then. $H_t = \frac{l_d}{g} \times \frac{A}{a_d} \omega^2 r \cos \theta$

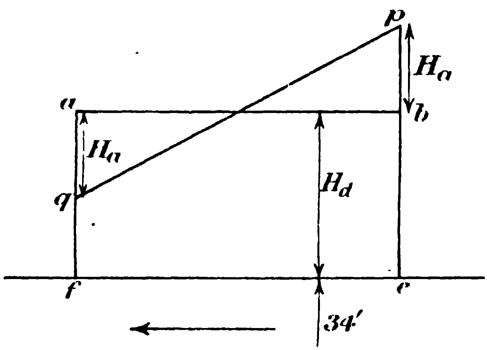


Fig. 111

In Fig. 111 the indicator diagram of the delivery stroke of Fig. 108 is shown with the acceleration head added. The work done is the area $efqp$, and is not affected by the acceleration of the water. The minimum pressure head in the cylinder is represented by the point q and equals

$$H_d - H_a$$

above atmosphere. In absolute units this becomes—

$$34 + H_d - H_a$$

This amount must not be less than 8 ft. of water, otherwise separation will take place at the end of the stroke. The limiting condition is, therefore, when

$$34 + H - H_a = 8$$

Or, when

$$H_a = 26 + H_d$$

If the delivery pipe of the pump is vertical, both sides of this equation will increase with the length of the pipe; in which case it is highly improbable that H_a would be greater than H_d.

Consider the delivery pipe of a pump to be bent either to the form shown in Fig. 112 or to that of Fig. 113. Let the length of pipe and height lifted be the same in both cases. The conditions at points a in both figures will be the same, there being no difference in the values of H_a and H_d in either

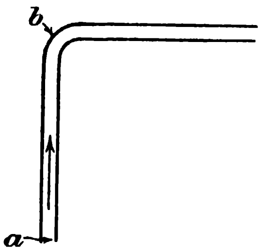


FIG. 112

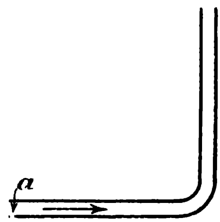


FIG. 113

case. If separation takes place it would do so at the point b of Fig. 112; for at this point H_d is zero and there is still a considerable length of pipe beyond b which affects H_a.

EXAMPLE.

A single acting pump has a piston diameter of 6 in. and a crank radius of 1 ft. The delivery pipe is 3 in. diameter and 100 ft. long. The water is lifted 100 ft. above the centre of the pump. Find the maximum speed at which the pump may be run so that no separation takes place during the delivery stroke. Neglect the velocity head in the delivery pipe and assume separation occurs at an absolute pressure of 8 ft. of water.

Referring to Fig. 111, separation takes place when

$$H_d + 34 - H_a = 8$$

or,

$$H_a = H_d + 26$$
$$= 100 + 26 = 126 \text{ ft. of water.}$$

But, at end of stroke, $H_a = \dfrac{l_d}{g} \times \dfrac{A}{a} \omega^2 r$

Therefore, $$126 = \frac{100}{32{\cdot}2} \times \left(\frac{6}{3}\right)^2 \omega^2 \times 1$$

From which, $$\omega = 3{\cdot}22$$

Let n = number of revolutions per minute

Then, $$\omega = \frac{2\pi n}{60}$$

Hence, $$n = \frac{3{\cdot}22 \times 60}{2\pi}$$

$$= 30{\cdot}6 \text{ revolutions per minute.}$$

100. **Work done against Friction in Pipes.** There will be a frictional resistance to the flow in the suction and delivery

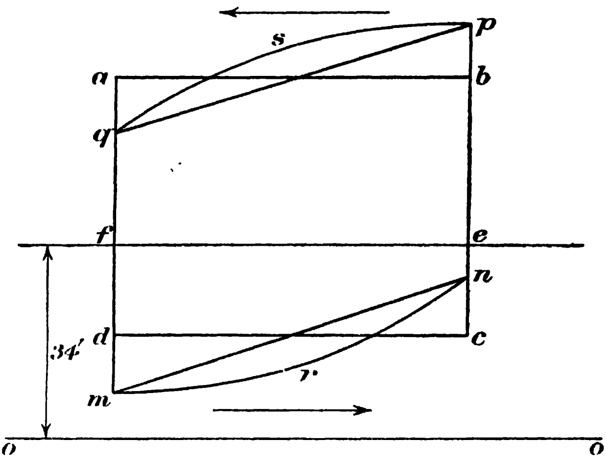


FIG. 114

pipes which follows the ordinary friction laws dealt with in Chapter VI.

For any point of the stroke the velocity in the pipe is given by Equation (3), Art. 97—

$$v = \frac{A}{a}\,\omega r \sin\theta$$

Head lost in friction $= h_f = \dfrac{4fl}{d} \times \dfrac{v^2}{2g}$

$$= \frac{4fl}{d\,2g}\left(\frac{A}{a}\,\omega r \sin\theta\right)^2$$

where f = coefficient of friction.

At the two ends of the stroke, $\sin\theta = 0$, therefore the velocity in the pipe is zero, and there will be no loss of head due to friction.

h_f has its maximum value when $\theta = 90$, that is at the middle of the stroke, when

$$h_f = \frac{4fl}{d\,2g}\left(\frac{A}{a}\omega r\right)^2$$

The equation for h_f is a parabola. If the frictional head is added to the indicator diagrams of Figs. 110 and 111 the combined indicator diagram will be as shown in Fig. 114, the parabola $m\,r\,n$ being the work done against friction in the suction pipe, and the parabola $q\,s\,p$ being that of the delivery pipe.

Total work done during suction stroke $=$ area $e\,f\,m\,r\,n$

$=$ area $e\,f\,d\,c$ + area $m\,r\,n$

Total work done during delivery stroke $=$ area $e\,f\,q\,s\,p$

$=$ area $a\,b\,e\,f$ + area $q\,s\,p$

As the mean ordinate of a parabola is equal to two-thirds of the maximum ordinate,

mean ordinate of suction pipe parabola $= \frac{2}{3} h_{f_s}$

$$= \frac{2}{3} \times \frac{4fl_s}{d_s\,2g}\left(\frac{A}{a_s}\omega r\right)^2$$

where the suffix s applies to the suction pipe.

Work done against friction during suction stroke $=$ area of parabola $m\,r\,n$

Work done against friction per sec. $= \frac{2}{3} \times \frac{4fl_s}{d_s\,2g}\left(\frac{A}{a_s}\omega r\right)^2 \times W$

In the same way, work done against friction during delivery stroke per sec. $= \frac{2}{3} h_{f_d} \times W$

$$= \frac{2}{3} \times \frac{4fl_d}{d_d\,2g}\left(\frac{A}{a_d}\omega r\right)^2 \times W \text{ ft. lb.}$$

where the suffix d refers to the delivery pipe and W is weight of water pumped per sec.

Total work done per second $= W\left(h_s + h_d + \frac{2}{3} h_{f_s} + \frac{2}{3} h_{f_d}\right)$

The vacuum pressure on the piston during the suction stroke for any angle θ of the crank

$$= H_s + H_a + h_{f_s}$$

$$= H_s + \frac{l_s A \omega^2 r \cos\theta}{g a_s} + \frac{4 f l_s}{d_s 2g}\left(\frac{A}{a_s}\omega r \sin\theta\right)^2 \text{ ft. of water}$$

The pressure on the piston, above atmosphere, during the delivery stroke is equal to

$$H_d + \frac{l_d A \omega^2 r \cos\theta}{g a_d} + \frac{4 f l_d}{d_d 2g}\left(\frac{A}{a_d}\omega r \sin\theta\right)^2 \text{ ft. of water}$$

It will be noticed that in both these equations the acceleration head is a maximum at the ends of the stroke and zero at the centre, whilst the frictional head is zero at the ends and a maximum at the centre of the stroke.

EXAMPLE.

A single acting pump has a stroke of 1 ft. and a piston diameter of 6 in. The centre of the pump is 15 ft. above level of water in sump and 100 ft. below delivery water level. The lengths of the suction and delivery pipes are 20 ft. and 120 ft. respectively, and their diameters are 3 in. The coefficient of friction for these pipes is ·01. If the pump is working at 30 revs. per min., find the pressure head on the piston at the beginning, middle, and end of both strokes, and find the horse-power required to drive the pump. (Ignore the velocity head of the discharge water.)

(1) SUCTION STROKE.

At ends of stroke $H_a = \frac{l_s}{g} \times \frac{A}{a_s}\omega^2 r$

$$= \frac{20}{32{\cdot}2} \times \left(\frac{6}{3}\right)^2 \left(\frac{2\pi\, 30}{60}\right)^2 \times \tfrac{1}{2}$$

$$= 12{\cdot}3 \text{ ft. of water.}$$

At middle of stroke, $h_{f_s} = \frac{4 f l_s}{d_s 2g}\left(\frac{A}{a_s}\omega r\right)^2$

$$= \frac{4 \times {\cdot}01 \times 20}{\frac{1}{4} \times 64{\cdot}4}\left(4 \times \frac{2\pi\, 30}{60} \times \tfrac{1}{2}\right)^2$$

$$= 1{\cdot}96 \text{ ft. of water.}$$

Pressure at beginning of stroke $= H_s + H_a$

$$= 15 + 12{\cdot}3$$

$$= 27{\cdot}3 \text{ ft. of water (vacuum)}$$

Pressure at end of stroke $= H_s - H_a$

$= 15 - 12{\cdot}3$

$= 2{\cdot}7$ ft. of water (vacuum)

Pressure at middle of stroke $= H_s + h_{f_s}$

$= 15 + 1{\cdot}96$

$= 16{\cdot}96$ ft. of water (vacuum)

(2) DELIVERY STROKE.

At ends of stroke $H_a = \dfrac{l_d}{g} \times \dfrac{A}{a_d}\omega^2 r$

$$= \frac{120}{32{\cdot}2} \times \left(\frac{6}{3}\right)^2 \left(\frac{2\pi\, 30}{60}\right)^2 \times \tfrac{1}{2}$$

$= 73{\cdot}8$ ft. of water.

At middle of stroke $h_{f_d} = \dfrac{4\,f\,l_d}{d_d\,2g}\left(\dfrac{A}{a}\,\omega r\right)^2$

$$= \frac{4 \times {\cdot}01 \times 120}{\frac{1}{4} \times 64{\cdot}4}\left(4 \times \frac{2\pi\, 30}{60} \times \tfrac{1}{2}\right)^2$$

$= 11{\cdot}75$ ft. of water

Pressure at beginning of stroke $= H_d + H_a$

$= 100 + 73{\cdot}8$

$= 173{\cdot}8$ ft. of water (above atmos.)

Pressure at end of stroke $= 100 - 73{\cdot}8$

$= 26{\cdot}2$ ft. of water (above atmos.)

Pressure at middle of stroke $= H_a + h_f$

$= 100 + 11{\cdot}75$

$= 111{\cdot}75$ ft. of water (above atmos.)

Work done per stroke $= p \times$ area $\times$ length

$= wH \times$ volume of cylinder

$=$ Weight of water per stroke $\times H$

Weight of water per stroke $= W = 62{\cdot}4 \times \dfrac{\pi}{4}({\cdot}5)^2 \times 1$

$= 12{\cdot}25$ lb.

Work done during suction stroke

$$= W\left(H_s + \frac{2}{3}h_{f_s}\right)$$

$$= W\left\{15 + \left(\frac{2}{3} \times 1{\cdot}96\right)\right\}$$

$$= 16{\cdot}31\ W \text{ ft. lb.}$$

Work done during delivery stroke

$$= W\left(H_d + \frac{2}{3}h_{f_d}\right)$$

$$= W\left\{100 + \left(\frac{2}{3} \times 11{\cdot}75\right)\right\}$$

$$= 107{\cdot}8\ W \text{ ft. lb.}$$

Total work done per revolution $= W(107{\cdot}8 + 16{\cdot}31)$

$$= 12{\cdot}25 \times 124{\cdot}1 = 1520 \text{ ft. lb.}$$

Horse-power required $= \dfrac{1520 \times 30}{33{,}000}$

$$= 1{\cdot}38$$

101. **Maximum Vacuum Pressure during Suction Stroke.** It is not quite clear from Fig. 114 which part of the suction stroke will have the maximum vacuum pressure. A part of the curve $m\,r\,n$ may fall below m; in which case separation may occur at some point other than the beginning of the stroke.

The velocity head of the water in the suction pipe is converted into pressure head on entering the cylinder, therefore the maximum vacuum pressure will occur just inside the suction pipe at the section where it enters the cylinder.

Let the total vacuum pressure in the pipe at this section $= H$, and v_s = velocity in suction pipe.

Then, $$H = H_s + H_a + \frac{v_s^2}{2g} + h_f$$

Substituting the values of H_a, v_s, and h_f in terms of θ,

$$H = H_s + \frac{l_s}{g}\frac{A}{a_s}\omega^2 r\cos\theta + \frac{v_s^2}{2g}\left(1 + \frac{4fl_s}{d_s}\right)$$

$$= H_s + \frac{l_s}{g}\frac{A}{a_s}\omega^2 r\cos\theta + \left(\frac{A}{a_s}\right)^2\frac{\omega^2 r^2\sin^2\theta}{2g}\left(1 + \frac{4fl_s}{d_s}\right)$$

Differentiating and equating to zero for a maximum,

$$\frac{dH}{d\theta} = -\frac{l_s}{g}\frac{A}{a_s}\omega^2 r \sin\theta + \left(\frac{A}{a_s}\right)^2 \frac{\omega^2 r^2 \sin\theta\cos\theta}{g}\left(1 + \frac{4 f l_s}{d_s}\right) = 0$$

From which, $$\cos\theta = \frac{l_s a}{Ar\left(1 + \frac{4f l_s}{d_s}\right)}$$

The maximum value cos θ can have is unity. If the right-hand half of the above equation is applied to any actual pump

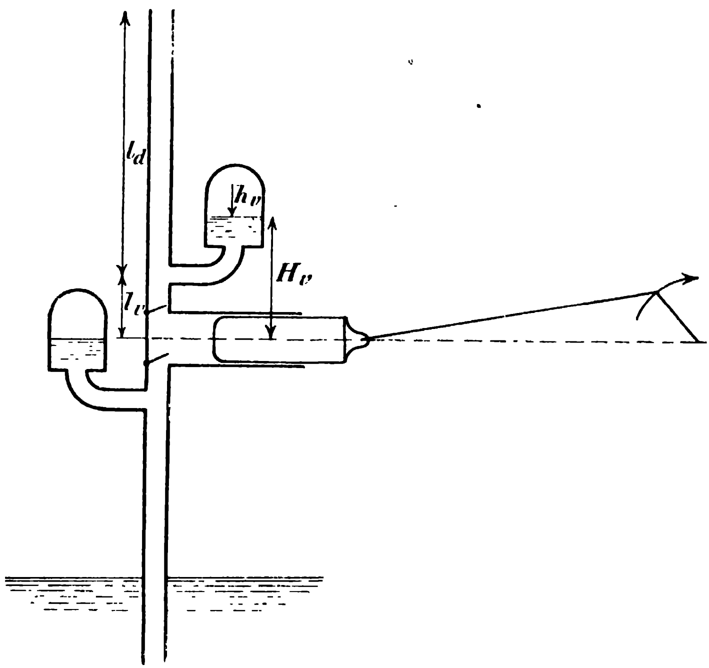


Fig. 115

it will be found to be much greater than unity. The vacuum pressure is, therefore, a maximum at the beginning of the stroke.

Hence, if separation occurs during the suction stroke, it will do so at the beginning.

102. **The Reduction of the Acceleration Head by means of an Air Vessel.** As the pressure in the pump must not fall below the separation pressure of the water, the maximum speed of a

pump is limited by the acceleration head. The acceleration head depends on the length of the suction or delivery pipe, and it may be considerably reduced by fitting an air vessel on these pipes as near to the cylinder as possible. Suppose an air vessel be fitted on the delivery pipe of the pump in Fig. 115. The air vessel is a cast-iron chamber having an opening at the base, through which the water may flow. As the level of the water in the chamber rises, the air trapped in the upper portion of the chamber is compressed, and will force the water out as soon as the pressure of the latter falls.

The water in the delivery pipe beyond the air vessel is assumed to flow with a uniform velocity throughout the cycle. During the middle portion of the delivery stroke, when the piston is forcing the water into the delivery pipe with a velocity greater than the mean, the additional water will flow into the air vessel. At the ends of the stroke, when the water is forced into the delivery pipe with a velocity less than the mean, the water will flow out of the air vessel and so make up the deficiency. The constant flow in the delivery pipe beyond the air vessel is thus maintained. The only volume of water which is now accelerated is that in the delivery pipe between the air vessel and cylinder ; this is made small by fitting the air vessel as near the cylinder as possible.

The pressure of the air in the air vessel will vary as the water flows in and out ; this variation is reduced by making the air vessel large compared with the area of the delivery pipe. In order to simplify the problem, it is assumed that the air vessel is so large that the change of water level in it may be neglected. This is the same as assuming the air pressure in the air vessel to be constant.

Let l_d = length of delivery pipe beyond air vessel

l_v = length of delivery pipe between cylinder and air vessel

v_d = constant velocity of water in delivery pipe beyond air vessel

Then, $H_a = \dfrac{l_v}{g} \dfrac{A}{a_d} \omega^2 r \cos\theta$

Head lost in friction in delivery pipe beyond air vessel

$$= \frac{4 f l_d v_d^2}{d_d \, 2g}$$

Head lost in friction in delivery pipe between air vessel and cylinder

$$= \frac{4 f l_v}{d_d 2g}\left(\frac{A}{a_d}\omega r \sin\theta\right)^2$$

Also, $$v_d = \frac{\text{volume of water per sec.}}{\text{area of delivery pipe}}$$

If pump is single acting,

$$v_d = \frac{2r\,A\,n}{60\,a_d}$$

where n is the number of revolutions per minute.

If pump is double acting,

$$v_d = \frac{4\,r\,A\,n}{60\,a_d}$$

Total pressure head at beginning of delivery stroke

$$= H_d + \frac{v_d^2}{2g} + \frac{4 f l_d v_d^2}{d_d 2g} + \frac{l_v A}{g\,a_d}\omega^2 r$$

Total pressure head at end of stroke

$$= H_d + \frac{v_d^2}{2g} + \frac{4 f l_d v_d^2}{d_d 2g} - \frac{l_v A}{g\,a_d}\omega^2 r$$

Total pressure head at middle of stroke,

$$= H_d + \frac{v_d^2}{2g} + \frac{4 f l_d v_d^2}{d_d 2g} + \frac{4 f l_v}{d_d 2g}\left(\frac{A}{a_d}\omega r\right)^2$$

The last term in each of these equations is small and may usually be neglected.

The same reasoning applies if an air vessel is fitted on the suction pipe, the water accelerated being reduced to the amount between the air vessel and cylinder. The above formula will hold for the suction pipe if the suffix s is substituted for the suffix d. In this case the pressure head will be below atmosphere.

Total work done per sec. $= W\left(H_s + H_d + \frac{4fl_d v_d^2}{d_d 2g} + \frac{4fl_s v_s^2}{d_s 2g}\right)$

The total pressure head in the air vessel reckoned above the centre of the cylinder will be approximately equal to

the total pressure head in the delivery pipe above the same datum.

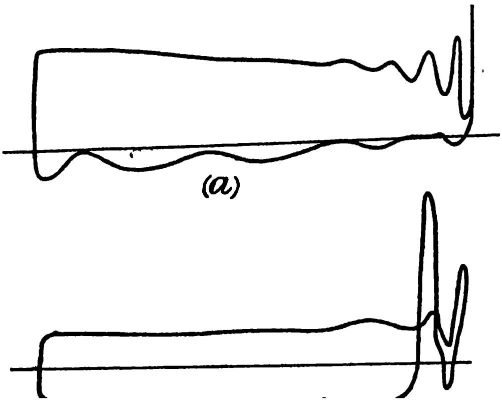
Let h_v = pressure of air in air vessel in feet of water

and H_v = height of water level in air vessel above centre of cylinder

Then,

$$h_v + H_v = H_d + \frac{4 f l_d v_d^2}{d_d \, 2g} + 34$$

if all small quantities are neglected.



(a)

(b) Knocking

FIG. 116

Actual indicator diagrams taken from a Tangye pump are shown in Fig. 116. In this pump an air vessel was fitted on the delivery pipe only. Fig. 116 (a) shows the indicator diagram taken when the pump was running normally; a diagram taken when the pump was knocking is shown in Fig. 116 (b), the sudden pressure rise caused by the knock is noticeable.

EXAMPLE.

A reciprocating pump draws water from a sump through a suction pipe 6 in. diameter and 40 ft. long. the water level being 10 ft. below the level of the cylinder. The cylinder diameter is 9 in., stroke 15 in., and the length of the connecting rod 5 ft. The driving crank rotates at 20 revs. per min. Determine the pressure in the cylinder at the beginning of the stroke (a) when no air vessel is fitted; (b) when an air vessel is fitted at the cylinder level and distance 5 ft. from it. (London Univ.)

(*a*) (1) Assuming simple harmonic motion,

$$H_a = \frac{A\,l_s}{a_s\,g}\,\omega^2 r$$

$$= \left(\frac{9}{6}\right)^2 \times \frac{40}{32\cdot2}\left(2\pi\frac{20}{60}\right)^2 \frac{7\cdot5}{12}$$

$$= 7\cdot67 \text{ ft. of water.}$$

Total pressure head in cylinder

$$= H_s + H_a$$

$$= 10 + 7\cdot67 = 17\cdot67 \text{ ft. of water less than atmos.}$$

(2) If harmonic motion is not assumed,

$$H_a = \frac{A\,l_s}{a_s\,g}\,\omega^2 r\left(1 + \frac{r}{L}\right)$$

$$= 7\cdot67 \times 1\cdot125 = 8\cdot67 \text{ ft. of water}$$

Total pressure head in cylinder

$$= 10 + 8\cdot67 \quad = 18\cdot67 \text{ ft. of water below atmos.}$$

(*b*) Assume pump is single acting.

Then, $$v_s = \frac{A}{a_s} \times 2\,r\,\frac{n}{60}$$

$$= \left(\frac{9}{6}\right)^2 \times 2 \times \frac{7\cdot5}{12} \times \frac{20}{60} = \cdot937 \text{ ft. per sec.}$$

$$h_f = \frac{4\,f\,l_s\,v_s^2}{d_s\,2g}$$

$$= \frac{4 \times \cdot01 \times 35 \times \cdot937^2}{\cdot5 \times 64\cdot4} = \cdot0382 \text{ ft. of water}$$

(1) Assuming simple harmonic motion,

$$H_a = \frac{A\,l_v}{a_s\,g}\,\omega^2 r$$

$$= \left(\frac{9}{6}\right)^2 \times \frac{5}{32\cdot2} \times \left(2\pi\frac{20}{60}\right)^2 \frac{7\cdot5}{12}$$

$$= \cdot959 \text{ ft. of water}$$

Total pressure head in cylinder

$$= H_s + H_a + h_f$$

$$= 10 + \cdot959 + \cdot0382$$

$$= 10\cdot9972 \text{ ft. of water below atmos.}$$

(2) If simple harmonic motion is not assumed,

$$H_a = \frac{A\,l_v}{as\,g}\,\omega^2 r\left(1 + \frac{r}{L}\right)$$

$$= \cdot 959 \times 1\cdot 125 = 1\cdot 084 \text{ ft. of water}$$

Total pressure head in cylinder

$$= 10 + 1\cdot 084 + \cdot 0382$$

$$= 11\cdot 1222 \text{ ft. of water below atmos.}$$

103. **Work Saved by Fitting Air Vessel.** The following applies to either suction or delivery strokes. Consider in the first case the pump to be single acting. If there is no air vessel on the pipe, the diagram representing the work lost in friction during the revolution is a parabola (Art. 100), the area of which equals

$$W \times \frac{2}{3} \times \frac{4\,f\,l}{d\;2g}\left(\frac{A}{a}\,\omega r\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where W = weight of water pumped per revolution.

Suppose an air vessel is now fitted just outside the cylinder. The velocity of flow in the pipe is now constant ; the frictional loss will, therefore, also be constant and acts over both strokes. The diagram showing the work done against friction during the revolution will now be a rectangle of area

$$W \times \frac{4\,f\,l\,v^2}{d\;2g}$$

where v = mean velocity of flow in pipe

But, $v = \dfrac{A}{a}\,\dfrac{2r\,\omega}{2\pi} = \dfrac{A}{a}\,\dfrac{\omega r}{\pi}$

Therefore, work done against friction

$$= W \times \frac{4\,f\,l}{d\;2g}\left(\frac{A}{a}\,\frac{\omega r}{\pi}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Subtracting Equation (2) from Equation (1), work saved by fitting air vessel

$$= W \times \frac{4\,f\,l}{d\;2g} \times \left(\frac{A}{a}\,\omega r\right)^2 \left(\frac{2}{3} - \frac{1}{\pi^2}\right)$$

$$\text{Percentage of work saved} = \frac{\frac{2}{3} - \frac{1}{\pi^2}}{\frac{2}{3}} \times 100$$

$$= 84{\cdot}8 \text{ per cent of frictional work.}$$

If pump is double acting,

$$v = 2\frac{A}{a}\frac{\omega r}{\pi}$$

$$\text{Percentage saved} = \frac{\frac{2}{3} - \frac{4}{\pi^2}}{\frac{2}{3}} = 39{\cdot}2 \text{ per cent.}$$

104. **Rate of Flow Into and From Air Vessel.** Consider first the case of a single acting pump. The water in the pipe beyond the air vessel will have a constant velocity during the whole cycle, whilst the water enters or issues from the cylinder during one stroke only.

Velocity of flow to or from cylinder $= \dfrac{A}{a}\omega r \sin\theta$ ft. per sec.

Rate of flow to or from cylinder $= A\,\omega\, r \sin\theta$ cu. ft. per sec.

Velocity of flow beyond air vessel $= \dfrac{A}{a}\dfrac{\omega r}{\pi}$ ft. per sec.

Rate of flow beyond air vessel $= \dfrac{A\,\omega r}{\pi}$ cu. ft. per sec.

Rate of flow from air vessel $= \dfrac{A\,\omega r}{\pi} - A\,\omega r \sin\theta$

$$= A\,\omega r\left(\frac{1}{\pi} - \sin\theta\right) \text{ cu. ft. per sec.}$$

If this equation is negative the water is flowing into the air vessel. It will be noticed that there are two points on the delivery stroke at which $\sin\theta = \dfrac{1}{\pi}$; at these points there will be no flow either into or from the air vessel.

Next, suppose the pump to be double acting.

Then,

$$\text{velocity of flow beyond air vessel} = \frac{2A}{a}\frac{\omega r}{\pi}$$

$$\text{and rate of flow beyond air vessel} = 2A\frac{\omega r}{\pi}$$

$$\text{Rate of flow from air vessel} = A\omega r\left(\frac{2}{\pi} - \sin\theta\right) \text{cu. ft. per sec.}$$

105. **Pump Duty.** The "duty" of a pump is a practical way of expressing the overall efficiency. For a pump driven by a steam engine the duty is the number of foot pounds of work given out by the pump for every 1,000,000 British thermal units supplied to the engine by the boiler. Hence it takes into account the efficiency of the pump and of the steam engine.

If the pump is delivering W lbs. of water per sec. against a head of H ft.,

$$\text{work done by pump} = WH \text{ ft. lb. per sec.}$$

No. of British thermal units supplied to engine per sec.

$$= \begin{pmatrix}\text{weight of steam} \\ \text{used per sec.}\end{pmatrix} \times \begin{pmatrix}\text{total heat of 1 lb.} \\ \text{of steam supplied}\end{pmatrix}$$

Duty of pump

$$= \frac{WH \times 1{,}000{,}000}{(\text{Wgt. of steam per sec.}) \times (\text{total heat 1 lb. steam})}$$

Formerly, the term "duty" was the number of foot pounds of work given out by the pump per bushel of coal burned in the boiler. In this case the efficiency of the boiler is also included.

If the term "duty" is applied to a pump driven by an electric motor, it is based on 1,000,000 British thermal units supplied to the motor.

EXAMPLES 8.

(1) Water is raised to a height of 60 ft. by a single acting pump having a bore of 6 in. and a stroke of 12 in. If the pump has a speed of 40 revs. per min., find the theoretical horse-power required and the theoretical discharge. Neglect all losses.

Ans.—H.p. = ·89 ; Q = 48·9 gallons per minute.

(2) If the pump in Question (1) has an actual discharge of 47 gallons per min., find the percentage slip and the coefficient of discharge.

Ans.—3·88 per cent ; ·962.

(3) If the pump in Question (1) has a delivery pipe of 4 in. diameter, and a length of 50 ft., find the acceleration head at the beginning of the stroke when no air vessel is fitted.

Ans.—30·6 ft. of water.

(4) If a large air vessel is fitted on the delivery pipe of Question (3), close to the cylinder, find the theoretical velocity of flow in delivery pipe and the pressure head in the cylinder necessary to overcome friction in the delivery pipe. (f = ·01.)

Ans.—1·5 ft. per sec.; ·21 ft. of water.

(5) A double acting reciprocating pump (cylinder 4 in. diameter, stroke 6 in.) makes 120 strokes per minute. It draws water from a sump, the surface of which is 6 ft. below the centre of the pump cylinder. If the total length of the suction pipe is 18 ft., and the diameter 2 in., determine the absolute pressure, in pounds per square inch, of the water in the cylinder (*a*) at the beginning, (*b*) at middle, and (*c*) at the end of the suction stroke, there being no air vessel on the suction pipe. Sketch the probable diagram for the stroke. State if separation is likely to occur, and give reasons. Assume the piston has simple harmonic motion. (London Univ.)

Ans.—(*a*) 2·56 lb. per sq. in.
(*b*) 12·1 " "
(*c*) 21·7 " "

(6) Discuss the conditions under which "separation" and "negative slip" occur in reciprocating pumps.

Sketch the form of indicator card obtained when (*a*) separation only, (*b*) separation, and also opening of the delivery valve occur during the suction stroke.

The bore and stroke of a single acting reciprocating pump are 4 in. and 8 in. respectively, and the plunger has simple harmonic motion. The suction pipe is $3\frac{1}{2}$ in. in diameter and 14 ft. long, and the centre of the pump is 12 ft. above the water in the sump. Determine the theoretical speed, in revolutions per minute, at which there will be separation, assuming it to occur when the pressure falls below 4 lb. per sq. in. (London Univ.)

Ans.—73 revs. per min.

(7) Explain fully the functions of air vessels when they are introduced on the suction and delivery pipes of pumps. (London Univ.)

(8) A single acting reciprocating pump, 12 in. diameter, 20 in. stroke, with a large air chamber on the suction side, has a suction head of 8 ft. The suction pipe is 6 in. diameter, and 14 ft. long. The pump makes 40 working strokes per minute, and discharges at its own level.

Neglecting all losses except those due to friction in the suction pipe (f = ·01), find the horse-power of the pump.

If the plunger has simple harmonic motion, determine the rate of flow from the air chamber when the plunger is at the centre of its stroke. (London Univ.)

Ans.—H.p. = ·828; Q = ·874 cu. ft. per sec.

(9) Briefly explain the reasons for placing air vessels on the suction and delivery pipes of a reciprocating pump.

A single acting reciprocating pump has a plunger diameter of 10 in. and a stroke of 18 in. The delivery pipe is $4\frac{1}{2}$ in. diameter and 160 ft long. If the motion of the plunger is simple harmonic, find the horse-power saved in overcoming friction in the delivery pipe by the provision of a large air vessel when the speed of the pump is 60 revs. per min. Assume that f = ·01. (London Univ.)

Ans.—7·52 h.p.

(10) A plunger is fitted in a vertical pipe which is full of water, and whose lower end is submerged in a suction tank. It is moved upwards with an acceleration of 5 ft. per sec. If air is liberated from the water when the absolute pressure falls below 4 ft. of water, and if the barometric height is 32 ft. of water, what is the maximum height above the level in the suction tank at which the plunger can operate without cavitation? (A.M.I. Mech. E.)

Ans.—24·2 ft.

(11) What is meant by "separation" in a reciprocating pump? The plunger of such a pump moves with simple harmonic motion. The diameter is 12 in. and the stroke 2 ft. The suction pipe line is 9 in. in diameter and 80 ft. long and the suction lift 14 ft. Calculate the maximum speed at which the pump can operate without separation occurring at the beginning of the stroke. Take the effective height of the barometer as 28 ft. of water. (A.M. Inst. C.E.)

Ans.—17 revs. per min.

(12) What is meant by "separation" in a reciprocating pump? In such a pump the cylinder diameter is 9 in.; the suction pipe is 9 in. diameter and 60 ft. long; the height of the pump above the level of the water in the suction sump is 15 ft. If the stroke is 18 in., and if the motion is simple harmonic, at what speed will separation occur at the beginning of the stroke? Take the effective height of the barometer as 30 ft. of water. (A.M.I. Mech. E.)

Ans.—31·2 revs. per min.

CHAPTER IX

IMPACT OF WATER

106. **Pressure on Stationary Flat Plate.** When a jet of water impinges normally on a flat plate (Fig. 117) the force on the plate is equal to the rate of change of the momentum of the jet, or to the change of momentum per second.

Let a = cross-sectional area of jet in square feet,

V = velocity of jet in feet per second,

and W = weight of water striking plate per second.

Then, $W = w\,a\,V.$

The jet strikes the plate and leaves it tangentially, so that all its momentum in a direction normal to plate is destroyed.

Force on plate = change of momentum per second

$$= \left\{\begin{matrix}\text{mass of water striking} \\ \text{plate per second}\end{matrix}\right\} \times \left\{\begin{matrix}\text{change of velocity} \\ \text{normal to plate}\end{matrix}\right\}$$

$$= \frac{W}{g} \times V$$

$$= \frac{w\,a\,V^2}{g}\ \text{lb.}$$

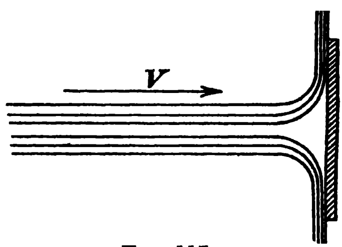


Fig. 117

If the plate is inclined to the jet, as in Fig. 118, the force of the jet may be resolved into a normal component.

Let θ = angle of inclination of plate to jet.

Normal force on plate = (change of momentum per sec.) $\sin\theta$

$$= \frac{W}{g} \times V \times \sin\theta$$

$$= \frac{w\,a\,V^2}{g} \sin\theta \ \text{lb.}$$

EXAMPLE.

A jet of water 2 in. diameter impinges on a fixed plate and has a velocity of 100 ft. per sec. Find the normal force on the plate (1) when the jet is normal to the plate; (2) when the jet is inclined at 60° to the plate.

$$\left.\begin{array}{l}\text{Weight of water per}\\ \text{sec. striking plate}\end{array}\right\} = w\,a\,V$$

$$= 62{\cdot}4 \times \frac{\pi}{4}\left(\frac{2}{12}\right)^2 \times 100$$

$$= 136 \text{ lb.}$$

(1) Force $= \dfrac{W\,V}{g} = \dfrac{136 \times 100}{32{\cdot}2}$

$$= 422{\cdot}0 \text{ lb.}$$

(2) Normal force $= \dfrac{W\,V}{g} \sin\theta$

$$= 422 \times {\cdot}866 = 366 \text{ lb.}$$

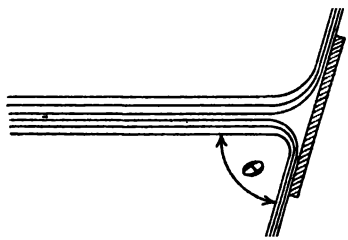


FIG. 118

107. Pressure on Moving Flat Plate. If a jet of water impinges on a plate which is moving in the same direction as the jet, the velocity with which the jet strikes the plate will be the relative velocity between the jet and the plate.

Referring to Fig. 119,

let v = velocity of plate.

Weight of water striking plate per sec. $= W = w\,a\,(V - v)$

Force on plate $= \dfrac{W}{g}(V - v)$

$$= \frac{w\,a\,(V - v)^2}{g}$$

This case would not be possible in practice as there would be a continually lengthening jet, the distance between the plate and nozzle increasing by v ft. every second.

If, instead of a single plate, there is a continuous series of plates at a fixed distance apart and all moving in the same direction as the jet with a velocity v, the weight of water striking the plates is now equal to $w\,a\,V$. This condition

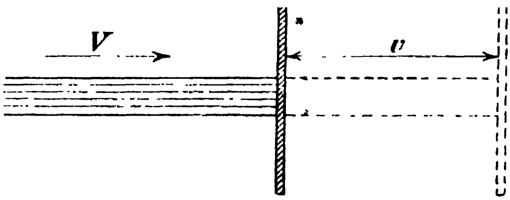


Fig. 119

would be obtained if the plates are all fixed radially around the circumference of a large wheel on which the jet impinged tangentially. (Fig. 120.)

Then, force on plates $= \dfrac{W}{g}(V - v)$

$$= \frac{w\,a\,V}{g}(V - v)$$

Work done per sec. on plates $= \dfrac{w\,a\,V}{g}(V - v)v$

$$= \frac{(V-v)}{g}v \text{ per lb. of water}$$

Energy supplied by jet = kinetic energy of jet per sec.

$$= \frac{W\,V^2}{2g}$$

$$= \frac{V^2}{2g} \text{ per lb. of water}$$

Efficiency of plates $= e = \dfrac{\text{work done per lb. of water}}{\text{kinetic energy of jet per lb.}}$

$$= \frac{\dfrac{(V-v)v}{g}}{\dfrac{V^2}{2g}} = \frac{2(V-v)v}{V^2}$$

Differentiating and equating to zero for maximum efficiency,

$$\frac{de}{dv} = V - 2v = 0$$

From which,
$$v = \frac{V}{2}$$

Then, maximum efficiency
$$= \frac{2\left(V - \frac{V}{2}\right)\frac{V}{2}}{V^2} = \frac{1}{2}$$

Flat plates used in this manner are called vanes, and a wheel of the type shown in Fig. 120 is known as an undershot water wheel.

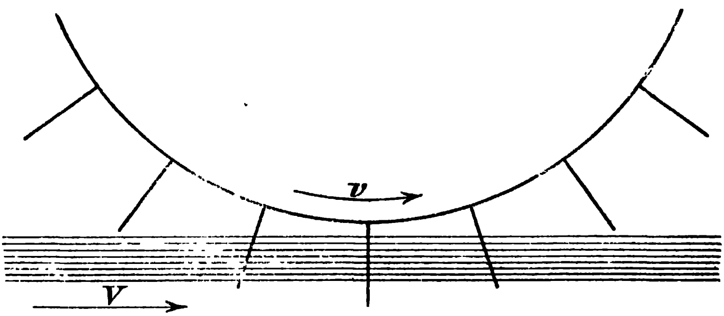


FIG. 120

EXAMPLE.

A jet of water 3 in. diameter and moving with a velocity of 40 ft. per sec. strikes a series of flat plates normally. If the plates are moving in the same direction as the jet with a velocity of 30 ft. per sec., find the pressure on the plates, the work done per second, and the efficiency.

Pressure on plates
$$= \frac{waV}{g}(V - v)$$
$$= \frac{62 \cdot 4}{32 \cdot 2} \times \frac{\pi}{4} \times \left(\frac{1}{4}\right)^2 \times 40(40 - 30)$$
$$= 38 \text{ lb.}$$

Work done per sec.
$$= 38 \times 30$$
$$= 1140 \text{ ft. lb.}$$

Efficiency
$$= \frac{2(V - v)v}{V^2}$$
$$= \frac{2(40 - 30)30}{1600}$$
$$= 37 \cdot 5 \text{ per cent.}$$

108. Pressure on a Fixed Curved Vane. Consider the curved fixed vane of Fig. 121, and let $a\ b$ be the normal at the centre of the vane. The jet strikes the vane at an angle of α to $a\ b$ and leaves at an angle of β, the vane deflecting the jet through an angle of $180° - (\alpha + \beta)$. The velocity of the jet is not changed in magnitude as it flows over the vane; it is the direction only which is changed. The velocity of the entering jet in the direction $a\ b$ is $V \cos \alpha$, and it leaves the vane with a velocity component of $-V \cos \beta$ in the direction $a\ b$.

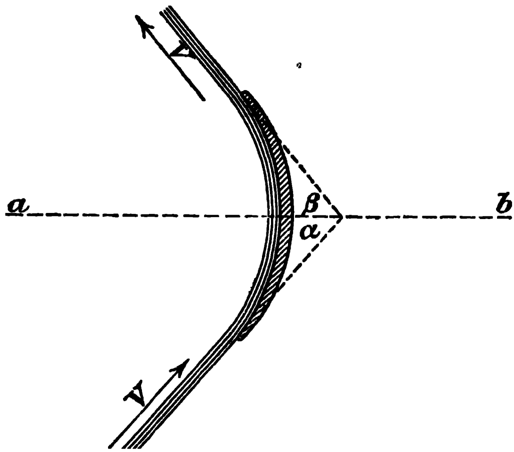


FIG. 121

Force on vane in direction ab = change of momentum per sec.

$$= \frac{W}{g} \text{ (change of velocity in direction } ab)$$

$$= \frac{W}{g}[V \cos \alpha - (-V \cos \beta)]$$

$$= \frac{W}{g}(V \cos \alpha + V \cos \beta)$$

where $W = w\,a\,V$

If the vane is semicircular, the angles α and β are each equal to 0, then,

$$\text{force on vane in direction } ab = \frac{2W}{g}V$$

The force of a jet on a semicircular vane is thus twice as great as that on a flat plate. This is due to the fact that, with

a semicircular vane, use is made of the reaction of the leaving water which exerts the same force on the vane in leaving as in entering. This principle is made use of in the Pelton wheel.

There will be a tangential force on the vane at right angles to *a b*. This will be equal to mass of water per sec. × change of velocity in a direction at right angles to *a b*.

Or, tangential force $= \frac{W}{g}(V \sin \alpha - V \sin \beta)$

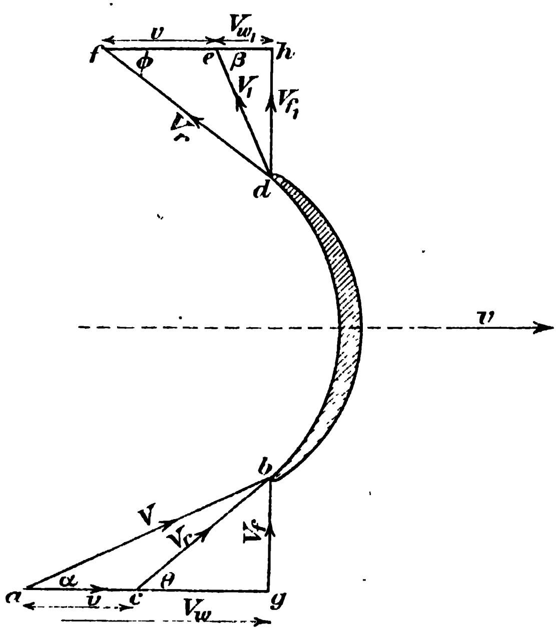


Fig. 122

109. **Pressure on a Moving Curved Vane.** Suppose the curved vane of Fig. 121 is moving in the direction *a b* with a velocity *v*, and let the jet impinge on the vane with a velocity *V*, as before. The velocity of the water over the vane will be equal to the relative velocity of the jet to the vane, and may be found by subtracting the vectors of *V* and *v*.

Let V_r = relative velocity between jet and vane at entrance.

Referring to Fig. 122, draw *a b* to represent the velocity of the jet at entrance in magnitude and direction. Next draw *a c* to represent the velocity of the vane in magnitude and direction.

Then $c\,b$ represents the relative velocity between the jet and the vane. If the water is to enter without shock, the vane at entrance must be parallel to $c\,b$.

The water will pass over the vane and leave with the velocity V_r. The absolute velocity of the leaving water may be found by drawing the triangle of velocities at exit.

Let V_1 = absolute velocity with which water leaves vane.

Draw $d\,f$ to represent the relative velocity V_r; if the water leaves the vane without shock, V_r will be parallel to the vane at exit.

Draw $f\,e$ to represent v in magnitude and direction. Then $d\,e$ gives the absolute velocity of the leaving water.

The velocity of the entering water may be resolved into two components, one parallel to the direction of motion of the vane and known as the velocity of whirl, the other perpendicular to the direction of motion of the vane and known as the velocity of flow. The same terms are also applied to the components of the velocity of the leaving water.

Let V_w = velocity of whirl at entrance
V_{w_1} = velocity of whirl at exit
V_f = velocity of flow at entrance
V_{f_1} = velocity of flow at exit

These are represented in Fig. 122 by $a\,g$, $h\,e$, $g\,b$, and $d\,h$ respectively.

Let θ = angle between relative velocity and direction of motion at inlet

and ϕ = angle between relative velocity and direction of motion at outlet

Then, if the water is to enter and leave the vane without shock, the angles of the blade at inlet and outlet must be made equal to θ and ϕ respectively.

The force on the vane in the direction of motion is equal to the change of momentum per second of the water in this direction.

Or,

$$\text{force on vane} = \frac{W}{g}(V_w + V_{w_1}) \quad . \quad . \quad . \quad . \quad (1)$$

where W = weight of water flowing per second.

If the friction between the water and vane be neglected, the relative velocity at exit equals the relative velocity at entrance.

Or, $V_{r_1} = V_r$

From Equation (1),

$$\text{Work done on vane per sec.} = \frac{W}{g}(V_w + V_{w_1})v \quad . \quad . \quad (2)$$

If V_{w_1} is in the same direction as the velocity of the vane, the equation then becomes

$$\text{Work done per second} = \frac{W}{g}(V_w - V_{w_1})v$$

The work done is also equal to the change of kinetic energy of the jet per second.

$$\text{Or, work done per second} = \frac{WV^2}{2g} - \frac{W\,V_1^2}{2g}$$

$$= \frac{W}{2g}(V^2 - V_1^2)$$

$$\text{Then,} \qquad \text{efficiency} = \frac{\frac{W}{2g}(V^2 - V_1^2)}{\frac{W}{2g}V^2} = \frac{(V^2 - V_1^2)}{V^2}$$

$$= 1 - \left(\frac{V_1}{V}\right)^2 \quad . \quad . \quad . \quad (3)$$

It follows from this equation that, for a given angle α, the efficiency is a maximum when V_1 is a minimum. This occurs when the angle ϕ is zero, in which case,

$$V_1 = V_{w_1} = V_r - v$$

If α also equals zero,

$$V_r = V - v$$

Then, $\qquad V_1 = V - 2v$

Therefore, $\qquad V_1 = 0 \text{ when } v = \frac{V}{2},$

in which case the efficiency is unity; also the vane is semicircular.

Example.

A vane has a velocity of 40 ft. per sec. Water impinges on the vane at an angle of 30° and leaves at an angle of 160° to the direction of motion. If the entering water has an absolute velocity of 80 ft. per sec., find (1) the angles of the blade tips at inlet and outlet; (2) the work done on the vane per pound of water; and (3) the efficiency.

(1) Referring to Fig. 122,

$$V = 80 \text{ ft. per sec.}, \quad v = 40 \text{ ft. per sec.}, \quad \alpha = 30^\circ$$

and $\beta = 20^\circ$.

From triangle of velocities at inlet,

$$V_w = 80 \cos 30 = 69{\cdot}3 \text{ ft. per sec.}$$

$$V_f = 80 \sin 30 = 40 \text{ ft. per sec.}$$

$$\tan \theta = \frac{V_f}{V_w - v} = \frac{40}{69{\cdot}3 - 40} = 1{\cdot}36$$

and $\theta = 53{\cdot}7^\circ$

$$V_r = \frac{V_f}{\sin \theta} = \frac{40}{\sin 53{\cdot}7} = 49{\cdot}6 \text{ ft. per sec.}$$

From triangle of velocities at outlet,

$$V_r = V_r = 49{\cdot}6 \text{ ft. per sec.}$$

$$\tan \beta = \frac{V_{r_1} \sin \phi}{V_{r_1} \cos \phi - v}$$

Or, $$\tan 20 = \frac{49{\cdot}6 \sin \phi}{49{\cdot}6 \cos \phi - 40}$$

From which $$\tan \phi = {\cdot}364 - \frac{{\cdot}294}{\cos \phi}$$

Therefore, $$\phi = 4^\circ$$

Also $$V_1 = \frac{V_{r_1} \sin 4^\circ}{\sin 20^\circ}$$

$$= \frac{49{\cdot}6 \times {\cdot}0698}{{\cdot}342} = 10{\cdot}12 \text{ ft. per sec.}$$

These results might also have been obtained by drawing the velocity triangles to scale.

(2) Work done per lb. of water $\Big\} = \dfrac{1}{g}(V_w + V_{w_1})v$

$$= \frac{1}{32 \cdot 2}(69 \cdot 3 + 10 \cdot 12 \cos 20)\,40$$

$$= 97 \cdot 9 \text{ ft. lb. per sec.}$$

(3) Efficiency $= \dfrac{\text{work done per sec.}}{\text{kinetic energy supplied per sec.}}$

$$= \frac{97 \cdot 9}{\frac{V^2}{2g}} = \frac{97 \cdot 9 \times 64 \cdot 4}{(80)^2}$$

$$= 98 \cdot 5 \text{ per cent.}$$

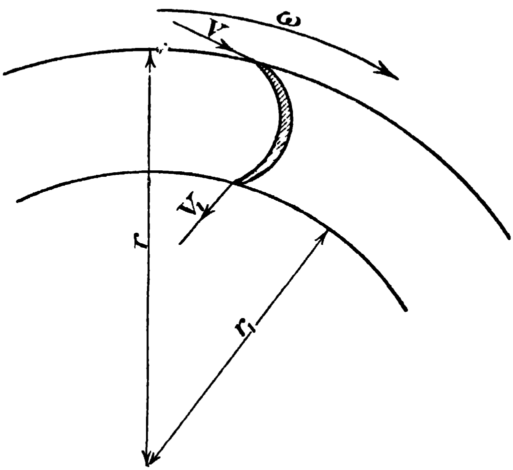


Fig. 123

The work done might also have been found from the change of kinetic energy.

Or, work done per lb. of water $\Big\} = \dfrac{V^2}{2g} - \dfrac{V_1^2}{2g}$

Efficiency $= \dfrac{V^2 - V_1^2}{V^2}$

110. **Flow over a Radial Vane.** Suppose the blade of Fig. 123 to be one of a series of blades fixed radially to the rim of a rotating wheel.

Let r = radius of wheel at entrance

r_1 = radius of wheel at exit

ω = angular velocity of wheel

v = tangential velocity of blade tip at entrance

v_1 = tangential velocity of blade tip at exit.

Treat all velocities in direction of motion of wheel as positive.

Tangential momentum of water striking blade at entrance $\Big\}$ $= \dfrac{V_w}{g}$ per lb. of water per sec.

Moment of momentum at entrance $\Big\}$ $= \dfrac{V_w}{g} r$ per lb. of water per sec.

Tangential momentum of water leaving blade $\Big\}$ $= \dfrac{V_{w_1}}{g}$ per lb. of water per sec.

Moment of momentum at exit $= \dfrac{V_{w_1}}{g} r_1$ per lb. of water per sec.

Change of moment of momentum per lb. of water per sec. $\Big\}$ $= \dfrac{V_w r}{g} - \dfrac{V_{w_1} r_1}{g}$ = torque on wheel

Work done by torque per lb. of water $\Big\}$ $= \left(\dfrac{V_w r}{g} - \dfrac{V_{w_1}}{g}\right)\omega$

But, $v = \omega r$

and $v_1 = \omega r_1$

Then,

Work done on wheel per lb. of water $\Big\}$ $= \dfrac{V_w v}{g} - \dfrac{V_{w_1} v_1}{g}$ . . (1)

If the water leaves against the direction of motion of the wheel, V_{w_1} will be negative, and Equation (1) becomes

$$\frac{V_w v}{g} + \frac{V_{w_1} v_1}{g}$$

This equation is very important in problems dealing with turbines.

EXAMPLE.

A wheel having radial blades is 2 ft. radius at the outer tip of the blades and 1 ft. at the inner. Water enters the blades at the outer tip with a velocity of 100 ft. per sec. at an angle of 30° to the tangent, and leaves the blade with a velocity of flow of 14 ft. per sec. The blade has an angle of 40° at entrance and 35° at exit. Find the work done per pound of water entering the wheel, the speed of the wheel, and the efficiency.

The triangles of velocities are shown in Fig. 124.

Consider the triangle at inlet.

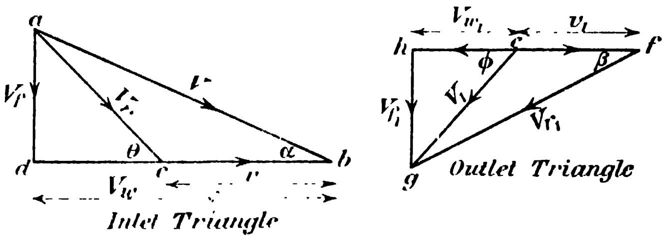


FIG. 124

$$V = 100 \text{ ft. per sec.}$$

$$V_f = 100 \sin 30 = 50 \text{ ft. per sec.}$$

$$V_w = 100 \cos 30 = 86{\cdot}6 \text{ ft. per sec.}$$

$$dc = V_w - v = \frac{50}{\tan 40} = 59{\cdot}6$$

Then, $v = 86{\cdot}6 - 59{\cdot}6 = 27$ ft. per sec.

Also, $\frac{v}{v_1} = \frac{r}{r_1} = 2$

Therefore, $v_1 = \frac{27}{2} = 13{\cdot}5$ ft. per sec.

Consider the triangle at outlet.

$$h_f = \frac{14}{\tan 35} = 20$$

$$= v_1 + V_{w_1}$$

And $V_{w_1} = 20 - 13{\cdot}5 = 6{\cdot}5$ ft. per sec.

and is negative, as it is against the direction of motion of the wheel.

From Equation (1)
work done per lb. of water

$$= \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g}$$

$$= \frac{(86{\cdot}6 \times 27) - (-6{\cdot}5 \times 13{\cdot}5)}{32{\cdot}2}$$

$$= 74{\cdot}8 \text{ ft. lb.}$$

$$\omega = \frac{v}{r} = \frac{27}{2} = 13{\cdot}5 \text{ radians per sec.}$$

Speed $$= \frac{13{\cdot}5 \times 60}{2\pi}$$

$$= 129 \text{ revs. per min.}$$

Efficiency $$= \frac{\text{work done}}{\text{kinetic energy supplied}}$$

$$= \frac{74{\cdot}8}{\frac{V^2}{2g}} = \frac{74{\cdot}8 \times 64{\cdot}4}{100^2}$$

$$= 48{\cdot}2 \text{ per cent.}$$

111. **Propulsion of Ships by Jet.** A ship may be driven through the water by the reaction of a jet of water issuing from the back or stern of the ship. The water is pumped into a tank carried by the ship; the whole of the pressure head in the tank is converted into velocity head as it flows from the ship's stern.

Let v = velocity of ship in feet per second
V = absolute velocity of issuing jet
V_r = relative velocity between jet and ship

Then $V_r = v + V$ . . . . . . (1)

Head of water in tank $= \dfrac{V_r^2}{2g}$

= Energy supplied per lb. of water

Weight of water issuing from orifice $= W = w\,a\,V_r$
where a is the area of jet.

The momentum of the issuing jet relative to the surrounding water is $\dfrac{WV}{g}$ per second. This will be the change of momentum as the water had no momentum before entering the ship.

Therefore,

force propelling ship $= \dfrac{WV}{g}$ lb.

Work done per sec. $= \dfrac{WV}{g} \times v$ ft. lb.

Substituting for V from Equation (1)

Work done per sec. $= \dfrac{W(V_r - v)v}{g}$ ft. lb.

Energy supplied per sec. $= \dfrac{W\,V_r^2}{2g}$

$$\text{Efficiency} = e = \frac{\dfrac{W(V_r - v)v}{g}}{\dfrac{W\,V_r^2}{2g}} = \frac{2\,(V_r - v)v}{V_r^2}$$

Differentiating and equating to zero for a maximum,

$$\frac{de}{dv} = V_r - 2v = 0$$

from which $v = \dfrac{V_r}{2}$

Then,

$$\text{maximum efficiency} = \frac{2\,(2v - v)v}{(2v)^2}$$

$$= 50 \text{ per cent.}$$

If the entrance to the inlet pipe of the pump is facing the direction of motion of the ship, the water will enter the pipe with a velocity v relative to the ship. This will reduce the energy to be supplied by the pump by the amount $\dfrac{W\,v^2}{2g}$.

Then, energy supplied $= \dfrac{W}{2g}(V_r^2 - v^2)$ ft. lb. per sec.

The work done by issuing jet is the same as before.

The efficiency now equals $\dfrac{\dfrac{W(V_r - v)v}{g}}{\dfrac{W}{2g}(V_r^2 - v^2)}$

$$\text{Or, efficiency} = \frac{2(V_r - v)v}{(V_r + v)\ (V_r - v)}$$

$$= \frac{2v}{V_r + v} \quad . \quad . \quad . \quad . \quad (2)$$

If $v = \frac{V_r}{2}$ which was the value for maximum efficiency in the first case,

$$\text{efficiency} = \frac{2v}{2v + v} = 66{\cdot}6 \text{ per cent.}$$

It will be noticed from Equation (2) that the efficiency equals unity when $V_r = v$. This would be impossible in practice as there would then be no propelling force on the ship.

The main advantage of jet propulsion is that it overcomes the difficulty of the racing of the propeller in rough seas. Several naval ships have been fitted with jet propulsion as an experiment. It was found that although the actual jet was more efficient than the screw propeller, the mechanical efficiency of the pumps reduced the overall efficiency of the whole plant to a much lower value than that of the screw propeller plant. Jet propulsion is sometimes used in lifeboats.

It should be noticed that by fitting the entrance of the suction pipe to face the direction of the ship's motion, a vacuum pressure is produced in front of the pipe entrance which causes an increased resistance to the ship. If the suction pipe entrance is placed at a suitable position at the side of the ship, the effect of the suction may act on the boundary layer and prevent break-away (Art. 216) ; this should reduce the ship's resistance.

EXAMPLE.

A jet-propelled boat has a velocity of 12 miles an hour when the jet has a velocity of 35 ft. per sec. relative to the boat. If the area of the jet is 25 sq. in., find the brake horse-power required to work the pumps. Assume that full advantage of the boat's motion is obtained when scooping in the water.

$$\text{Weight of water discharged per sec.} = w\, a\, V_r$$

$$= 62{\cdot}4 \times \frac{25}{144} \times 35$$

$$= 379 \text{ lb.}$$

$$v = \frac{12 \times 88}{60} = 17{\cdot}6 \text{ ft./sec.}$$

$$\text{Work done by pumps per sec.} = \frac{W(V_r^2 - v^2)}{2g}$$

$$= \frac{379}{64{\cdot}4}(35^2 - 17{\cdot}6^2)$$

$$= 5380 \text{ ft. lb.}$$

$$\text{Horse-power} = \frac{5380}{550}$$

$$= 9{\cdot}8.$$

EXAMPLES 9.

(1) A jet of water 2 in. diameter, having a velocity of 60 ft. per sec., impinges normally on a flat plate. Find the pressure on the plate (1) when the plate is at rest; (2) when the plate is moving in the same direction as the jet with a velocity of 20 ft. per sec. Find, also, the work done per second in the second case.

Ans.—(1) 152 lb.; (2) 67·5 lb. 1,350 ft. lb.

(2) A 3-in. diameter jet, having a velocity of 80 ft. per sec., strikes a flat plate, the normal of which is inclined at 30° to the jet. Find the normal pressure on plate (1) when plate is stationary; (2) when plate has a velocity of 40 ft. per sec. away from jet.

Ans.—(1) 526 lb.; (2) 127 lb.

(3) A jet of water impinges on a series of hemispherical cups and is deflected through 180°. If the velocity of the jet is 100 ft. per sec., and that of the cups 40 ft. per sec., find the work done per pound of water striking the cups.

Ans.—149 ft. lb.

(4) A jet of water having a velocity of 100 ft. per sec. impinges on a series of vanes moving with a velocity of 50 ft. per sec. The jet makes an angle of 30° to the direction of motion of the vanes when entering, and leaves at an angle of 120°. Draw the triangle of velocities for inlet and outlet and find (1) the angles of the vane tips so that the water enters and leaves without shock; (2) the work done per pound of water entering the vanes; and (3) the efficiency.

Ans.—(1) 53°, 15½°; (2) 149 ft. lb.; (3) 96 per cent.

(5) Water flows inwards over a series of curved vanes which are fixed to the rim of a revolving wheel. The outer diameter of the vanes is 4 ft. and the inner diameter 2 ft. The angle between the jet and the wheel tangent at inlet is 30°, and the water leaves the wheel with a velocity of 10 ft. per sec. at an angle of 120° to wheel tangent. Draw the velocity triangles at inlet and outlet, and find the best angles of the blades and the work done per pound of water if the jet has a velocity of 120 ft. per sec. and the wheel makes 300 revs. per min.

Ans.—55°; 14°; 208 ft. lb.

(6) A vessel provided with a jet propeller is driven at a speed of v ft. per sec. The water is discharged astern with a relative exit velocity of V ft. per sec. and the total jet area is A sq. ft. Find in terms of these quantities (*a*) the propelling force on the vessel; (*b*) the energy expended by the jet in propulsion; (*c*) the efficiency of the jet. State what conclusion can be drawn from these results, and why a vessel can be more efficiently driven by means of a screw propeller. In a jet-propelled vessel the water is discharged through two 9-in. orifices. The jet efficiency is 73 per cent, and the combined efficiency of the engine and pumps 45 per cent. Find the indicated horse-power required to drive the vessel at 13 knots. [Weight of sea water, 64 lb. per cu. ft.; 1 knot = 1·69 ft. per sec.] (London Univ.)

Ans.—133·2 h.p.

(7) A 42-in. pipe is deflected through 90°, the ends being anchored by tie rods at right angles to the pipe at the ends of the bend. If the pipe is delivering 63 cu. ft. per sec., find the tension in each tie rod. (London Univ.)

Ans.—796 lb.

(8) A jet of water having a velocity of 50 ft. per sec., and making an angle of 45° with the horizontal impinges on a vane moving horizontally with a velocity of 25 ft. per sec. Find the shape of vane to give the best results and the angles at the entering and leaving tips.

Find the horizontal pressure on the vane per pound of water striking per second. (London Univ.)

Ans.—73·6°; 0; 1·47 lb.

(9) A square plate weighing 28 lb., and of uniform thickness and 12 in. edge, is hung so that it can swing freely about the upper horizontal edge. A horizontal jet ¾ in. diameter and having a velocity of 50 ft. per sec. impinges on the plate. The centre line of the jet is 6 in. below the upper edge of the plate, and when the plate is vertical the jet strikes the plate normally and at its centre. Find what force must be applied at the lower edge of the plate in order to keep the plate vertical.

If the plate is allowed to swing freely, find the inclination to the vertical which the plate will assume under the action of the jet. (London Univ.)

Ans.—7·425 lb.; 32°.

(10) A motor-boat with jet propulsion draws 10 cu. ft. per sec. through orifices amidships and discharges it astern through orifices having an effective area of ·5 sq. ft. If the boat travels at 10 miles per hour, find the propelling force. (A.M.I. Mech. E.)

Ans.—103 lb.

(11) A circular jet of water delivers 2 cu. ft. per sec. with a velocity of 80 ft. per sec., and impinges tangentially on a vane moving in the direction of the jet with a velocity of 40 ft. per sec. The vane is so shaped that, if stationary, it would deflect the jet through an angle of 45°. Through what angle will it deflect the jet? What driving force will be exerted on the vane in its direction of motion? (A.M.I. Mech. E.)

Ans.—22·5°; 45·5 lb.

(12) A tank from which water is discharging under a constant head H, is mounted on frictionless wheels, so that the direction of motion is opposite to that of the jet, which issues from an orifice A sq. ft. in area in one side. What force in pounds applied horizontally would just prevent movement of the tank? If the tank moved with velocity v, and the jet issued with velocity V, what would then be the force causing motion, the work done per second. and the efficiency? For maximum efficiency, what will be the ratio of V to v? (A.M.I. Civil E.)

(13) A free jet, whose sectional area is 3 sq. in., and whose velocity is 80 ft. per sec., impinges tangentially on a smooth vane which diverts its direction through 120°. What is the magnitude and direction of the resultant force on the vane. (A.M.I. Mech. E.)

Ans.—448 lb. at 30°.

(14) A vessel is propelled by the reaction of jets discharged astern, the water being drawn in initially at the side. Establish expressions for the theoretical efficiency and for the input of power to the pumps in terms of the speed of the ship, s; the velocity through the jets, v; the weight of water pumped per sec., W, and the combined efficiency of the pump and pipe system, η.

In an actual case a small ship is fitted with jets of total outlet area 7 sq. ft. The velocity through the jets is 30 ft./sec. and the ship speed is 10 knots. The engine efficiency is 85 per cent, the pump efficiency is 65 per cent, and the pipe losses may be taken as equivalent to 10 per cent of the kinetic energy at the jets. Determine the propelling force and the overall efficiency. (1 knot = 6,080 ft./hr.; sea water, 64 lb./cu. ft. (London Univ.)

Ans.—$\dfrac{2(v-s)s}{v^2}$; $\dfrac{Wv^2}{2g \times 550\eta}$; 548 lb.; 24·43 per cent.

CHAPTER X

WATER TURBINES

112. **Classification of Turbines.** Power was formerly obtained from water by means of water wheels which were revolved either by the weight of the water or by the impulse of the stream. Such wheels are now obsolete and have been replaced by the water turbine.

The rotation of the turbine wheel or runner is caused by water flowing over curved vanes fixed to the rim. The action of these curved blades is to change the velocity of the water, both in magnitude and direction. The impulse given to the wheel is entirely due to this change of velocity of the water flowing through it. Actually, the force tending to rotate the wheel is due to the centrifugal force of the water as it passes over the curved vane. In principle, it is the same as the outward force on a railway curve due to a train passing round the curve. No rotating force is obtained from the static pressure of the water.

Turbines may be divided into two main classes : (1) reaction or pressure turbines, and (2) impulse or velocity turbines. In the reaction turbine the water enters the wheel under pressure and flows over the vanes. In passing over the vanes the pressure head is converted to velocity head and is finally reduced to atmospheric pressure before leaving the wheel. The water leaves the wheel with a large relative velocity but a small absolute velocity, practically the whole of its original energy having been given to the wheel.

Let H = total head of entering water

and V_1 = velocity of leaving water.

Then, energy given to wheel per pound of water $= H - \dfrac{V_1^2}{2g}$.

In the reaction turbine the total head H consists partly of pressure head and partly of velocity head. As the water is under pressure, the wheel must run full and may, therefore, be entirely submerged below the tail race ; it may also discharge into the atmosphere or it may be placed 30 ft. above the foot of the fall and discharge into a suction or draught tube. The

water must be admitted into a reaction turbine over the whole circumference of the wheel; the power is difficult to regulate without loss.

In the impulse turbine all the energy of the water is converted into velocity before entering the wheel by expanding through a nozzle or guide vanes. The pressure of the water is atmospheric, hence the wheel must not run full; in which case, it must be placed at the foot of the fall and above the tail race. The water may be admitted over part of the circumference only or over the whole circumference.

Let V = velocity of entering water

then, $H = \dfrac{V^2}{2g}$

Energy absorbed by wheel per pound of water $= H - \dfrac{V_1^2}{2g}$

$$= \frac{V^2}{2g} - \frac{V_1^2}{2g}$$

Both types of turbines may be sub-divided into classes based on the direction of flow of the water through the wheel. If the flow of the water is radial the turbine is known as a radial flow turbine and may be an inward flow or an outward flow, depending on whether the water enters at the outer circumference and flows inwards towards the centre, or enters at the centre and flows outwards. If the water flows parallel to the axis of the turbine it is known as an axial flow or parallel flow turbine. In some of the latest types of turbines the flow is partly radial and partly axial; such turbines are known as mixed flow turbines.

113. **Notation.** The following notation will be used for all types of turbines—

V = absolute velocity of entering water
V_1 = absolute velocity of leaving water
v = tangential velocity of wheel at inlet
v_1 = tangential velocity of wheel at outlet
V_r = velocity of water relative to wheel at inlet
V_{r_1} = velocity of water relative to wheel at outlet
V_w = velocity of whirl at inlet (Art. 109)
V_{w_1} = velocity of whirl at outlet

V_f = velocity of flow at inlet (Art. 109)
V_{f_1} = velocity of flow at outlet
r = radius of wheel at inlet
r_1 = radius of wheel at outlet
α = angle entering water makes with wheel's tangent
β = angle leaving water makes with wheel's tangent
θ = angle of blade tip at inlet
ϕ = angle of blade tip at outlet
W = weight of water entering wheel in pounds per second
H = total head of water supplied
e = hydraulic efficiency of turbine
n = number of revolutions per minute
N = number of blades in wheel
t = thickness of blades
b = breadth of wheel at inlet
b_1 = breadth of wheel at outlet

114. **Reaction Turbines.** (*a*) OUTWARD FLOW TURBINE. The outward radial flow turbine consists of a wheel in the shape of a cylindrical disc mounted on a shaft and having blades around the perimeter (Fig. 125). The water flows into the wheel at the centre and passes through fixed radial guide blades into the moving blades. The object of the fixed guide blades is to guide the water into the moving blades at the correct angle α. The water passes through the moving blades, causing them to rotate, and is discharged at the outer edge. The wheel is surrounded by a water-tight casing and may run in a vertical or horizontal position. It may be submerged below the tail race or placed in a suction or draught tube* above the foot of the fall. The latter position is the more convenient, as the wheel is then more accessible. Being a reaction turbine, the water in the wheel is under pressure; the wheel must, therefore, run full.

The flow of water through the wheel may be regulated by a cylindrical sluice gate situated between the moving blades and the guide blades. This is very unsatisfactory owing to the loss of head due to contraction when the gate is partly closed.

* See Fig. 141.

The revolving wheel causes a centrifugal head to be impressed on the water passing through it. This increases the relative velocity of the water in the outward flow type and consequently tends to increase the quantity of water passing through the wheel. If there is a slight increase in speed, the centrifugal head is increased and the wheel tends to race.

The efficiency is increased by discharging the water radially, in which case the velocity of whirl at outlet is zero.

The triangles of velocity for inlet and outlet are shown in Fig. 126.

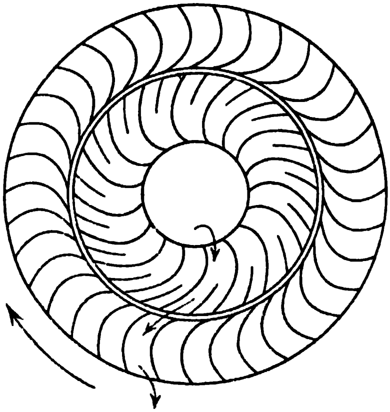


FIG. 125

It should be noted that—

$$V_w = V \cos \alpha$$
$$V_f = V \sin \alpha$$
$$V_r \sin \theta = V \sin \alpha$$
$$V_r \cos \theta = V_w - v$$
$$v = \omega r = \frac{2\pi n r}{60}$$

Also, $\quad \dfrac{v}{v_1} = \dfrac{r}{r_1}$

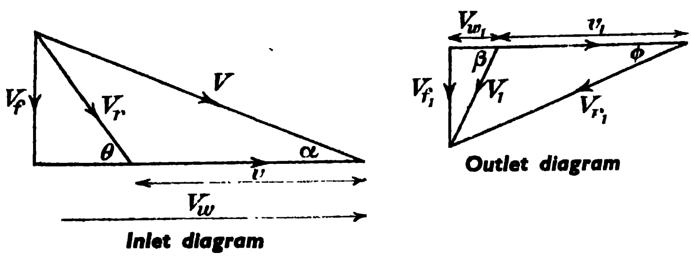


FIG. 126

From velocity diagram at outlet,

$$V_{r_1} \cos \phi = v_1 + V_{w_1}$$
$$V_{w_1} = V_1 \cos \beta$$
$$V_{f_1} = V_1 \sin \beta$$
$$V_1 \sin \beta = V_{r_1} \sin \phi$$

If discharge is radial $\beta = 90°$, then $V_{w_1} = 0$, and $V_1 = V_{f1}$

From Equation (1), Art. 110,

$$\left.\begin{array}{l}\text{Work done on wheel per pound}\\ \quad\text{of water}\end{array}\right\} = \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g}$$

$$\left.\begin{array}{l}\text{Energy lost per pound of water}\\ \quad\text{passing through wheel}\end{array}\right\} = H - \frac{V_1^2}{2g}$$

Therefore,

$$\frac{V_w v}{g} - \frac{V_{w_1} v_1}{g} = H - \frac{V_1^2}{2g} \qquad . \quad . \quad . \quad . \qquad (1)$$

It should be noted that in a reaction turbine H does not equal $\frac{V^2}{2g}$.

$$\text{Hydraulic efficiency} = \frac{H - \frac{V_1^2}{2g}}{H}$$

$$= \frac{V_w v - V_{w_1} v_1}{gH}$$

If the discharge is radial, Equation (1) becomes

$$\frac{V_w v}{g} = H - \frac{V_1^2}{2g}$$

Radial area of flow at inlet $= (2\pi r - Nt)b = k\ 2\pi r\ b$, where k is a factor which allows for area of blades.

$$\left.\begin{array}{l}\text{Volume of water flowing}\\ \text{through wheel per second}\end{array}\right\} = (2\pi r - Nt)b\ V_f$$

Radial area of flow at outlet $= (2\pi r_1 - Nt)b_1 = k_1\ 2\pi r_1\ b_1$, k_1 being the blade factor at outlet.

As quantity of water flowing through wheel at inlet equals quantity flowing at outlet,

$$\frac{V_f}{V_{f_1}} = \frac{(2\pi r_1 - Nt)b_1}{(2\pi r - Nt)b} = \frac{k_1\ 2\pi r_1\ b_1}{k\ 2\pi r\ b}$$

(*b*) Inward Flow Turbine. The inward radial flow reaction turbine is similar in principle to the outward flow, except that the water enters the wheel at the outer periphery and flows radially towards the centre; it then leaves the wheel in a direction parallel to the axis. The fixed guide blades surround the revolving blades externally, and the whole

is surrounded by an outer casing. The centrifugal head impressed on the water by the revolving wheel is now acting against the radial flow of the water, so that any increase in speed of the wheel will tend to reduce the quantity of flow through the wheel, and consequently reduce the power. This is an advantage, as it tends to prevent racing. The wheel may be placed below the level of the tail race or in a suction tube above the foot of the fall. The highest efficiency is obtained when the discharge is radial and when the velocity of the leaving water is as small as possible.

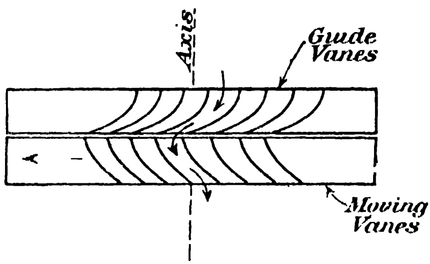


FIG. 127

The method of solution and the equations for an inward flow turbine are the same as given for the outward flow turbine.

(c) AXIAL FLOW TURBINE. In this type of reaction turbine the water enters the wheel at the side and flows parallel to the axis (Fig. 107). It is sometimes known as a parallel flow turbine. The diagrams of velocity and equations for this type of turbine are the same as for the radial flow types, except that the radius of flow is now constant. Therefore,

$$v = v_1, \text{ and } V_f = V_f'$$

Then,

$$\text{work done per pound of water} = \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g}$$

$$= \frac{v(V_w - V_{w_1})}{g}$$

It is usual for the water to leave in a direction parallel to the axis.

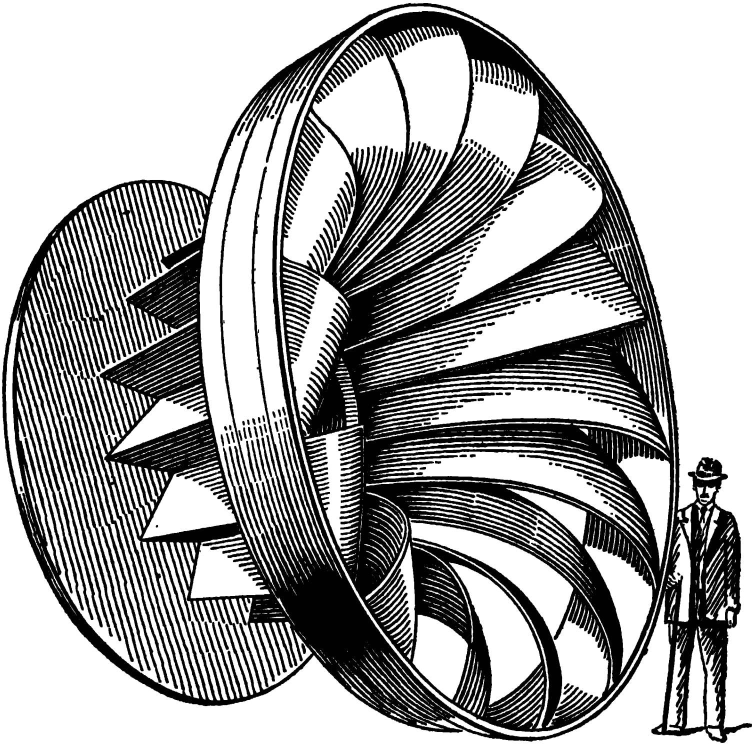
(d) MIXED FLOW TURBINE. The mixed flow reaction turbine is a combination of the inward radial flow and the axial flow, and is obtained by curving the blades in two directions. The type of blades for this turbine is shown in Fig. 128.* The water flows into the wheel radially and leaves at the centre axially.

* By courtesy of Messrs. Boving & Co.

Example 1.

Determine the hydraulic efficiency of a low head inward flow reaction turbine in which the guide blades make angles of 25° with the tangents to the blade circle and the receiving tips of the runners are inclined 105° to the tangents. The discharge is radial, the velocity of flow constant, and the water passes on to the moving blades without shock.

Calculate the velocity of flow if the supply head is 15 ft. (London Univ.)



(*Boving*)

Fig. 128.—Runner of Mixed Flow Reaction Turbines

Referring to Fig. 126,

$$\alpha = 25°$$
$$\theta = 105°$$
$$\beta = 90°$$
$$V_f = V \sin 25 = \cdot 4226\ V$$
$$V_w = V \cos 25 = \cdot 9063\ V$$
$$v = \cdot 9063\ V - \frac{\cdot 4226\ V}{\tan 105°} = 1\cdot 0195\ V$$

As $\qquad V_{w_1} = 0,$

$$\text{Work done} = \frac{V_w v}{g} = \frac{\cdot 9063\,V \times 1\cdot 0195\,V}{g}$$

$$= \frac{\cdot 924\,V^2}{g}$$

$$\text{Energy rejected} = \frac{V_1^2}{2g} = \frac{V_f^2}{2g} - \frac{(\cdot 4226\,V)^2}{2g}$$

$$= \frac{\cdot 0895\,V^2}{g}$$

$$\text{Efficiency} = \frac{\dfrac{\cdot 924\,V^2}{g}}{\dfrac{\cdot 924\,V^2}{g} + \dfrac{\cdot 0895\,V^2}{g}} = \frac{\cdot 924}{\cdot 924 + \cdot 0895}$$

$$= \frac{\cdot 924}{1\cdot 0135} = 91\cdot 15 \text{ per cent}$$

$$H = \frac{V_w\,v}{g} + \frac{V_f^2}{2g}$$

Or, $$15 = \frac{1\cdot 0135\,V^2}{g}$$

Then, $$V = \sqrt{\frac{15 \times 32\cdot 2}{1\cdot 0135}}$$

$$= 21\cdot 8 \text{ ft. per sec.}$$

and, $$V_f = \cdot 4226\,V$$
$$= \cdot 4226 \times 21\cdot 8$$
$$= 9\cdot 24 \text{ ft. per sec.}$$

EXAMPLE 2.

An inward flow turbine works under a total head of 90 ft. The velocity of the wheel periphery at inlet is 50 ft. per sec. The outlet pipe of the turbine is 1 ft. diameter, and the turbine is supplied with 50 gallons of water per second. The radial velocity of flow through the wheel is the same as the velocity in the outlet pipe.

Neglecting friction, determine (*a*) the vane angle at inlet; (*b*) the guide blade angle; (*c*) the horse-power of the turbine. (London Univ.)

As $V_1 = V_{f_1}$ the discharge is radial.

Then, $\qquad V_{w_1} = 0.$

Assume the turbine to be a reaction turbine.

$$V_f = V_1 \quad \frac{\text{discharge}}{\text{pipe area}} \quad \frac{50}{6{\cdot}24 \times {\cdot}785} = 10{\cdot}21 \text{ ft. per sec.}$$

$$\frac{V_w v}{g} = H - \frac{V_f^2}{2g}$$

Then, $$\frac{V_w 50}{32{\cdot}2} = 90 - \frac{(10{\cdot}21)^2}{64{\cdot}4}$$

and, $$V_w = 56{\cdot}9 \text{ ft. per sec.}$$

The diagram of velocity at inlet may now be drawn to scale and the values of θ and α measured. Or, they may be calculated from Fig. 126 as follows,

$$\tan \alpha = \frac{V_f}{V_w} = \frac{10{\cdot}21}{56{\cdot}9} = {\cdot}1795$$

Then, $$\alpha = 10{\cdot}2^\circ$$

$$\tan \theta = \frac{V_f}{V_w - v} = \frac{10{\cdot}21}{6{\cdot}9} = 1{\cdot}47$$

Then, $$\theta = 55{\cdot}8^\circ.$$

Work done per pound of water $= \dfrac{V_w v}{g}$

$$= \frac{56{\cdot}9 \times 50}{32{\cdot}2} = 88{\cdot}38 \text{ ft. lb.}$$

$$\text{H.P.} = \frac{W \times \text{work done}}{550}$$

$$= \frac{50 \times 10 \times 88{\cdot}38}{550} = 80{\cdot}2$$

EXAMPLE 3.

An inward flow reaction turbine is supplied with 21 cu. ft. of water per second under a head of 50 ft. It develops 100 h.p. at 375 revs. per min.; the inner and outer diameters of the wheel are 20 in. and 30 in. respectively. The velocity of the water at exit is 10 ft. per sec., and it leaves the wheel radially. Determine the actual and theoretical hydraulic efficiencies of the wheel.

If the actual hydraulic efficiency of the wheel were 84 per cent, find the most suitable angles for the guide and wheel vanes at inlet. Assume the width of wheel constant. (London Univ.)

$$v = \frac{30}{12} \times \pi \times \frac{375}{60} = 49{\cdot}1 \text{ ft. per sec.}$$

$$v_1 = 49{\cdot}1 \times \frac{20}{30} = 32{\cdot}7 \text{ ft. per sec.}$$

As width of wheel is constant,

$$\frac{V_f}{V_{f_1}} = \frac{r_1}{r}$$

Then, $V_f = 10 \times \frac{20}{30} = 6{\cdot}67$ ft. per sec.

As discharge is radial,

$$V_1 = V_{f_1}$$

Theoretical work done per pound of water $= H - \frac{V_1^2}{2g}$

$$= 50 - \frac{(10)^2}{2g} = 48{\cdot}45 \text{ ft. lb}$$

Theoretical hydraulic efficiency $= \frac{48{\cdot}45}{50} = 96{\cdot}9$ per cent

Actual work done per pound of water $= \frac{\text{H.P.} \times 550}{W}$

$$= \frac{100 \times 550}{21 \times 62{\cdot}4} = 42 \text{ ft. lb.}$$

Actual efficiency $= \frac{42}{50} = 84$ per cent

As discharge is radial,

theoretical work done per pound $= \frac{V_w v}{g} = 48{\cdot}45$

Then, $V_w = \frac{48{\cdot}45 \times 32{\cdot}2}{49{\cdot}1}$

$$= 31{\cdot}8 \text{ ft. per sec.}$$

The values of α and θ may be found by drawing the velocity diagram at inlet to scale, or,

$$\tan \alpha = \frac{V_f}{V_w} = \frac{6{\cdot}67}{31{\cdot}8} = {\cdot}2097$$

Hence, $\alpha = 11{\cdot}9°$

$$\tan (180 - \theta) = \frac{V_f}{v - V_w} = \frac{6{\cdot}67}{(49{\cdot}1 - 31{\cdot}8)} = {\cdot}385$$

Hence, $\theta = 158{\cdot}9°$.

Example 4.

An outward flow reaction turbine has a speed of 200 revs. per min., and a constant breadth of 9 in. The diameter of the wheel at inlet and outlet are 5 ft. and 6 ft. respectively. The wheel works under a total head of 120 ft. and the quantity of water passing through the wheel is 200 cu. ft. per sec. If the hydraulic efficiency is 90 per cent, find the angles of the blades and guide vanes.

Work done per pound of water $= H - \frac{V_1^2}{2g} = \cdot 9H$

From which, $V_1 = \sqrt{\cdot 1 \times 64\cdot 4 \times 120}$

$= 27\cdot 8$ ft. per sec.

$$v = \pi d \frac{n}{60}$$

$$= \pi \times 5 \times \frac{200}{60} = 52\cdot 4 \text{ ft. per sec.}$$

$$v_1 = 52\cdot 4 \times \frac{6}{5} = 62\cdot 8 \text{ ft. per sec.}$$

$$V_f = \frac{\text{Quantity per second}}{\text{radial area of flow}} = \frac{200}{\pi d \times \frac{9}{12}}$$

$$= \frac{200}{\pi \times 5 \times \cdot 75} = 17 \text{ ft. per sec.}$$

$$V_{f_1} = 17 \times \frac{5}{6} = 14\cdot 2 \text{ ft. per sec.}$$

Referring to outlet diagram of Fig. 126,

$$\sin \beta = \frac{V_{f_1}}{V_1} = \frac{14\cdot 2}{27\cdot 8} = \cdot 511$$

Hence, $\beta = 30\cdot 5°$

$V_w = V_1 \cos \beta = 27\cdot 8 \cos 30\cdot 5$

$= 23\cdot 9$ ft. per sec.

$$\tan \phi = \frac{V_{f_1}}{v_1 + V_{w_1}} = \frac{14\cdot 2}{62\cdot 8 + 23\cdot 9} = \cdot 164$$

Then, $\phi = 9\frac{1}{2}°$

$$\frac{V_w v}{g} - \frac{V_{w_1} v_1}{g} = eH$$

That is, $\frac{V_w 52\cdot 4}{32\cdot 2} + \frac{(23\cdot 9 \times 62\cdot 8)}{32\cdot 2} = \cdot 9 \times 120$

From which, $V_w = 37\cdot 4$ ft. per sec.

Referring to inlet diagram of Fig. 126,

$$\tan\alpha = \frac{V_f}{V_w} = \frac{17}{37\cdot4} = \cdot455$$

Then, $\alpha = 24\cdot5°$

$$\tan\theta = \frac{V_f}{V_w - v} = \frac{17}{37\cdot4 - 52\cdot4} = -1\cdot132$$

Therefore, $\theta = 131\cdot5°$

115. **Impulse Turbines.** The problems dealing with impulse turbines may be solved in a similar way to the reaction turbine problems, but the following points should be noted—

1. The turbine must not run full; the pressure is atmospheric throughout.

2. The total head is converted to velocity before entering the wheel. Then $V = \sqrt{2gH}$. This is sometimes stated as $V = k\sqrt{2gH}$, where k is a coefficient which takes into account losses in the nozzle or guide vanes.

3. As $V = \sqrt{2gH}$, the hydraulic efficiency may be stated as $\dfrac{V^2 - V_1^2}{V^2}$.

4. The velocities of flow V_f and V_{f_1} depend on the radial area and on the amount the wheel is full.

(*a*) Radial Flow Turbine. The flow may be inwards or outwards. The water enters the wheel through fixed guide blades as in the reaction turbine. As the water flows over the moving vanes, a centrifugal head is impressed on it by the revolving wheel, which is immediately converted to velocity head. This increases the relative velocity of the water in an outward flow and decreases it in an inward flow. The centrifugal head given to the water was proved in Art. 30 to be

$$\frac{v^2}{2g} - \frac{v_1^2}{2g}$$

Then,
$$\frac{V_r^2}{2g} = \frac{V_{r_1}^2}{2g} + \left(\frac{v^2}{2g} - \frac{v_1^2}{2g}\right) \quad . \quad . \quad . \quad . \quad (1)$$

If the flow is inward, the relative velocity is thus reduced by the centrifugal force. This makes the speed of the inward flow turbine easier to control than that of the outward flow.

A small increase of speed of the wheel due to a temporary lightening of the load, increases the centrifugal force, which decreases the flow through the wheel and consequently decreases the power. The wheel thus tends to automatically adjust itself to the load. With the outward flow turbine, the centrifugal force increases the flow and the wheel tends to race.

The radial flow impulse turbine is not suitable for very low falls, as the wheel must be placed above the foot of the fall in order that it does not run full. A certain amount of the fall is thus lost. In a high fall this amount is not noticeable. The efficiency is greatest when V_1 is as small as possible.

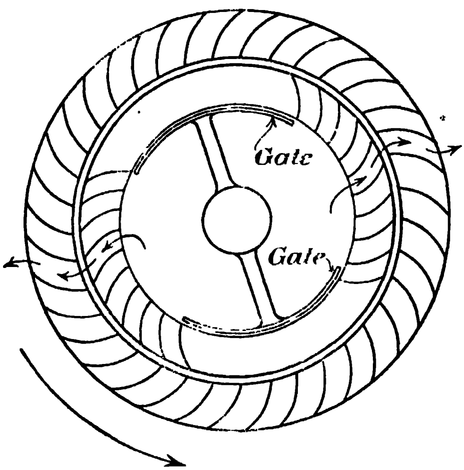


FIG. 129

In an impulse turbine it is possible to regulate the flow through the wheel without loss. The water need not be admitted over the whole circumference, as the pressure is atmospheric. The flow may therefore be regulated by means of a revolving gate (Fig. 129), which, when turned, completely shuts off the flow in the vanes covered by it, without interfering with the flow in the remaining vanes.

This method of regulating the flow is used in the Girard impulse turbine. In this type, the water is admitted to two opposite quadrants of the wheel when the revolving gate is fully open, the remaining two quadrants being covered by the gate.

(*b*) AXIAL FLOW TURBINE. The same conditions governing the radial flow impulse turbine apply to the axial flow impulse turbine. Except that in this type there is no centrifugal head impressed on the water as $v = v_1$; therefore the relative velocity is constant.

Or, $$V_r = V_{r_1}$$

The maximum efficiency occurs when V_1 is as small as possible.

The chief types of axial flow impulse turbines are the Girard and the Pelton wheel. The latter type differs from the ordinary turbine and is dealt with separately in Art. 117.

EXAMPLE 1.

The mean blade circle diameter of the runner of an axial flow impulse turbine of the Girard type is 4½ ft. The guide blade angle is 24°, the receiving and discharging angles of the runner blades being 48° and 23° respectively. The breadth of the moving blades at inlet is 4 in.

Calculate the speed of the turbine so that the water may pass smoothly on to the blades when the turbine is working under a head of 280 ft., and find the horse-power developed if, with full circumferential admission, the passages are 85 per cent full at inlet. (London Univ.)

The velocity diagrams are shown in Fig. 130.

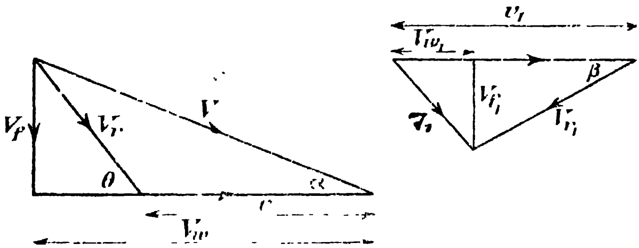


FIG. 130

$$V = \sqrt{2gH} = \sqrt{64 \cdot 4 \times 280} = 134 \text{ ft. per sec.}$$

$$V_f = V \sin 24^\circ = 134 \times \cdot 4067 = 54 \cdot 5 \text{ ft. per sec.}$$

$$V_w = V \cos 24^\circ = 134 \times \cdot 9135 = 122 \cdot 3 \text{ ft. per sec.}$$

$$v = V_w - \frac{V_f}{\tan 48} = 122 \cdot 3 - \frac{54 \cdot 5}{1 \cdot 1106} = 73 \cdot 2 \text{ ft. per sec.}$$

$v_1 = v$, as the turbine is an axial flow

$$v = \pi d \frac{n}{60}$$

That is, $\quad 73 \cdot 2 = \pi \times 4 \cdot 5 \times \frac{n}{60}$

From which, $\quad n = 311$ revs. per min.

$$V_r = \frac{V_f}{\sin 48} = \frac{54.5}{\cdot 7431} = 73 \cdot 2 \text{ ft. per sec.}$$

$$V_{r_1} = V_r = 73 \cdot 2 \text{ ft. per sec.}$$

as the turbine is an axial flow impulse.

$$V_{w_1} = v_1 - V_{r_1} \cos 23$$
$$= 73{\cdot}2 - (73{\cdot}2 \times {\cdot}9205) = 5{\cdot}8 \text{ ft. per sec.}$$

$$\text{Work done per pound} = \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g}$$
$$= \frac{(122{\cdot}3 \times 73{\cdot}2)}{32{\cdot}2} - \frac{(5{\cdot}8 \times 73{\cdot}2)}{32{\cdot}2}$$
$$= 265 \text{ ft. lb.}$$

$$\text{Quantity of water per second} = b \times \pi d \times V_f \times {\cdot}85$$
$$= \frac{4}{12} \times \pi \times 4{\cdot}5 \times 54{\cdot}5 \times {\cdot}85$$
$$= 218 \text{ cu. ft. per sec.}$$

$$\text{Horse-power} = \frac{W \times \text{work done per lb.}}{550}$$
$$\frac{218 \times 62{\cdot}4 \times 265}{550}$$
$$= 6550$$

EXAMPLE 2.

In an outward-flow impulse turbine the available head is 81 ft. The rim speed of the wheel at inlet is $\cdot 4\sqrt{2gH}$, guide vane angle and wheel vane angle at outlet are 20°, inlet radius ·75 ft., outlet radius 1 ft. The velocity of the water in the guides is 95 per cent of the theoretical velocity due to total head. The losses in the wheel to be taken as 6 per cent of total head. Find the hydraulic efficiency.

If the depth of the guides is ·25 ft. what would be the horse-power of the turbine if used with admission over one-quarter of the circumference, allowing 10 per cent loss of area due to vanes? (London Univ.)

Referring to velocity diagrams of Fig. 126,

$$v = {\cdot}4\sqrt{2gH} = {\cdot}4\sqrt{64{\cdot}4 \times 81} = 28{\cdot}95 \text{ ft. per sec.}$$
$$v_1 = \frac{vr_1}{r} = 28{\cdot}95 \times \frac{1}{{\cdot}75} = 38{\cdot}6 \text{ ft. per sec.}$$
$$V = {\cdot}95\sqrt{2gH} = {\cdot}95\sqrt{64{\cdot}4 \times 81} = 68{\cdot}6 \text{ ft. per sec.}$$
$$V_w = V \cos 20 = 68{\cdot}6 \times {\cdot}9397 = 64{\cdot}5 \text{ ft. per sec.}$$
$$V_f = V \sin 20 = 68{\cdot}6 \times {\cdot}342 = 23{\cdot}45 \text{ ft. per sec.}$$
$$\tan\theta = \frac{V_f}{V_w - v} = \frac{23{\cdot}45}{64{\cdot}5 - 28{\cdot}95} = {\cdot}66$$

Then, $\theta = 33{\cdot}4°$

$$V_r = \frac{V_f}{\sin\theta} = \frac{23{\cdot}45}{{\cdot}5505} = 42{\cdot}6 \text{ ft. per sec.}$$

Using Equation (1), Art. 115, and allowing for 6 per cent of total head loss in vanes,

$$\frac{V_{r_1}^2}{2g} = \frac{V_r^2}{2g} - \left(\frac{v^2}{2g} - \frac{v_1^2}{2g}\right) - \cdot 06H$$

That is, $V_{r_1}^2 = 42 \cdot 6^2 - (28 \cdot 95^2 - 38 \cdot 6^2) - (\cdot 06 \times 2g \times 81)$

From which, $V_{r_1} = 46 \cdot 5$ ft. per sec.

$$V_{w_1} = V_{r_1} \cos 20 - v_1$$

$$= (46 \cdot 5 \times \cdot 9397) - 38 \cdot 6$$

$$= 5 \text{ ft. per sec.}$$

$$\text{Work done per pound} = \frac{V_w v}{g} + \frac{V_{w_1} v_1}{g}$$

$$= \frac{(64 \cdot 5 \times 28 \cdot 95)}{32 \cdot 2} + \frac{(5 \times 38 \cdot 6)}{32 \cdot 2}$$

$$= 64 \text{ ft. lb.}$$

$$e = \frac{\text{Work done per pound}}{\frac{V^2}{2g}} = \frac{64 \times 64 \cdot 4}{(68 \cdot 6)^2}$$

$$= 87 \cdot 5 \text{ per cent}$$

$$\left.\begin{array}{c}\text{Radial area of flow} \\ \text{at inlet}\end{array}\right\} = \pi\, d\, b \times \frac{1}{4} \times \frac{90}{100}$$

$$= \pi \times 1 \cdot 5 \times \cdot 25 \times \frac{1}{4} \times \frac{90}{100}$$

$$= \cdot 265 \text{ sq. ft.}$$

$$\left.\begin{array}{c}\text{Quantity of water} \\ \text{per second}\end{array}\right\} = \cdot 265\, V_f$$

$$= \cdot 265 \times 23 \cdot 45 = 6 \cdot 22 \text{ cu. ft.}$$

$$\text{Horse-power} = \frac{W \times \text{work done per pound}}{550}$$

$$= \frac{6 \cdot 22 \times 62 \cdot 4 \times 64}{550}$$

$$= 45 \cdot 1$$

116. Summary of Equations for Turbine Problems. The following is a tabulated summary of equations and conditions governing all classes of turbines, which will be useful for reference when solving problems on turbines—

	IMPULSE.	REACTION.
Radial and axial flow	$V = \sqrt{2gH}$ Work done per pound $= \dfrac{V_w v}{g} - \dfrac{V_{w_1} v_1}{g}$ $= \dfrac{V^2}{2g} - \dfrac{V_1^2}{2g}$ Hydraulic eff. $= \dfrac{V^2 - V_1^2}{V^2}$ Wheel must not run full. V_f depends on area of flow and on amount full Pressure is atmospheric.	Work done per pound $= \dfrac{V_w v}{g} - \dfrac{V_{w_1} v_1}{g}$ $= H - \dfrac{V_1^2}{2g}$ Hydraulic eff. $= \dfrac{H - \dfrac{V_1^2}{2g}}{H}$ Wheel must run full. V_f depends on area of flow. Pressure varies throughout.
Radial flow only	$\dfrac{v_1}{v} = \dfrac{r_1}{r}$ $\dfrac{V_{r_1}^2}{2g} = \dfrac{V_r^2}{2g} - \left(\dfrac{v^2}{2g} - \dfrac{v_1^2}{2g}\right)$	$\dfrac{v_1}{v} = \dfrac{r_1}{r}$
Axial flow only	$v = v_1$ $V_{r_1} = V_r$	$v = v_1$

117. Types of Turbines. (*a*) BARKER'S MILL OR SCOTCH TURBINE. This is a simple type of reaction turbine which is now obsolete.* It consists of a revolving cylindrical tank having arms through which the water is discharged backwards, as shown in Fig. 131. Problems on this type of turbine may be solved from the ordinary methods applied to reaction turbines. The velocity diagrams for inlet and outlet may be drawn in the same manner as in Fig. 126. It should be noted that the arm corresponds to the moving vane. As the water enters the vane radially and at the centre,

$$\alpha \text{ and } \theta = 90^\circ$$
$$v = 0$$
$$V_f = V_r = V$$

The diagram at inlet thus becomes a radial line as shown in Fig. 132.

* It has been revived as a lawn sprinkler.

Consider the velocity diagram at outlet. As the water leaves tangentially,

$$\beta \text{ and } \phi = 0$$

$$V_{w_1} = V_1$$

$$V_{f_1} = 0$$

The velocity diagram at outlet is thus a horizontal line (Fig. 132), from which

$$V_{r_1} = v_1 + V_1 \quad . \quad . \quad (1)$$

$$\left.\begin{array}{l}\text{Work}\\ \text{done per}\\ \text{pound of}\\ \text{water}\end{array}\right\} = \frac{V_w v}{g} + \frac{V_{w_1} v_1}{g}$$

$$= 0 + \frac{(V_{r_1} - v_1)v_1}{g}$$

(From Eq. 1)

$$= H - \frac{V_1^2}{2g}$$

where H = head of water in tank.

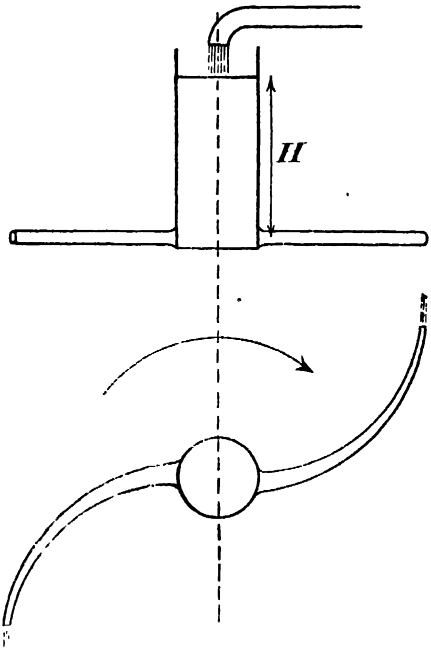


Fig. 131

$$\left.\begin{array}{r}\text{Total energy supplied}\\ \text{per pound of water}\end{array}\right\} = H = \frac{(V_{r_1} - v_1)v_1}{g} + \frac{V_1^2}{2g}$$

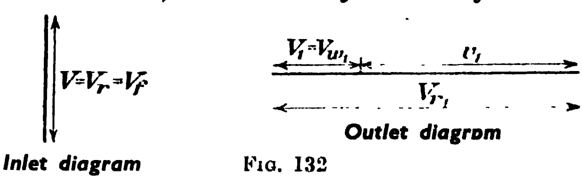


Fig. 132

Substituting for V_1 from Equation 1,

$$H = \frac{(V_{r_1} - v_1)v_1}{g} + \frac{(V_{r_1} - v_1)^2}{2g}$$

$$= \frac{V_{r_1}^2 - v_1^2}{2g}$$

Efficiency

$$= \frac{2(V_{r_1} - v_1)v_1}{V_{r_1}^2 - v_1^2} = \frac{2v_1}{V_{r_1} + v}$$

(*b*) Fourneyron Turbine. This is an outward radial flow reaction type and was the first successful reaction turbine to be made. It has been used for heads of 1 ft. to 360 ft. and has an efficiency of about 75 per cent. It is governed by a cylindrical sluice gate which fits between the moving and fixed blade rings. As throttling the supply in this way causes a loss of head due to sudden contraction, transverse diaphragms are fitted through the blades which divide the wheel into four sections, as shown in Fig. 133. The cylindrical gate may then close one or more sections completely without causing any loss of head.

(*c*) Francis Turbine. The Francis turbine is an inward flow radial reaction type and was the first type of inward

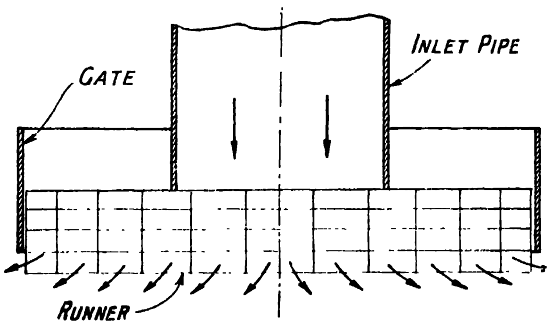


Fig. 133

flow to be constructed. This has the advantage of the centrifugal force acting against the flow, which reduces the tendency to race. The flow is regulated by a sliding cylindrical sluice gate placed inside or outside of the blade ring.

(*d*) Thomson Turbine. This is an inward flow reaction turbine. The turbine wheel is surrounded by an eccentric chamber called a vortex chamber (Fig. 134). The water enters the wheel at the largest part of the chamber and is guided to the moving blades by four pivoted guide blades. The flow may then be regulated by closing up the guide blades, which varies the supply to the whole of the wheel's circumference. No loss of head then occurs, as there is no contraction of section. The width of the wheel is varied, so that the radial area of discharge equals the radial area at inlet; the velocity of flow is, therefore, constant.

(e) Jonval Turbine. The Jonval is an axial flow impulse turbine. The simplest type consists of one horizontal ring of moving blades into which the water is directed by guide vanes placed above. The flow is regulated by a horizontal sluice which closes parts of the wheel.

A later type of Jonval wheel consisted of several concentric rings of moving blades. The power may then be regulated by closing one or more rings completely.

The invention of the suction tube is also due to Jonval.

(f) Girard Turbine. There are two types of Girard turbines, an axial flow and a radial flow; both are impulse

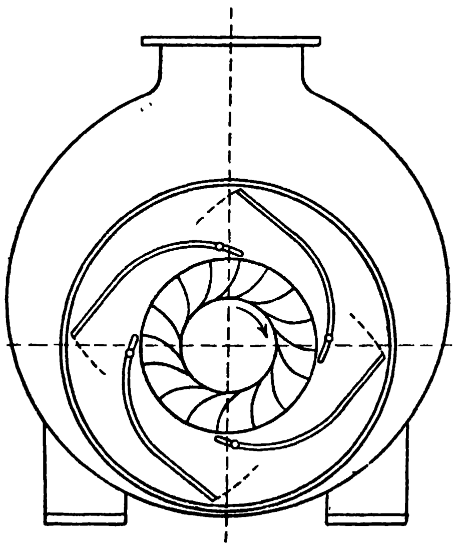


Fig. 134

wheels. They may be used for heads up to 1,700 ft. and have an over-all efficiency of about 75 per cent. The guide passages do not extend over the whole circumference but over two opposite quadrants. The water supply is varied by a sliding circular sluice gate (Fig. 129) which completely shuts off the flow through the vanes it covers. By turning this sluice the flow may be stopped through as many vanes as required. This prevents any loss of head due to contraction when running at "part gate." The wheel of the axial flow is usually placed vertical, the radial flow may be horizontal or vertical.

(*g*) Pelton Wheel. The Pelton wheel is a special type of axial flow impulse turbine and is used for very high heads. It is the most efficient type of impulse wheel, having an overall efficiency of 84 per cent. This type of wheel has been evolved from an earlier type of water wheel used in the mines of California.

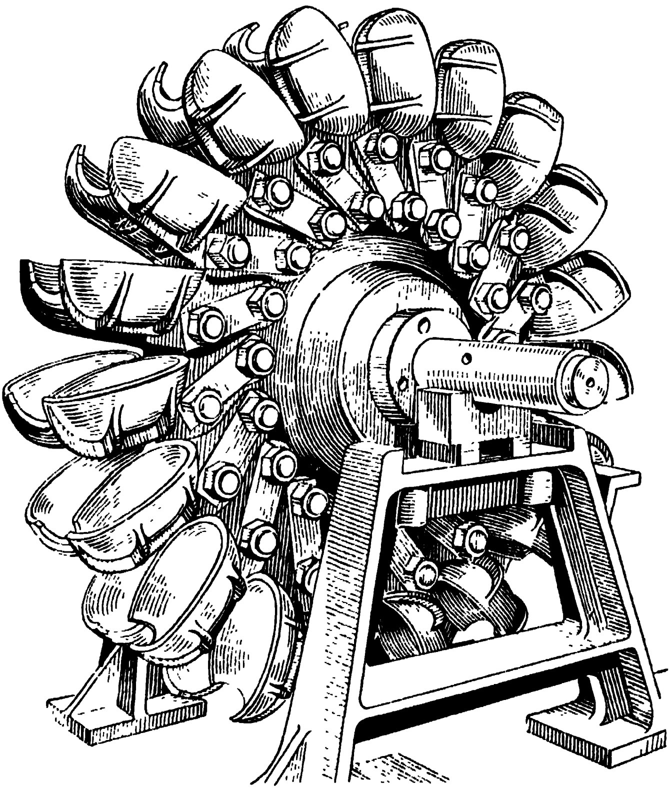


Fig. 135.—Pelton Wheel

The jet impinges on the wheel from one or more nozzles and strikes the blade at the centre (Fig. 135), flowing axially in both directions. The blades are known as buckets and consist of a double hemispherical cup (Fig. 136). As the water flows axially in both directions, there is no axial thrust on the wheel.

The flow of water through the wheel may be regulated by a throttle valve in the supply pipe or by a needle valve in the

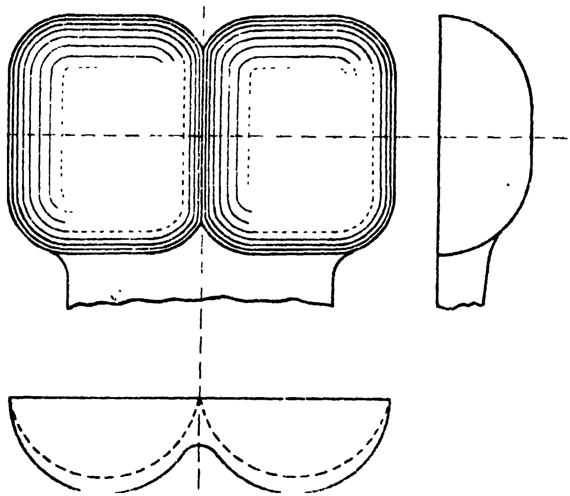


FIG. 136.—PELTON WHEEL BUCKETS (EXTERNAL VIEWS)

nozzle. The buckets are so shaped that the jet is discharged backwards. Usually, the total deflection of the bucket is 160°

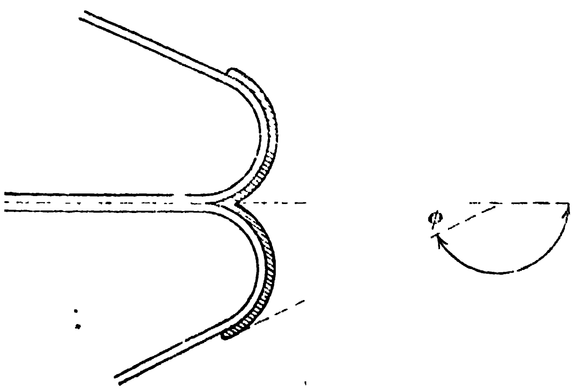


FIG. 137

(Fig. 137). An arrangement of a Pelton wheel, showing nozzles, made by Sir W. G. Armstrong, Whitworth & Co., is shown in Fig. 138.

The work done and efficiency of the Pelton wheel may be obtained from the velocity diagrams as in the case of an ordinary axial flow impulse turbine.

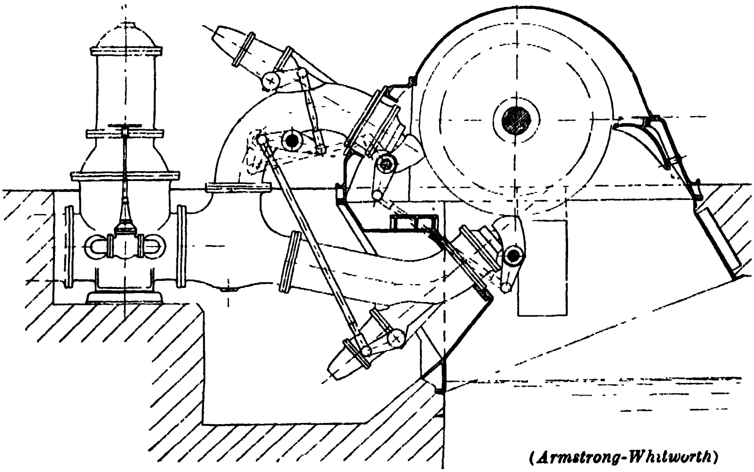


FIG. 138.—ARRANGEMENT OF PELTON WHEEL

For a Pelton wheel,

$$\theta = 0$$

Also, $$\alpha = 0$$

Then, velocity diagram at inlet is a horizontal straight line, as shown in Fig. 139.

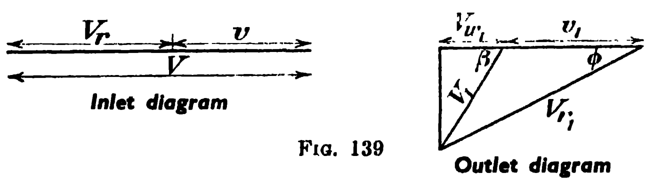


FIG. 139

Hence, $$V_r = V - v$$

And, $$V_w = V = \sqrt{2gH}$$

$$V_f = 0$$

From triangle at outlet (Fig. 139),

$$v_1 = v$$

$$V_{r_1} = V_r = V - v$$

$$V_{w_1} = V_{r_1} \cos \phi - v_1$$

$$= (V - v) \cos \phi - v$$

$$\left.\begin{array}{l}\text{Work done per pound}\\ \text{of water}\end{array}\right\} = E = \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g}$$

$$= \frac{V v}{g} + \frac{[(V - v)\cos\phi - v]v}{g}$$

(as V_{w_1} is negative)

$$= \frac{1}{g}[Vv + v(V - v)\cos\phi - v^2] \quad . \quad (1)$$

$$= eH$$

$$= H - \frac{V_1^2}{2g}$$

$$\text{Efficiency} = e = \frac{\frac{1}{g}[Vv + v(V - v)\cos\phi - v^2]}{\frac{V^2}{2g}}$$

The speed of the wheel for maximum efficiency can be found by differentiating this equation in terms of v and equating to zero.

$$\frac{d.e}{dv} = V + (V - 2v)\cos\phi - 2v = 0$$

From which, $\qquad V(1 + \cos\phi) - 2v(1 + \cos\phi) = 0$

Hence, $\qquad v = \frac{V}{2}$

Therefore, the speed of the wheel for maximum efficiency will be equal to half the speed of the jet.

In practice it is found that the maximum efficiency is when the speed of the wheel is $\cdot 46V$.

Putting $\qquad v = \frac{V}{2}$

$$\text{Maximum efficiency} = \frac{2\left[\frac{V^2}{2} + \frac{V^2}{4}\cos\phi - \frac{V^2}{4}\right]}{V^2}$$

$$= \frac{1}{2}(1 + \cos\phi)$$

When $\phi = 0$, the efficiency is equal to unity.

It will be noticed that the deviation of the jet is $180 - \phi$.

The following rules are used for the proportions of the buckets—

Let $\qquad d$ = diameter of jet

Depth of bucket $= 1\cdot 2d$

Width of bucket $= 5d$

The number of buckets may be obtained by arranging them so that the jet is always completely intercepted by a bucket.

Let R be the mean radius of bucket circle and γ be the angle subtended by two adjacent buckets (Fig. 140). If the jet is to be always intercepted, one bucket will be just about to move out of the jet as another has just moved in.

Let b, c, and e be adjacent buckets. As jet is moving at twice the speed of the buckets, a section of jet will move from c to e

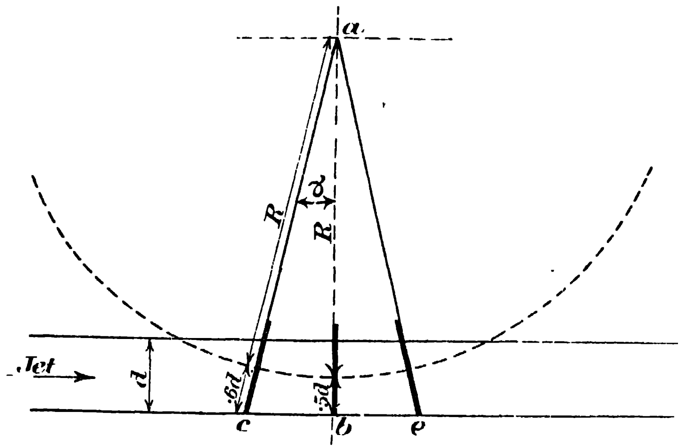


FIG. 140

in same time as bucket b moves to e. Hence, for jet to be always intercepted, the buckets will be as shown in Fig. 140.

Consider triangle abc,

$$ac = R + \tfrac{1}{2} \text{ depth of bucket } = R + \cdot 6d$$

$$ab = R + \tfrac{1}{2} \text{ diameter of jet } = R + \cdot 5d$$

Then, $$\cos \gamma = \frac{R + \cdot 5d}{R + \cdot 6d}$$

From which equation γ is obtained.

Then, number of buckets* $= \dfrac{360}{\gamma}$

Pelton wheels are in use with heads as large as 5,000 ft.

* This equation does not hold in practice as there is not sufficient space around the wheel perimeter for this number of buckets to be inserted. Actually, the number of buckets is about half of that given by the equation.

EXAMPLE 1.

A cup, similar to that in a Pelton wheel, deflects a jet of water through an angle of 120°. Determine the speed of the cup in terms of the velocity of the jet so that the work done by the jet on the cup shall be a maximum and express this work as a percentage of the energy of the jet.

Show how the speed necessary for maximum efficiency would be affected if the friction of the water in passing over the surface of the cup were considerable. (London Univ.)

Referring to Figs. 137 and 138,

$$\phi = 180 - 120 = 60°$$

$$\left.\begin{array}{l}\text{Work done per pound}\\ \text{of water}\end{array}\right\} = E = \frac{V_w v}{g} - \frac{V_{w_1} v_1}{g}$$

$$= \frac{Vv}{g} + \frac{v(V - v)\cos 60 - v^2}{g}$$

Differentiating for a maximum,

$$\frac{dE}{dv} = V + V\cos 60 - 2v\cos 60 - 2v = 0$$

Hence, $$V(1 + \cos 60) - 2v(1 + \cos 60) = 0$$

Therefore, $$v = \frac{V}{2}$$

Then,

$$\left.\begin{array}{l}\text{maximum work done}\\ \text{per pound of water}\end{array}\right\} = \frac{\frac{1}{2}V^2 + \frac{1}{4}V^2\cos 60 - \frac{1}{4}V^2}{g}$$

$$= \frac{\frac{3}{8}V^2}{g}$$

Energy supplied $$= \frac{V^2}{2g}$$

Efficiency $$= \frac{\dfrac{\frac{3}{8}V^2}{g}}{\dfrac{V^2}{2g}} = \cdot 75$$

If there is no friction over the cup, the relative velocity at exit equals relative velocity at entrance. Let friction reduce relative velocity at exit to $k \times V_r$,

Then, relative velocity at exit $= k(V - v)$,

and, $\quad V_{w_1} = k(V - v)\cos 60 - v$

$$\left.\begin{array}{l}\text{Work done per pound}\\ \text{of water}\end{array}\right\} = E = \frac{Vv}{g} + \frac{[k(V - v)\cos 60 - v]v}{g}$$

$$= \frac{Vv}{g} + \frac{vk(V - v)\cos 60 - v^2}{g}$$

Differentiating for a maximum,

$$\frac{d.E}{dv} = V + Vk\cos 60 - 2\,vk\cos 60 - 2v = 0$$

Or, $$V(1 + k\cos 60) - 2v(1 + k\cos 60) = 0$$

From which, $$v = \frac{V}{2}$$

Therefore, the speed for maximum efficiency is not affected by the friction of the water passing over the cup.

Example 2.

A Pelton wheel is required to work under a head of 130 ft., and to develop 100 h.p. at 250 revs. per min. Assuming an efficiency of 80 per cent and a coefficient of velocity of ·98, find the jet diameter, the diameter of the bucket circle, the size of the buckets, and the number of buckets required. (London Univ.)

$$V = \cdot 98\sqrt{2gH}$$

$$= \cdot 98\sqrt{64{\cdot}4 \times 130} = 89{\cdot}5 \text{ ft. per sec.}$$

For maximum efficiency,

$$v = \cdot 46V \quad \text{(Practical value)}$$

$$= \cdot 46 \times 89{\cdot}5 = 41{\cdot}3 \text{ ft. per sec.}$$

Horse-power $$= \frac{\cdot 8WH}{550}$$

Hence, $$W = \frac{100 \times 550}{130 \times \cdot 8} = 530 \text{ lb. per sec.}$$

Let d = diameter of jet

and D = diameter of bucket circle

Then, $$v = \pi D\frac{n}{60}$$

From which, $$D = \frac{41{\cdot}3 \times 60}{\pi \times 250} = 3{\cdot}16 \text{ ft.}$$

$$\left.\begin{array}{l}\text{Quantity of water}\\ \text{flowing per second}\end{array}\right\} = \frac{\pi}{4}d^2\,V = \frac{W}{w}$$

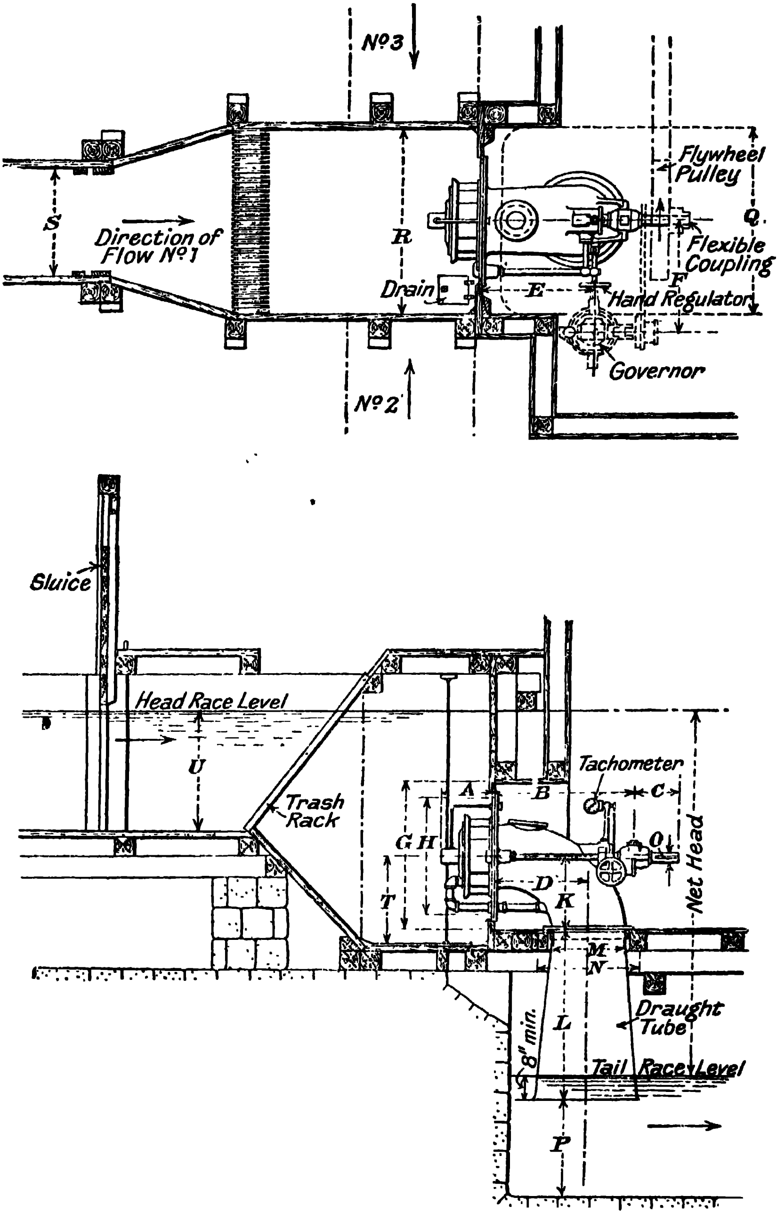


FIG. 141 (Armstrong-Whitworth)

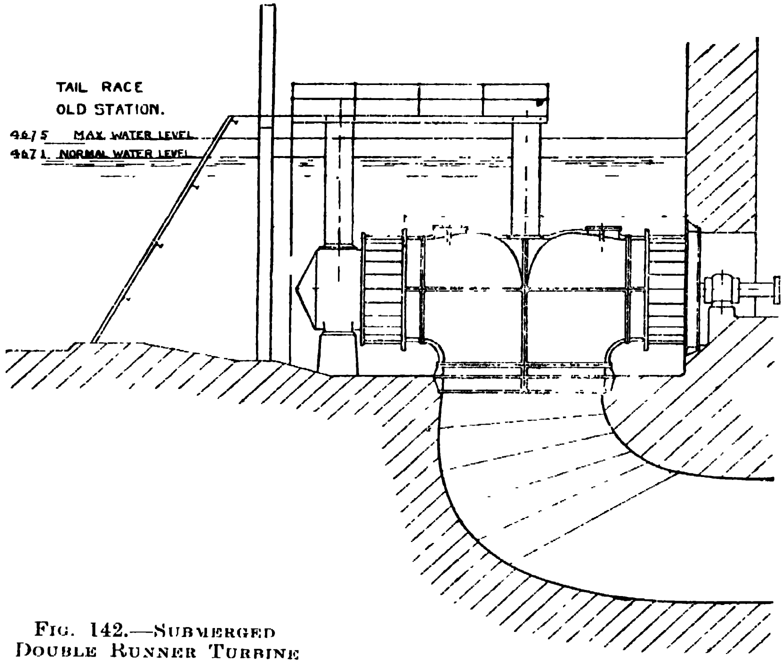
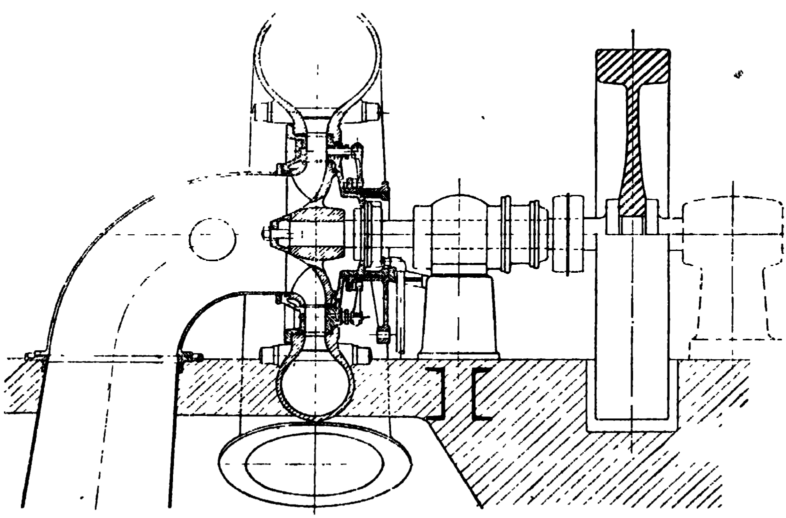


Fig. 142.—Submerged Double Runner Turbine

(*Armstrong-Whitworth*)



(*Armstrong-Whitworth*)

Fig. 143.—Francis Turbine

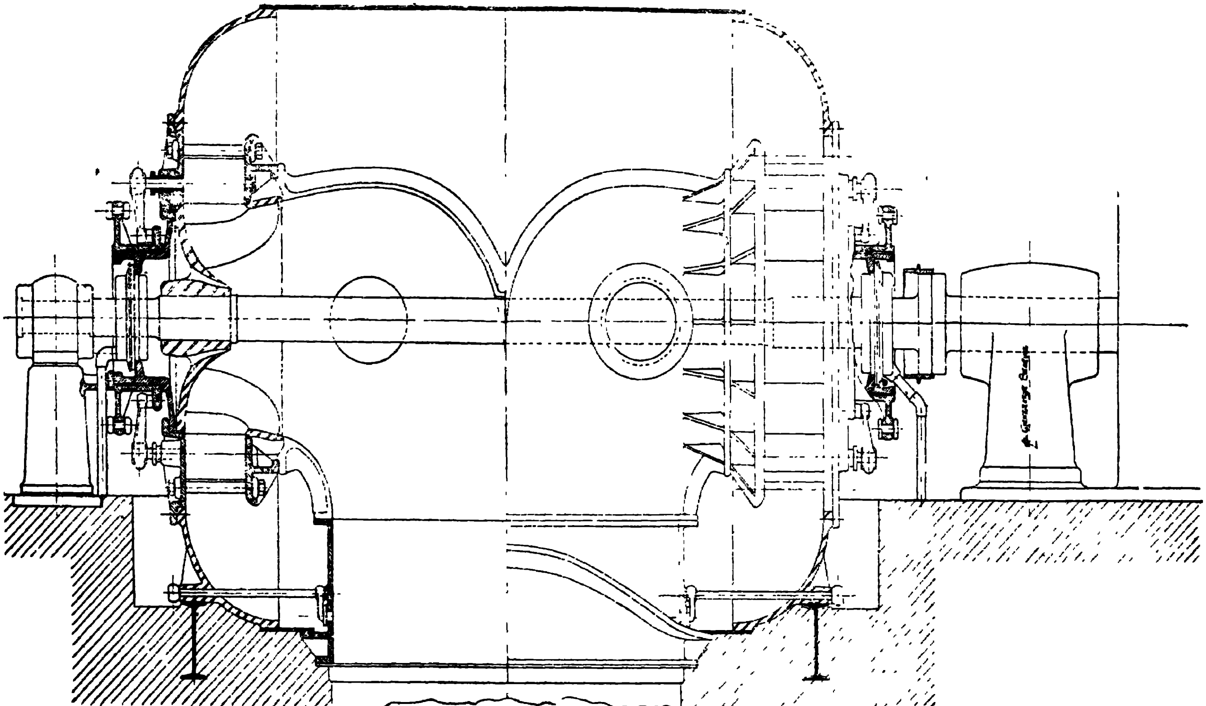


FIG. 144.—TWIN FRANCIS IN CYLINDRICAL CASING (Armstrong-Whitworth)

Hence, $$d^2 = \frac{530 \times 4}{\pi \times 62{\cdot}4 \times 89{\cdot}5}$$

$$d = {\cdot}347 \text{ ft.}$$
$$= 4{\cdot}17 \text{ in.}$$

Depth of bucket $= 1{\cdot}2d$

$$= 1{\cdot}2 \times 4{\cdot}17 = 5 \text{ in.}$$

Width of bucket $= 5d$

$$= 5 \times 4{\cdot}17 = 20{\cdot}8 \text{ in.}$$

For number of buckets,

$$\cos \gamma = \frac{R + {\cdot}5d}{R + {\cdot}6d}$$

$$= \frac{18{\cdot}96 + 2{\cdot}08}{18{\cdot}96 + 2{\cdot}5} = {\cdot}982$$

From which, $\gamma = 11°$

Then, number of blades $= \frac{360}{11} = 34$

118. **Some Modern Turbines.** A diagram of a turbine installation and views of some large modern turbines are shown in Figs. 141 to 144; these turbines were designed and constructed by the Hydro-electric Department of Sir W. G. Armstrong, Whitworth & Co., Ltd. Fig. 141 shows the installation of a small estate turbine for a low head; the suction or draught tube is clearly shown.

Fig. 142 shows a submerged double-runner turbine. A modern Francis turbine is shown in Fig. 143; and a twin Francis turbine in cylindrical casing in Fig. 144.

119. **Specific Speed of a Turbine.** The specific speed of a water turbine is the speed at which the turbine will run when producing one horse-power under a head of 1 ft. of water. This is sometimes called the Unit Speed or the Type Characteristic of the turbine. Within certain limits each type of turbine will have its own value for the specific speed; hence, if the specific speed is known it is possible to judge the type of turbine.

An equation for the specific speed of a turbine can be obtained by applying the principle of similarity to water turbines. It will be assumed that all turbines are geometrically similar; that is, that all their linear dimensions are in proportion, and the blade angles are constant.

Let D = diameter of turbine in feet

n_s = specific speed of turbine in revs. per min.

P = horse-power developed.

Then, using the notation of Art. 113,

$$v = \frac{\omega D}{2}$$

From which, $$D \propto \frac{v}{\omega}$$

But, $$\omega \propto n$$

And, from inlet triangle of any turbine,

$$v \propto V$$

That is, $$v \propto \sqrt{H} \qquad (\text{as } V \propto \sqrt{H})$$

Hence, $$D \propto \frac{\sqrt{H}}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Assuming linear dimensions of turbines to be similar,

$$b \propto D$$

Hence, from (1), $$b \propto \frac{\sqrt{H}}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From inlet triangle of any turbine.

$$V_f \propto V$$

That is, $$V_f \propto \sqrt{H} \qquad (\text{as } V \propto \sqrt{H}) \quad . \quad . \quad (3)$$

$$\left.\begin{array}{l}\text{Quantity per sec.}\\ \quad\text{passing through turbine}\end{array}\right\} = \text{radial area of flow} \times \text{vel. of flow}$$

$$= \pi D b \times V_f$$

Substituting from equations (1), (2), and (3),

Quantity per sec. $\propto \frac{\sqrt{H}}{n} \times \frac{\sqrt{H}}{n} \times \sqrt{H}$

$$\propto \frac{H^{\frac{3}{2}}}{n^2}$$

Weight of water per sec. $= W = w \times$ quantity per sec.,

Or,
$$W \propto \frac{H^{\frac{3}{2}}}{n^2} \quad . \quad . \quad . \quad (4)$$

Now, horse-power of turbine $= \frac{WH}{550}$

Hence, from Eq. (4),
$$P \propto \frac{H^{\frac{3}{2}}}{n^2} \times H$$

$$\propto \frac{H^{\frac{5}{2}}}{n^2}$$

Or,
$$n \propto \frac{H^{\frac{5}{4}}}{\sqrt{P}}$$

That is,
$$n = k \frac{H^{\frac{5}{4}}}{\sqrt{P}}$$

Where k is a constant depending on the type of turbine. When the turbine is developing 1 horse-power under a head of 1 ft., it will be noticed that k is equal to n which, under these conditions, is known as the specific speed n_s.

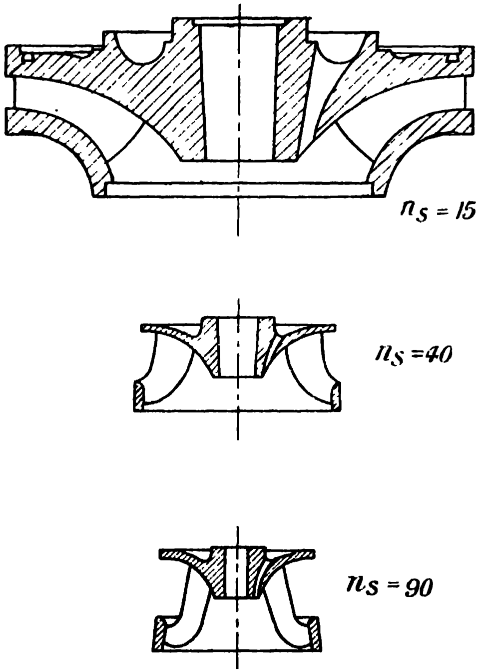
Hence,
$$k = n_s$$

Hence,
$$n_s = \frac{n\sqrt{P}}{H^{\frac{5}{4}}} \quad . \quad . \quad . \quad . \quad (5)$$

It is found that for impulse turbines n_s lies between 3 and 10, for reaction turbines the value of n_s is between 10 and 100. The n_s for a Pelton wheel is about 4.

Suppose it is required to install a water turbine to work at a given speed, under a given head, and to produce a given

horse-power. Then, putting these quantities in Equation 5 the specific speed is obtained. If this has a value of between 10 and 100, a reaction turbine should be used; if the value is less than 10, an impulse turbine or Pelton wheel should be



(*Armstrong-Whitworth*)

FIG. 145.—TYPES OF REACTION TURBINE RUNNERS

Showing comparative sizes of runners for same output under unit head

used. If the value is more than 100, then two or more reaction turbines would be required.

In Fig. 145* are shown the proportion of the runners of three reaction turbines, each of different specific speeds; the efficiency curves of the same turbines are shown plotted in Fig. 146.

EXAMPLE.

Deduce an expression for the specific speed of a reaction turbine. Under a head of 40 ft. the maximum feasible specific speed is 100. If, under this head, an installation of 20,000 h.p. is required, and if the speed is to be 150 revs. per min., how many units should be used? (A.M.I. Mech. E.)

* By courtesy of Sir W. G. Armstrong, Whitworth & Co.

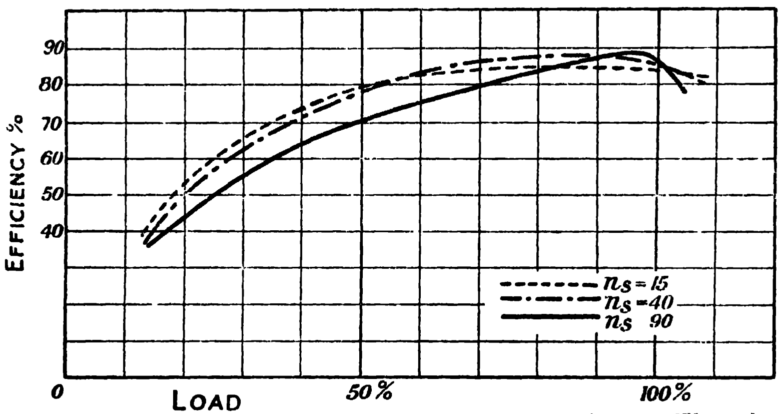
Using Equation 5,

$$n_s = \frac{n\sqrt{P}}{H^{\frac{5}{4}}}$$

That is, $$100 = \frac{150\sqrt{P}}{40^{\frac{5}{4}}}$$

From which, $$P = 4{,}500 \text{ per unit}$$

No. of units $$= \frac{20{,}000}{4{,}500}$$
$$= 5$$



(Armstrong-Whitworth

FIG. 146.—EFFICIENCY CURVES FOR REACTION TURBINE RUNNERS

120. **Governing of Turbines.** The speed of a water turbine is regulated by means of a centrifugal governor of a similar type to that used on a steam engine. The governor controls the water supply by operating sliding gates as explained in Arts. 114, 115, and 117, or by operating pivoted guide vanes, as shown in Fig. 134. The centrifugal governor is not powerful enough to move the gates unaided on account of their weight, and a mechanism is installed which, when operated by the centrifugal governor, is of sufficient power to move the heavy gates. This method is known as relay governing.

The earliest method of relay governing was by means of a system of open and crossed belts on fast and loose pulleys driven from the turbine shaft. The belts were moved from

the fast to the loose pulleys by the centrifugal governor. This is similar to the belt drive for the return motion of an engineer's planing machine.

Modern types of relay governors consist of a differential cylinder worked by water or oil pressure. A diagrammatic view of a differential cylinder is shown in Fig. 147. The relay piston is larger at one end than at the other, the relay cylinder being correspondingly shaped to suit. Oil or water is kept at a constant pressure in the annular space *B*. The pressure of the oil or water in the space *A* is regulated by the centrifugal governor. When the turbine is running steady, the total pressure at *A* equals the total pressure at *B* and the piston is then in equilibrium. If the turbine speeds up, the centrifugal governor operates the oil valve and admits high pressure oil

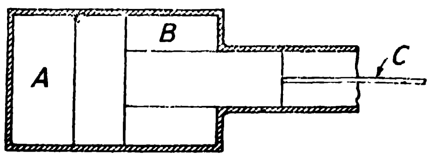


FIG. 147

in *A*; this forces the piston to the right, and the piston rod *C* will cause the gate to partly close. If the turbine slows down, the centrifugal governor exhausts the oil in *A*, the piston is then forced to the left by the oil pressure in *B*, and the gate will open farther.

In all modern makes of this type of relay governor oil is used, the oil pressure being obtained from an oil pump driven off the turbine shaft.

The following is a description of an automatic oil pressure governor made by Sir W. G. Armstrong, Whitworth & Co. A view of this relay governor is shown in Fig. 148.

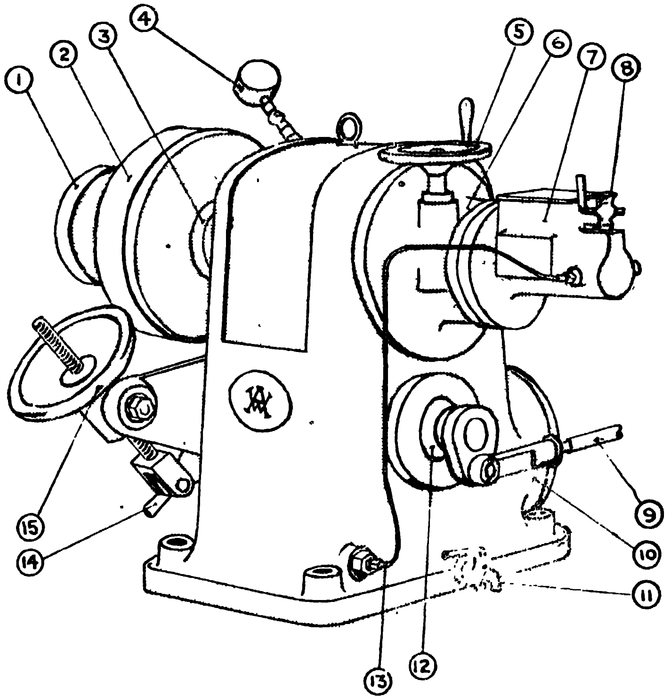
The governor is driven direct from the turbine shaft by belt or other means, and is of the automatic oil pressure sensitive type.

The oil pressure is obtained from a rotary gear pump, also driven from the turbine shaft, with accurately cut teeth to ensure the highest possible efficiency and noiseless rotation.

The servo-motor piston is connected direct to the actuating lever on the governor shaft, the piston being differential. The pressure oil operates on the servo-motor piston through a

rotating cylindrical distributing valve which is directly connected to the governor head, and moves axially when any speed variation takes place.

In the event of a speed rise, the rotating valve is moved away from the governor head, uncovering a port which opens



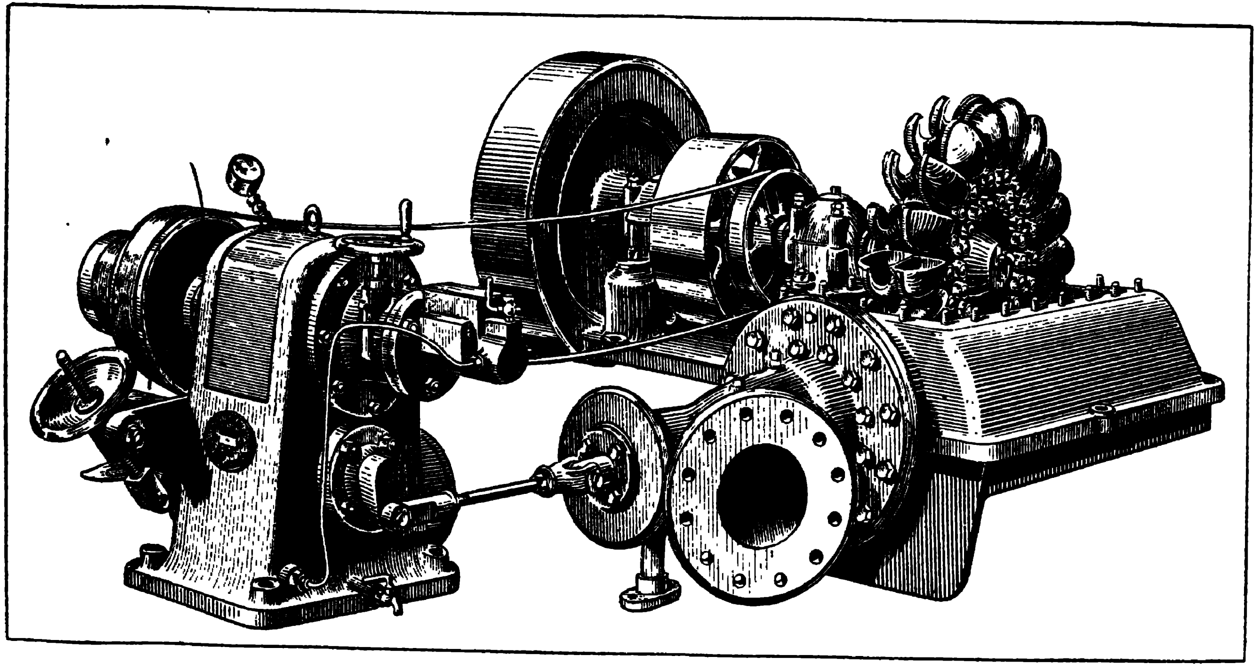
(*Armstrong-Whitworth*)

FIG. 148.—AUTOMATIC OIL PRESSURE GOVERNOR

1. Governor head
2. Driving pulley
3. Gear pump
4. Pressure gauge
5. Speed adjusting handwheel
6. Valve housing
7. Relay chamber
8. Air cock
9. Connecting rod
10. Servomotor cylinder
11. Drain cock for sump
12. Governor shaft bearing
13. Compensating pipe
14. Locking pin
15. Handgear

the large area side of the servo-motor cylinder to the sump. Constant pressure always being maintained on the small area side, the piston is moved back from the centre of the governor, thereby closing down the turbine.

Should the speed drop, the valve moves towards the governor head, uncovering a port which admits pressure oil into the large



(*Armstrong-Whitworth*)

Fig. 149.—Relay Governor Coupled to Pelton Wheel

area side of the servo-motor piston, moving the piston towards the centre of the governor owing to the greater volume of pressure oil on the back of the piston, thereby opening up the turbine.

The movement of the actuator lever operates a relay plunger pump which is connected by means of a pipe to an oil brake damping cylinder at the end of the valve spindle when the servo-motor piston is moved in the closing direction. The relay pump simultaneously forces oil to the back of the oil brake piston, thus restoring equilibrium, and preventing the governor from over closing. This arrangement makes hunting impossible.

The governor head is of the patent evolute type, where only rolling motion takes place, making friction losses practically negligible.

A synchronizing attachment is provided for varying the speed regulation of the set between full and no load. This mechanism is adjustable by hand. A hand control is provided for adjusting the running speed to $\pm$ 4 per cent. An electric remote control from the switchboard can also be installed if required.

Hand operating gear is provided for starting up and in case of emergency.

In Fig. 149 is shown a view of this governor coupled to a Pelton wheel. In this case the governor is operating the needle valve at the nozzle.

121. **Inertia of Water Column in Supply Pipe.** Another difficulty in water turbine governing is the regulation of the increase of pressure due to the inertia effect of the column of water in the supply pipe. On the governor partly closing the gate there will be a slowing down of the water in the supply pipe ; this will cause an increase of pressure at the guide vanes which may tend to speed up the turbine. In order to prevent this, a pressure regulator in the form of a spring relief valve is fitted at the turbine end of the supply pipe.

Another method of overcoming the inertia effect of the water column in the supply pipe is to fit a vertical pipe and tank, known as a "surge tank" or stand pipe, on the supply pipe as near to the turbine as possible (Fig. 150). This tank is open to the atmosphere at the top. When the turbine gates are closing, the slowing down of the water column in the supply

pipe will cause a rise of pressure, and water will flow into the surge tank, thus reducing the shock. When the turbine gates are opening, water will flow from the surge tank into the turbine whilst the water column in the supply pipe is accelerating.

For turbines with very large heads the surge tank is closed at the top, the air trapped in being compressed and expanded by the closing and opening of the turbine gates. This is the same in principle as the air vessel on a reciprocating pump.

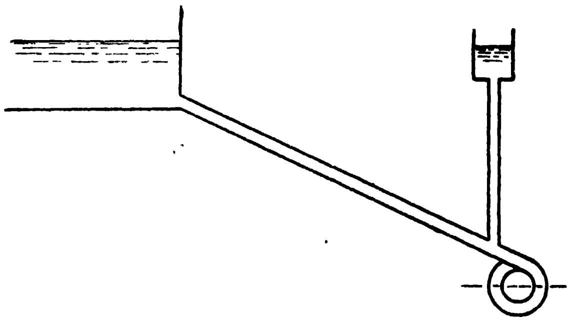


Fig. 150

122. **Characteristic Curves for Turbine.** There is a large variation in the efficiency of a turbine when the gate and speed are varied; for small gate openings and low speeds the efficiency is very low. To obtain the conditions for the maximum efficiency for a turbine a diagram is plotted showing the efficiencies for all conditions of running; from this diagram the condition for maximum efficiency may be obtained; such a diagram is known as a Characteristic Curve. The points on the diagram are obtained by testing the turbine for various gate openings and at various speeds; the diagram holds for that particular turbine only.

Before plotting the diagram certain characteristics for the turbine are calculated from the results of the tests; these characteristics are known as Unit Power, Unit Speed, and Unit Quantity.

Unit Power. The unit power of any particular turbine may be defined as the power developed under a head of 1 ft., or under unit head if any other system of dimensions be used.

Let P = horse-power developed

then, $P \propto W H$

but, $W = 62{\cdot}4\, a\, V$

and, $V \propto \sqrt{2gH}$

hence, $W \propto \sqrt{H}$

Then, $P \propto H^{\frac{3}{2}}$

Or, $P = k_1 H^{\frac{3}{2}}$

where k_1 is a coefficient which will vary with the efficiency of the turbine; that is, with the gate opening and speed.

When $H = 1$ ft.

$$P = k_1 = \text{unit power}$$

Hence, the unit power of a turbine $= k_1 = \dfrac{P}{H^{\frac{3}{2}}}$

UNIT SPEED. The unit speed for a particular turbine is the speed when running under a head of 1 ft. For a given turbine,

$$n \propto \sqrt{H}$$

or, $n = k_2\sqrt{H}$

where k_2 is a coefficient which will vary with the conditions of running.

When $H = 1$ ft., $n = k_2$ = unit speed

Hence, unit speed of a turbine $= k_2 = \dfrac{n}{\sqrt{H}}$

UNIT QUANTITY. This is the volume of water passing through the turbine when the head is 1 ft.

$$Q = a\, V$$

or, $Q \propto a\sqrt{H}$

hence, $Q \propto \sqrt{H}$

or, $Q = k_3\sqrt{H}$

where k_3 is a coefficient depending on the condition of running

When $H = 1$ ft., $Q = k_3$ = unit quantity,

then, unit quantity for a turbine $= k_3 = \dfrac{Q}{\sqrt{H}}$

THE CHARACTERISTIC CURVE. This is a chart showing the efficiencies of a particular turbine under all conditions of running. The turbine is first tested for a particular gate opening; the speed and head are varied, and the quantity and brake horse-power are measured. From these results values of the efficiency, unit power, and unit speed are calculated for the various speeds and heads at that gate opening. A

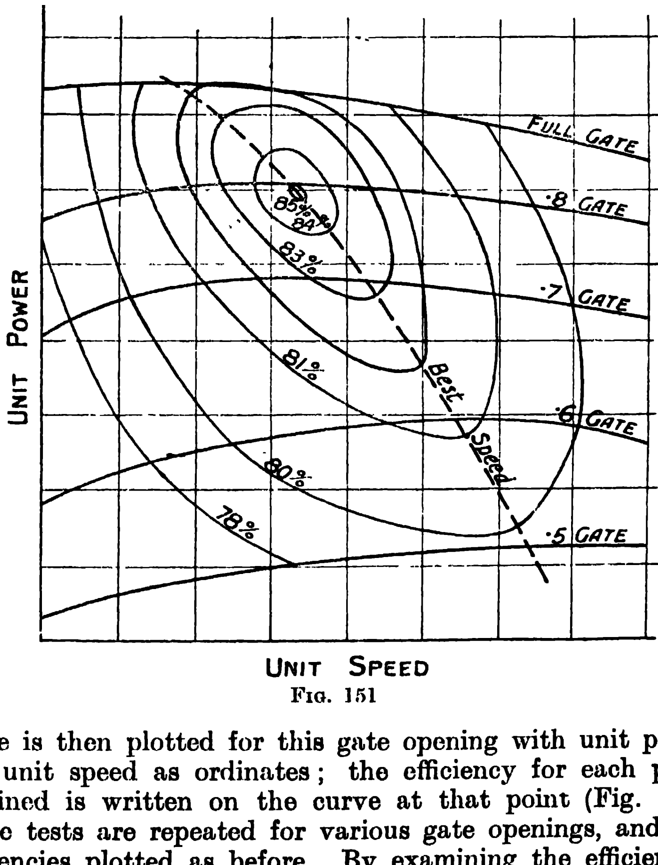


FIG. 151

curve is then plotted for this gate opening with unit power and unit speed as ordinates; the efficiency for each point obtained is written on the curve at that point (Fig. 151). These tests are repeated for various gate openings, and the efficiencies plotted as before. By examining the efficiencies written at each point, lines of equal efficiency can be drawn by interpolation; these lines correspond to the contour lines on a map.

From this chart it can be seen at a glance what the speed of the turbine should be, at any gate opening, in order to give

the best efficiency for that gate opening. It also shows clearly the maximum efficiency of the turbine for all conditions, and the gate opening and speed which produce this maximum efficiency can be read off the chart; this should be the normal condition of running for the turbine.

123. **Principle of Similarity Applied to Turbines.** The principle of similarity may be applied to turbines in order to predict the performance of a future design from the tests on a model. A small model is made similar to the actual turbine and, by means of a test, its horse-power is measured under a known head and at a known speed; the quantity of water supplied is also measured. From these results it is possible to calculate the performance of the actual turbine.

Let D = diameter of a turbine.

P = horse-power of turbine.

For all similar turbines all the velocities such as V, v, V_r, V_f, etc., will be proportional to $\sqrt{H}$. This is obvious from the velocity diagrams, the blade angles being constant.

Hence, $v \propto \sqrt{H}$

and, $V_f \propto \sqrt{H}$

but, $v = \pi D n$

hence, $\pi D n \propto \sqrt{H}$

from which, $$D \propto \frac{\sqrt{H}}{n} \quad \ldots \quad (1)$$

Also, $$P = \frac{WH}{550}$$

$$= \frac{w(\pi D b) V_f H}{550}$$

but, $b \propto D$

and, $V_f \propto \sqrt{H}$

hence, $P \propto D^2 H^{\frac{3}{2}}$

or, $$P = k_1 D^2 H^{\frac{3}{2}} \quad \ldots \quad (2)$$

where k_1 is a constant for the type of turbine considered. By combining Equations (1) and (2) the equation for specific speed may be obtained, as in Art. 119.

Also, $$Q = \pi D b \, V_f$$

then, $$Q \propto D^2 \sqrt{H}$$

or, $$Q = k_2 D^2 \sqrt{H} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where k_2 is a constant for the type considered.

It was shown in Art. 103 that

$$n = k \frac{H^{\frac{5}{4}}}{\sqrt{P}}$$

Hence, by measuring the values of P, H, n and D from the model test the values of k, k_1, and k_2 may be calculated. These values will also hold for the large turbine; hence, as its diameter is known, and as the head under which it will run is known, its horse-power, speed, and quantity may be calculated. It should be noted that the horse-power used for equation (1) is the water horse-power supplied, whilst that measured in the model test is the brake horse-power; hence, the efficiency of the runner has been assumed to be the same for both model and large turbine. This is not quite true, as the efficiency of the model is slightly less than the large turbine due to the friction of the water being greater in the small passages of the model.

EXAMPLE.

Tests on a model turbine, 1 ft. diameter, give a maximum efficiency of 82 per cent at 900 revs. per min. and at ¾ gate opening, under a head of 64 ft. The output was then 38·4 b.h.p. A similar turbine is required to develop 500 b.h.p. at ¾ gate under a head of 81 ft. Calculate its diameter and speed of rotation. How would you expect its efficiency to compare with that of the model? (London Univ.)

Assume efficiency is the same for model and turbine, and use the equation for specific speed given in Art. 119.

Then, $$n = k \frac{H^{\frac{5}{4}}}{\sqrt{P}}$$

From which, $$k = \frac{n\sqrt{P}}{H^{\frac{5}{4}}}$$

$$= \frac{900\sqrt{38 \cdot 4}}{64^{\frac{5}{4}}} = 30 \cdot 75$$

Next apply this equation to the large turbine, the value of k will be the same as for model.

$$n = 30 \cdot 75 \frac{H^{\frac{5}{4}}}{\sqrt{P}}$$

$$= 30 \cdot 75 \times \frac{81^{\frac{5}{4}}}{\sqrt{500}}$$

$$= 334 \text{ revs. per min.}$$

From Equation (1),

$$D = c \frac{\sqrt{H}}{n}$$

where c is a constant for model and large turbine.

Hence, applying this equation to model,

$$c = \frac{Dn}{\sqrt{H}}$$

$$= \frac{1 \times 900}{\sqrt{64}} = 112 \cdot 5$$

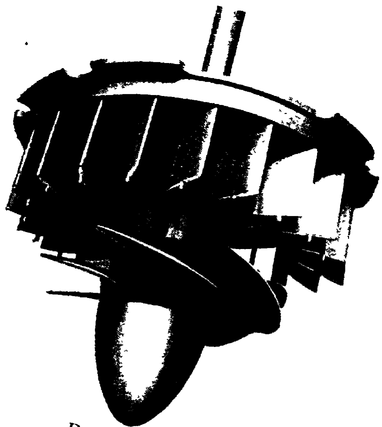
Using this value of c and applying the equation to the large turbine,

$$D = \frac{112 \cdot 5 \sqrt{H}}{n}$$

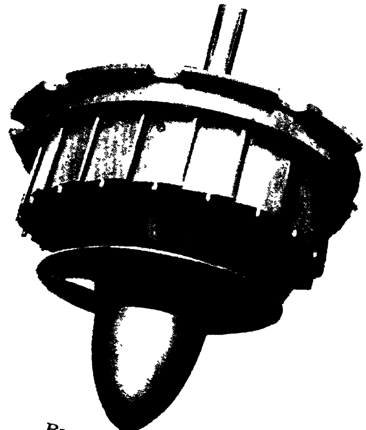
$$= \frac{112 \cdot 5 \sqrt{81}}{334}$$

$$= 3 \cdot 04 \text{ ft.}$$

124. **Propeller Turbines.** A type of water turbine which has attained great popularity within recent years is known as the propeller turbine. It is an axial flow reaction type having a small number of blades, usually from four to six. The runner is fitted horizontally (Fig. 152) and the blades resemble, in appearance, the propeller blades of a ship. Propeller turbines have a high specific speed, varying between 100 and 400.



Blades Open



Blades Closed

(T.5167)

Fig. 152.—Kaplan Turbine Runner

(Escher-Wyss Engineering Works, Ltd.)

A very efficient type of propeller turbine is the Kaplan turbine; this make is fitted with swivel blades by means of which the blade angle can be varied with the output. By this method a high efficiency is maintained at all gates.

Views of the runner and guide blades of a Kaplan turbine* are shown in Fig. 152. In the left-hand view the runner blades and guide vanes are shown fully open; in the right-hand

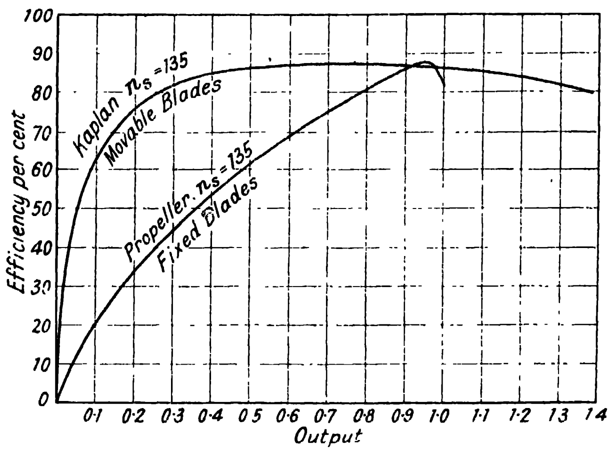


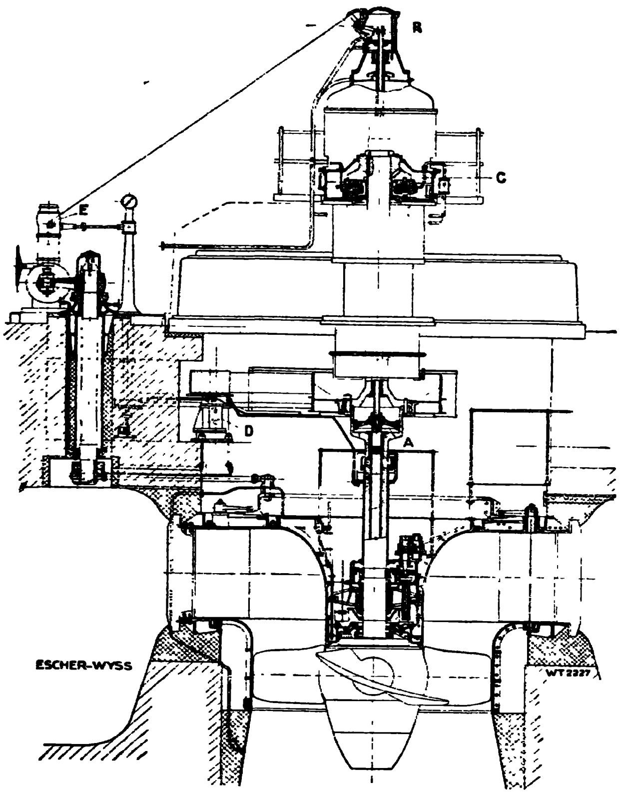
FIG. 153

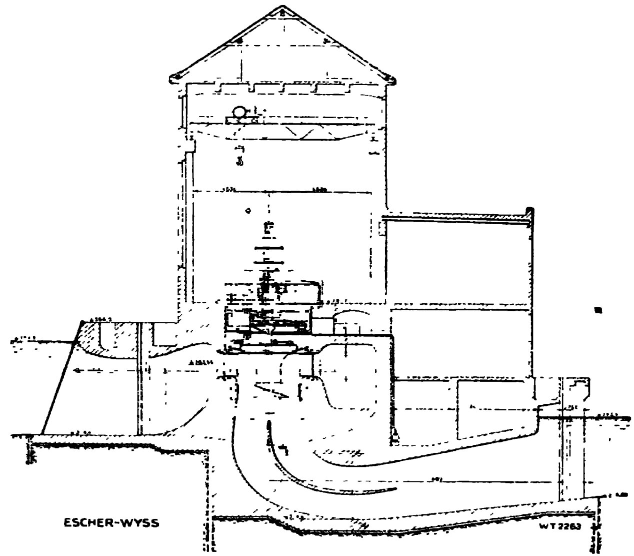
view they are closed. The blades are swivelled by a mechanism contained in the boss; this is operated by the turbine governor through the action of a servo-motor. By this means the blade angles are automatically adjusted, according to the power developed by the turbine. This prevents the falling off of the efficiency at part gate.

In Fig. 153 are shown the efficiency curves of a Kaplan turbine with swivel blades, and a propeller turbine with fixed blades, plotted to the same scale on a base representing the output or gate opening; both turbines have the same specific speed. It will be noticed from the curves how the Kaplan swivel-bladed turbine maintains its high efficiency through most of its range of output.

Kaplan turbines are pure axial flow, with four to six blades

* By courtesy of Messrs. Escher-Wyss Engineering Works, Ltd., Zürich, Switzerland.





(Escher-Wyss Engineering Works Ltd.)

having no outside rim; the blades are made of stainless steel. They are constructed to run at speeds varying between 60 and 220 r.p.m. and to work under maximum heads varying from 9 ft. to 50 ft. They are very efficient at low heads. The runner may have a very large diameter; one runner is 26 ft. 3 in. in diameter, and weighs 150 tons. Horse-powers of 7,000 per unit have been reached.

A sectional view through a Kaplan turbine installation is shown in Fig. 154. The oil pressure servo-motor is shown at *A*; the high-pressure oil admission is at *B*. *C* is the thrust bearing and *D* the geared oil pump. The speed control governor is denoted by *E*.

A cross-sectional view through the power station containing this turbine is shown in Fig. 155. The small difference of head between the two water levels should be noticed.

EXAMPLES 10.

(1) An inward-flow reaction turbine has an external diameter of 2 ft. If the breadth of the wheel at inlet is 6 in. and the velocity of flow at inlet is 5 ft. per sec., find the weight of water passing through the turbine per sec.

Ans.—980 lb.

(2) If the turbine of Question 1 has a speed of 192 revs. per min., and if the guide blade makes an angle of 10° to the wheel tangent, draw the velocity diagram at inlet and find the runner blade angle, the velocity of whirl, the absolute velocity of the water leaving the guide vane, and the relative velocity of the water entering the runner blade.

Ans.—$\theta = 31°$; $V_w = 28{\cdot}4$ ft. per sec.; $V = 28{\cdot}8$ ft. per sec.; $V_r = 9{\cdot}8$ ft. per sec.

(3) If the turbine of Question 1 has an inner diameter of 1 ft., find the breadth of the wheel at outlet in order to keep the velocity of flow 5 ft. per sec. Find also the runner blade angle at outlet if the discharge is radial and draw the velocity diagram at outlet.

Ans.—12 in.; 26°.

(4) Find the work done per lb. of water for the turbine in Question 1; find also the head supplied, the horse-power produced, and the hydraulic efficiency.

Ans.—17·7 ft. lb.; $H = 18{\cdot}12$ ft.; h.p. $= 31{\cdot}6$; eff. $= 98$ per cent.

(5) The lead-on angle of the guide vanes in an axial flow impulse turbine is 20°; the wheel vane angle at entrance is 30°; the head 400 ft. If the velocity at discharge is axial, and if the coefficient of velocity for the guide vanes is ·98, determine the work done per second when passing 10 cu. ft. per sec. and running under maximum efficiency conditions. [Sin 20° = ·342.] (A.M.I.Mech.E.)

Ans.—229,000 ft. lb.

(6) In an outward-flow turbine supplied with 180 cu. ft. per sec. and making 200 rev. per min., the internal and external diameters of the wheel are 6 ft. and 7 ft. 6 in. respectively and the effective width of the wheel-face at inlet and outlet is 9 in. The head on the wheel is 115 ft. and the discharge is free and radial. Neglecting the thickness of the vanes and friction losses, determine the angles of the vanes at entrance and exit, and sketch a vane showing these angles. (A.M.I. Civil E.) (Assume turbine to be a reaction.)

Ans.—109·8° ; 7·4°.

(7) The peripheral velocity of the wheel of an inward flow turbine is 70 ft. per sec. The velocity of whirl of the inflowing water is 55 ft. per sec., and the radial velocity of flow 7 ft. per sec. If the flow is 24 cusecs and the hydraulic efficiency 80 per cent, find the head on the wheel, the horse-power of the turbine, and, by drawing to scale, the triangle of velocities, the inlet angle of the vanes. The discharge is radial. (A.M.I. Civil E.)

Ans.—149·5 ft. ; 325 h.p. ; 155°.

(8) Describe the working of, and a method of governing, an axial-flow Girard turbine.

If for such a turbine the angle of the guide blades is 30°, and the angle of the rotor vanes is 25° at outlet, find the maximum hydraulic efficiency, and the best speed of the turbine.

The available head is 100 ft., and the mean diameter of the rotor 6 ft. (London Univ.) [Assume axial discharge for max. eff.]

Ans.—93·7 per cent ; 137 revs per min.

(9) An inward flow turbine, having an overall efficiency of 75 per cent., is required to give 175 h.p. The head H is 20 ft. ; velocity of periphery of the wheel is $\cdot 95\sqrt{2gH}$; and the radial velocity of flow is $\cdot 35\sqrt{2gH}$. The wheel is to make 230 revs. per min., and the hydraulic losses in the turbine are 20 per cent of the available energy. Determine (*a*) the angle of the guide blade at inlet ; (*b*) the wheel vane angle at inlet ; (*c*) the diameter of the wheel ; (*d*) the width of the wheel at inlet. [Assume turbine is reaction and radial discharge.] (London Univ.)

Ans.—(*a*) 39·8° ; (*b*) 146½° ; (*c*) 2·83 ft. ; (*d*) 11·02 in.

(10) A Pelton wheel has a mean bucket speed of 40 ft. per sec., and is supplied with water at the rate of 150 gallons per sec. under a head of 100 ft. If the buckets deflect the jet through an angle of 160°, find the horse-power and efficiency of the wheel.

Ans.—264·2 h.p. ; e = 97 per cent.

(11) Obtain an expression for the theoretical efficiency of a Pelton wheel when the angle of the bucket at exit makes an angle of θ with the direction of the jet. Show by a diagram how the efficiency of the wheel will vary as the relation of the velocity of the jet to the velocity of the bucket is varied.

Describe, with carefully drawn sketches, at least one method of governing a Pelton wheel. (London Univ.)

(12) Briefly describe an "Inward Flow Turbine." Show that in a turbine with radial vanes at the receiving circumference the theoretical hydraulic efficiency is $\dfrac{2}{2 + \tan^2 \alpha}$ where α is the angle made by the guide blade with a tangent to the point where it cuts the receiving circumference, the velocity of radial flow being constant. (London Univ.)

(Assume turbine is reaction with radial discharge.)

(13) In an outward flow reaction turbine the rim speed at inlet is 40 ft. per sec., and the ratio of the radii is ·8. The turbine is placed 3 ft. below the water surface in the tail race, and the wheel vane angles are 90° and 20° at inlet and outlet respectively. The radial velocity of flow at inlet is 14 ft. per sec. Neglecting frictional losses, and taking velocity of outflow as radial, find the guide vane angle, pressure head at inlet to the wheel, speed of flow from guides, the total head, and hydraulic efficiency. (London Univ.)

Ans.—19·3° ; 29·8 ft. of water (gauge) ; 42·3 ft. per sec. ; 54·5 ft. $e = 91{\cdot}2$.

(14) An inward flow turbine, having an overall efficiency of 75 per cent, is required to give 180 h.p. The head H is 30 ft. The velocity of the periphery of the wheel is $6\sqrt{H}$, and the radial velocity of flow is $2\sqrt{H}$ The wheel is to make 120 revs. per min. The hydraulic losses in the turbine are 20 per cent of the available energy. Determine : (*a*) the guide blade angle at inlet ; (*b*) the wheel vane angle at inlet ; (*c*) the diameter of the wheel ; (*d*) the width of the wheel at inlet. [Assume turbine is reaction and radial discharge.] (London Univ.)

Ans.—(*a*) 25·1° ; (*b*) 130·5° ; (*c*) 5·23 ft. ; (*d*) 4·8 in.

(15) An inward-flow pressure turbine has a runner whose vanes are radial at inlet and inclined backwards at 30° to the tangent at discharge. The diameter at entry is twice that at discharge, and the width at entry is one-half that at discharge. The guide vanes are inclined at 15° to the tangent. The velocity of the water leaving the guide is 80 ft. per sec. Determine the correct velocity for the runner, and the absolute velocity of the water at the point of discharge. (A.M. Inst. C.E.)

Ans.—$v = 77{\cdot}3$ ft. per sec. ; $V_1 = 20{\cdot}9$ ft. per sec.

(16) The following data were obtained from a test on a Pelton wheel—

Area of jet	=	12·0 sq. in.
Discharge	=	6·35 cu. ft. per sec.
Head at nozzle	=	100·0 ft.
Brake horse-power	=	56·0
H.P. absorbed in friction and windage	=	3·0

Determine the energy lost in the nozzle, and also the energy absorbed due to losses in the wheel at discharge. (A.M.I. Mech. E.)

Ans.—7·05 h.p. and 5·8 h.p.

(17) Given the formula $N_s = N\sqrt{P} \div H^{\frac{5}{4}}$ for the specific speed of a water-turbine, in which N is revs. per min., P is the brake horse-power, and H is the available head in feet, prove that the specific speed of a single jet Pelton wheel is $\frac{55d}{D}$, in which d is the diameter of the jet in feet, and D is the diameter of the mean bucket circle in feet. Assume that the coefficient of velocity of the jet is unity, and that when the maximum efficiency is 85 per cent the mean bucket speed is $\cdot 46\sqrt{2gH}$. (London Univ.)

(18) Define the terms "specific speed," "unit speed," and "unit power" as applied to a hydraulic turbine. Describe the method of preparation of, and the use of, the characteristic diagram for a turbine, the co-ordinates being "unit speed" and "unit power." (London Univ.)

(19) Show that in a given turbine the peripheral speed of the runner for maximum efficiency is proportional to $\sqrt{H}$ where H is the available head, and that under these conditions the quantity of water consumed is proportional to $\sqrt{H}$, and the power developed to $H^{\frac{3}{2}}$.

Hence, show how the performance of a turbine may be predicted from that of a geometrically similar model. (London Univ.)

(20) What factors determine whether a turbine of the Francis, Kaplan, or Pelton type would be used in a hydro-electric power scheme? Determine the horse-power of a machine suitable for a head of 500 ft., the quantity available being 30 cu. ft. per sec. If the speed is to be 375 r.p.m., what type of turbine would be used, and what would be its leading dimensions? (I. Mech. E.)

Ans.—1,700 h.p.; $n_s = 6{\cdot}55$ (use Pelton wheel), $d = 5{\cdot}5$ in. $D = 4{\cdot}6$ ft.

(21) Deduce an expression for the specific speed of a hydraulic turbine. A turbine develops 10,000 h.p. under a head of 81 ft. at 120 r.p.m. What is its specific speed? What would be its normal speed and output under a head of 64 ft? (I. Mech. E.)

Ans.—49·4; 106·7 r.p.m.; 7,024 h.p.

(22) Determine the diameter, speed, and specific speed of a Kaplan turbine runner to develop 7,500 h.p. under a head of 17 ft., given: velocity of flow = 20 ft. per sec.; diameter of boss = ·36 × external diameter; and mechanical efficiency = 87 per cent. Velocity of outer edges of blades, 66 ft. per sec. How is a Kaplan turbine governed? (I. Mech. E.)

Ans.—22·8 ft.; 55·3 r.p.m.; $n_s = 138{\cdot}5$.

CHAPTER XI

CENTRIFUGAL PUMPS

125. **Centrifugal Pumps.** The action of a centrifugal pump is that of a reversed turbine, except that special arrangements must be made in order to increase the efficiency. All centrifugal pumps are outward flow, as the radial velocity of the water in the pump is then increased by the centrifugal head impressed on it by the rotating vanes. The pump must be full when starting; for this reason, it should not be allowed to drain. The pump is driven by power from an external source, by which means the vanes are rotated. This gives a centrifugal head to the water in the pump, and the water will leave the vanes at the outer circumference with a high velocity and pressure. A partial vacuum will form in the centre, into which the water from the suction pipe flows. The high pressure of the leaving water is utilized in overcoming the delivery head of the pump. In earlier types of centrifugal pumps the high velocity of the leaving water was wasted in eddies in the circular chamber which surrounded the vanes; but this is now transformed into pressure head by causing the leaving water to flow through a passage of gradually increasing area. The kinetic energy of the leaving water is thus converted into pressure energy, which is utilized in increasing the delivery head of the pump. The efficiency is thus considerably increased.

Centrifugal pumps are usually of the radial flow type, but pumps having a mixed flow and axial flow are also made. Axial flow pumps are known as propeller pumps and are used for low heads.

The following are the methods adopted to convert the kinetic energy of the leaving water into pressure energy.

(*a*) Volute Chamber. The vane wheel or impeller is surrounded by a spiral casing known as a volute chamber (Fig. 156). The leaving water flows inside this chamber circumferentially, the velocity decreasing with the increasing area of flow. When the water reaches the delivery pipe, the velocity will be small and the pressure will have correspondingly increased. It has been found from tests that this type of chamber only slightly increases the efficiency of the pump; a considerable loss takes place in eddies due to the continually increasing quantity of water flowing through the chamber.

(*b*) VORTEX OR WHIRLPOOL CHAMBER. Professor James Thomson improved on the volute chamber by combining a circular chamber with a spiral chamber (Fig. 157); such a casing is known as a vortex or whirlpool chamber. An increased efficiency is obtained by means of this type of casing.

(*c*) GUIDE BLADES. Another method of converting the velocity head of the leaving water into pressure head is by causing the water to flow through passages of increasing area formed by guide vanes (Fig. 158). A pump fitted with such vanes is known as a turbine pump, and is similar in principle to a reversed inward flow turbine. The guide blades are placed at such an angle that the water enters without shock and is surrounded by a volute chamber, by which the water reaches the delivery pipe. The ring of guide blades is called a diffuser, and is found to be very efficient.

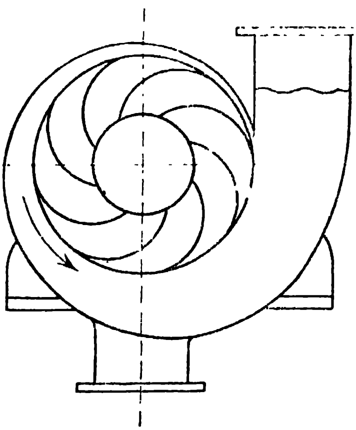


FIG. 156

There is much looseness in the use of the above terms. Some authorities apply the terms vortex chamber, whirlpool chamber, and diffuser to all types of casings surrounding the impeller.

126. Work Done and Efficiency of Centrifugal Pump. The blade angles and work done by a centrifugal pump may be found from the velocity triangles in the same way as for a turbine, except that the inlet diagram now becomes the outlet and *vice versa*. It is usual to assume the water enters the wheel radially.

Using the same notation as for turbines (Art. 113),

$$\left.\begin{array}{l}\text{Centrifugal head im-}\\ \quad\text{pressed on water}\end{array}\right\} = \frac{v_1^2}{2g} - \frac{v^2}{2g}$$

Let H = total theoretical lift of pump, or design head

Then,

$$\text{theoretical gross lift} \quad = H + \frac{v_d^2}{2g}$$

where v_d = velocity of discharge from delivery pipe.

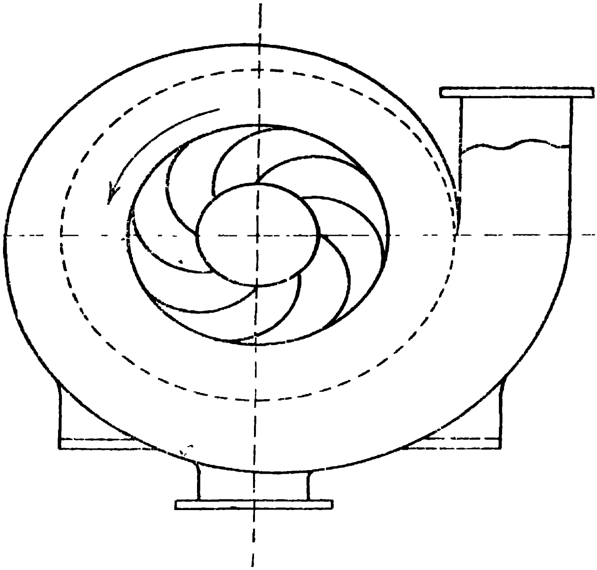


FIG. 157

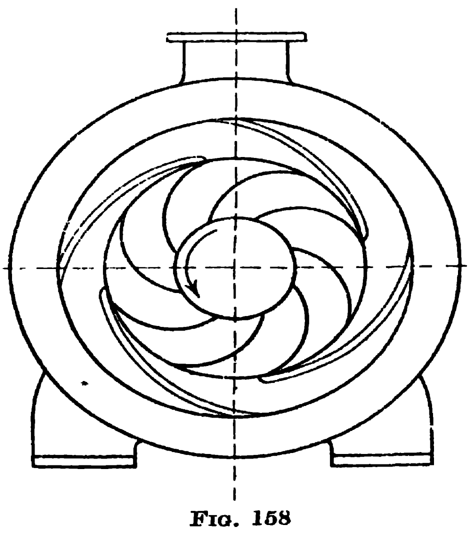


FIG. 158

The term $\frac{v_d^2}{2g}$ is small and may usually be neglected.

The velocity diagrams for inlet and outlet are shown in Fig. 159.

and, $$\frac{v}{v_1} = \frac{r}{r_1}$$

$$\left.\begin{array}{r}\text{work done by impeller}\\ \text{per pound of water}\end{array}\right\} = \frac{V_{w_1} v_1}{g}$$

$$= H + \frac{v_d^2}{2g}$$

Therefore, $$\frac{V_{w_1} v_1}{g} = H + \frac{v_d^2}{2g}.$$

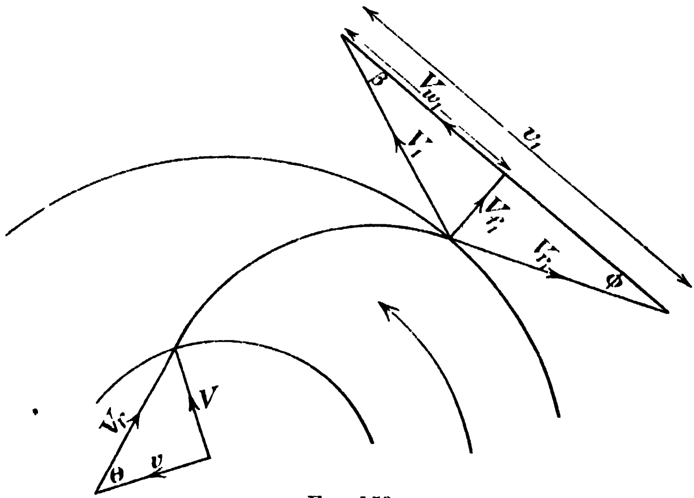


FIG. 159

Let h_f = head lost in friction in delivery and suction pipes
h = actual height water is lifted by pump
W = weight of water pumped per sec.

Then, gross lift $= h + h_f + \frac{v_d^2}{2g}$

Actual efficiency

$$= \frac{\text{actual lift}}{\text{energy supplied to pump shaft per lb. of water}}$$

$$= \frac{Wh}{\text{horse-power} \times 550}$$

The manometric efficiency $= \dfrac{\text{gross lift}}{\text{theoretical gross lift}}$

$$= \frac{h + h_f + \dfrac{v_d^2}{2g}}{H + \dfrac{v_d^2}{2g}} = \frac{h + h_f + \dfrac{v_d^2}{2g}}{\dfrac{V_{w_1} v_1}{g}}$$

Hydraulic efficiency

$$= \frac{\text{gross lift}}{\text{energy supplied to impeller per lb. of water}}$$

The energy supplied to impeller is less than that supplied to shaft by mechanical losses in bearings, etc.

127. **Minimum Starting Speed of Centrifugal Pump.** In starting a centrifugal pump, there will be no flow through the wheel until the pressure difference in the impeller is large enough to overcome the total lift. If the impeller is rotating and there is no flow, the pressure head caused by the centrifugal force on the rotating water will be $\dfrac{v_1^2}{2g} - \dfrac{v^2}{2g}$.

Flow will not commence until this amount is greater than H, as $v_d = 0$ when flow commences.

As $$H = \frac{V_{w_1} v_1}{g}$$

the least theoretical speed for flow to commence will be when

$$\frac{v_1^2}{2g} - \frac{v^2}{2g} = \frac{V_{w_1} v_1}{g}$$

Actually, flow will commence when

$$\frac{v_1^2}{2g} - \frac{v^2}{2g} = e\,\frac{V_{w_1} v_1}{g}$$

where e = manometric efficiency.

128. **Head Lost in Centrifugal Pump due to Reduced or Increased Flow.** A centrifugal pump will produce its maximum efficiency only when running and discharging at the speeds for which it was originally designed. If the normal discharge is increased or reduced, there will be a loss of head at entry due to shock.

Let triangle *abd* (Fig. 160) be the velocity triangle for the pump when running normally. The blades at inlet will be parallel to *ab*. If the radial flow through the pump is now reduced from *bd* to *cd*, whilst the speed of rotation remains the same, the triangle of velocity will be represented by *acd*, *ac* being the relative velocity. But the angle of the blade at inlet will be the same as before; the relative velocity, therefore, will no longer be parallel to the blade and shock will occur.

As the velocity of flow is fixed, and as the water must pass along the vane, it follows that the velocity diagram will be

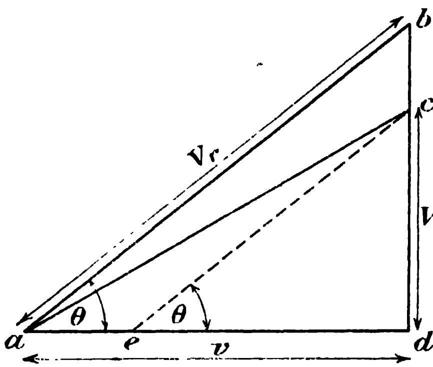


FIG. 160

triangle *ecd*, *ec* being parallel to *ab*. Therefore, a tangential change of velocity *ae* will suddenly take place, the shock causing the loss of head.

It was shown in Art. 45 that the loss of head due to a sudden change of velocity is equal to

$$\frac{(\text{change of velocity})^2}{2g}$$

Therefore, loss of head at entrance

$$= \frac{(ae)^2}{2g}$$

$$= \frac{\left(v - \dfrac{V}{\tan\theta}\right)^2}{2g}$$

where V is the velocity of flow *cd* through the pump.

The same equation applies if the flow is greater than the normal flow.

EXAMPLE.

Show how the triangle of velocities at inlet to the impeller of a centrifugal pump is affected by reducing the flow below the normal; and obtain an expression for the loss of head at inlet for any reduced value of the velocity of flow. State the assumptions made and the factors which affect the accuracy of the expression obtained.

A centrifugal pump has an impeller 20 in. outer diameter, and, when running at 520 revs. per min., discharges 1,700 gallons of water per minute against a head of 28 ft. At that discharge, the water enters the impeller without shock. The inner diameter is 10 in., the vanes are set back at outlet at an angle of 45°, and the area of flow, which is constant from inlet to outlet of the impeller, is ·65 sq. ft.

Determine (a) the manometric efficiency of the pump; (b) the vane angle at inlet; (c) the loss of head at inlet to the impeller when the discharge is reduced by 50 per cent., the speed of rotation being unchanged. (London Univ.)

The velocity diagrams are the same as shown in Fig. 159.

$$v_1 = \pi d_1 \frac{n}{60} = \pi \times \frac{20}{12} \times \frac{520}{60} = 45{\cdot}4 \text{ ft. per sec.}$$

$$v = v_1 \times \frac{10}{20} = 22{\cdot}7 \text{ ft. per sec.}$$

$$V = V_{f_1} = \frac{\text{quantity per sec.}}{\text{area of flow}} = \frac{1700}{60 \times 6{\cdot}24 \times {\cdot}65}$$

$$= 7 \text{ ft. per sec.}$$

$$V_{w_1} = v_1 - \frac{V_{f_1}}{\tan 45} = 45{\cdot}4 - 7 = 38{\cdot}4 \text{ ft. per sec.}$$

(a) $\left.\begin{array}{l}\text{Work done per}\\ \text{pound of water}\end{array}\right\} = \dfrac{V_{w_1} v_1}{g}$

$$= \frac{38{\cdot}4 \times 45{\cdot}4}{32{\cdot}2} = 54{\cdot}2 \text{ ft. lb.}$$

$$\text{Manometric efficiency} = \frac{\text{gross lift}}{\text{theoretical work done}}$$

$$= \frac{28}{54{\cdot}2} = 51{\cdot}7 \text{ per cent}$$

(b) From inlet velocity triangle,

$$\tan\theta = \frac{V}{v} = \frac{7}{22{\cdot}7} = {\cdot}308$$

Then, $\theta = 17{\cdot}1°$

(*c*) New velocity of flow at inlet $= \frac{7}{2} = 3 \cdot 5$ ft. per sec.

Head lost at inlet $= \dfrac{\left(v - \dfrac{V}{\tan\theta}\right)^2}{2g}$

$$= \frac{\left(22 \cdot 7 - \dfrac{3 \cdot 5}{\tan 17 \cdot 1}\right)^2}{2g}$$

$$= 2 \text{ ft.}$$

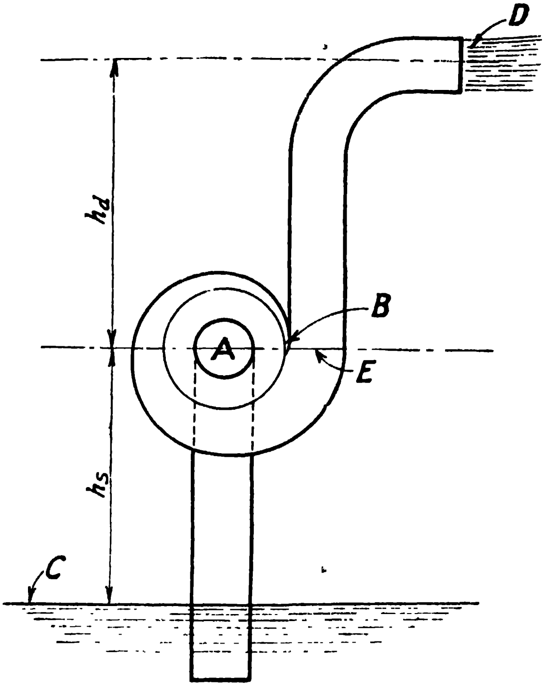


Fig. 161

129. **Water Pressure in Centrifugal Pumps.** The pressure of the water at any section of the stream in a pump may be found by applying Bernoulli's equation to various sections of the pump stream. Consider the pump and piping shown in Fig. 161. Let A be a point at the inlet edge of the impeller on the horizontal centre line of the pump, let B be a point at

the outlet edge of impeller on the same centre line, let C be a point on the water surface in the sump, and D be a point at the outlet of the discharge pipe. If all losses are neglected, the total energy of the water will be the same at all these points provided that the work done by the impeller is added to those points which are in the stream on the exit side of the impeller.

Let h_s = height of pump above sump in feet

h_d = height of outlet end of discharge pipe above pump in feet

p_A = pressure in lb. per sq. ft. absolute at A

p_B = pressure in lb. per sq. ft. absolute at B

Apply Bernoulli's equation to points A and C, let water level in sump be datum and neglect all losses.

Total energy at A = total energy at C

Hence, $h_s + \frac{V^2}{2g} + \frac{p_A}{w} = 34$

From this equation p_A may be found.

If the frictional loss in the suction pipe is taken into account the equation becomes—

$$h_s + \frac{V^2}{2g} + \frac{p_A}{w} + \mathrm{h}_f = 34$$

where h_f is the head lost in friction in the suction pipe.

Next apply Bernoulli's equation to points A and B, using centre line of pump as datum.

Total energy at A = total energy at B − useful work done by impeller

Hence, $$\frac{V^2}{2g} + \frac{p_A}{w} = \frac{V_1^2}{2g} + \frac{p_B}{w} - \left(\frac{V_{w_1} v_1}{g} \times \text{eff.}\right)$$

From this equation p_B may be found.

Next apply Bernoulli's equation to points B and D, using the centre line of pump as datum.

Total energy at B = total energy at D + losses.

Hence, $\frac{V_1^2}{2g} + \frac{p_B}{w} = 34 + h_d + \frac{v_d^2}{2g} + \text{loss in diffuser} + h_f$

where h_f is the head lost in friction in delivery pipe.

From this equation the loss in the diffuser may be found.

$$\left.\begin{array}{r}\text{Theoretical head saved by}\\ \text{fitting diffuser to pump}\end{array}\right\} = \frac{V_1^2}{2g} - \frac{v_d^2}{2g}$$

$$\begin{array}{c}\text{Efficiency of}\\ \text{diffuser}\end{array} = \frac{\text{Theoretical head saved} - \text{loss in diffuser}}{\text{Theoretical head saved}}$$

EXAMPLE.

A centrifugal pump has a total lift of 50 ft. from well to delivery tank. The wheel is 5 ft. above the well water surface. The velocity of delivery from the uptake is 5 ft. per sec.; the radial velocity of flow through the wheel is 10 ft. per sec.; the tangent to the vane at exit from the wheel makes an angle of 120° with the direction of motion; the water enters the wheel radially. Find (1) the velocity of the wheel at exit; (2) the pressure head at exit from the wheel; (3) the velocity head at exit from the wheel; (4) the desirable direction for the fixed guide vanes. Neglect friction and other losses. (London Univ.)

$$\text{Total head} = 50 + \frac{v_d^2}{2g}$$

$$= 50 + \frac{5^2}{2g} = 50{\cdot}39 \text{ ft.}$$

Referring to velocity diagrams in Fig. 159.

$$\phi = 180 - 120 = 60°$$

$$V_{w_1} = v_1 - \frac{10}{\tan\phi} = v_1 - 5{\cdot}77$$

(1) $$\frac{V_{w_1} v_1}{g} = H + \frac{v_d^2}{2g}$$

Or, $$\frac{v_1(v_1 - 5{\cdot}77)}{g} = 50{\cdot}39$$

From which, $$v_1 = 43{\cdot}23 \text{ ft. per sec.}$$

(2) $$V_{w_1} = v_1 - \frac{10}{\tan 60}$$

$$= 43{\cdot}23 - 5{\cdot}77$$

$$= 37{\cdot}46 \text{ ft. per sec.}$$

$$V_1 = \sqrt{V_{w_1}^2 + V_{f_1}^2}$$

$$= \sqrt{37{\cdot}46^2 + 10^2}$$

$$= 38{\cdot}8 \text{ ft. per sec.}$$

Apply Bernoulli's equation to the pump at outlet and to the outlet end of delivery pipe, and taking the level of the pump as datum,

$$H_p + \frac{V_1^2}{2g} = 50 + \frac{v_d^2}{2g} - 5$$

where H_p = pressure head at vane outlet.

Therefore,
$$H_p = 50 + \cdot 39 - 5 - \frac{(38\cdot 8)^2}{2g}$$
$$= 50 + \cdot 39 - 5 - 23\cdot 3$$
$$= 22\cdot 1 \text{ ft. of water.}$$

(3)
$$\frac{V_1^2}{2g} = \frac{(38\cdot 8)^2}{2g} = 23\cdot 3 \text{ ft. of water.}$$

(4) The fixed guide vanes will be parallel to the absolute velocity of water at outlet, i.e. will be parallel to V_1. From velocity diagram at outlet,

$$\tan \alpha = \frac{V_{f_1}}{V_{w_1}}$$

where β = inclination of guide vanes.

Then,
$$\tan \alpha = \frac{10}{37\cdot 46} = \cdot 267$$

From which,
$$\alpha = 15^\circ.$$

130. **The Specific Speed of a Centrifugal Pump.** The specific speed of a centrifugal pump is the speed at which the pump would deliver 1 gallon of water per minute under a head of 1 ft. It may be found by applying the principle of similarity to centrifugal pumps; the method is the same as that used for finding the specific speed of water turbines in Art. 119. The specific speed is useful as an index for denoting the type of pump; the value varies between 500 and 8,000 for a single impeller.

In working out the equation for the specific speed the assumption is made that all pumps are geometrically similar, then all linear dimensions will be in proportion to the diameter of the impeller. Also, the velocity diagrams for all pumps are

assumed to be similar, and all velocities are proportional to the square root of the total head.

Using the same notation as for turbines (Art. 113),

let d = external diameter of impeller

n = speed in revs. per min.

n_s = specific speed in revs. per min.

h = total head or lift in feet

Q = discharge in galls. per min.

Then, $b \propto d$

and as $v = \omega \frac{d}{2}$

and $\omega \propto n$

then, $v \propto nd$

Or $d \propto \frac{v}{n}$

But $v \propto \sqrt{h}$

Hence, $d \propto \frac{\sqrt{h}}{n}$ . . . . . (1)

Now, $Q \propto$ area of flow $\times$ vel. of flow

That is, $Q \propto \pi d\, b \times V_f$

But $V_f \propto \sqrt{h}$

Hence, $Q \propto d^2\sqrt{h}$

Or, substituting for d from Eq. (1),

$$Q \propto \frac{h}{n^2} \times \sqrt{h}$$

Or, $Q \propto \frac{h^{\frac{3}{2}}}{n^2}$

Hence, $n \propto \frac{h^{\frac{3}{4}}}{\sqrt{Q}}$

This may be written, $n = k\,\frac{h^{\frac{3}{4}}}{\sqrt{Q}}$

where k is a constant.

When h equals 1 ft. and Q is 1 gal. per min. it will be noticed that $n = k$, and also $n = n_s$ from definition of specific speed.

Hence, $$k = n_s = \frac{n\sqrt{Q}}{h^{\frac{3}{4}}}$$

As an example of the use of specific speed, suppose it is required to raise 30,000 gals. per min. through a height of 20 ft. at a pump speed of 2,000 revs. per min., and assume the value of n_s for a single impeller to be 8,000. Then,

$$8000 = \frac{2000\sqrt{Q}}{20^{\frac{3}{4}}}$$

From which, $Q = 1430$ gals. per min. per impeller.

$$\text{Least No. of impellers} = \frac{30{,}000}{1430} = 21$$

Hence, 21 impellers would be required, in parallel.

131. **Principle of Similarity Applied to Centrifugal Pumps.** This principle may be applied to centrifugal pumps in order to predict the performance of a future design from the results of tests on a model of the same proportion. A model is constructed and tested under known conditions; the horse-power required to drive it, the speed, the head, and the discharge being measured. From these results the horse-power, speed, and discharge of the large pump under a known head can be calculated.

In Art. 130 it was proved that

$$d \propto \frac{\sqrt{h}}{n} \quad \ldots \ldots (1)$$

$$Q \propto d^2\sqrt{h} \quad \ldots \ldots (2)$$

$$n \propto \frac{h^{\frac{3}{4}}}{\sqrt{Q}} \quad \ldots \ldots (3)$$

Also, $$\text{horse-power} = P = \frac{62{\cdot}4\,Qh}{550}$$

or, $$P \propto Qh \quad \ldots \ldots (4)$$

Substituting for Q from Equation (3)

$$P \propto \frac{h^{\frac{5}{2}}}{n^2} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From these five equations the performance of the large pump may be calculated from the results of the model test.

EXAMPLE.

It is required to predict the performance of a large centrifugal pump from that of a scale model one-fourth the diameter. The model absorbs 20 h.p. when pumping under the test head of 20 ft. at its best speed of 400 revs. per min. The large pump is required to pump against 60 ft. head. What will be its working speed, the horse-power required to drive it, and what will be the ratio of the quantities discharged by the larger pump and the model? (London Univ.)

From Equation (1), $\frac{nd}{\sqrt{h}}$ is a constant for both large pump and model; hence,

$$\frac{nd}{\sqrt{h}} \text{ for model} = \frac{nd}{\sqrt{h}} \text{ for large pump}$$

or,

$$\frac{400 \times d}{\sqrt{20}} = \frac{n \times 4d}{\sqrt{60}}$$

From which, $n = 173$ revs. per min.

From Equation (5), $\frac{Pn^2}{h^{\frac{5}{2}}}$ is a constant

hence,

$$\frac{Pn^2}{h^{\frac{5}{2}}} \text{ for model} = \frac{Pn^2}{h^{\frac{5}{2}}} \text{ for large pump}$$

Or,

$$\frac{20 \times 400^2}{20^{\frac{5}{2}}} = \frac{P \times 173^2}{60^{\frac{5}{2}}}$$

From which, $P = 1670$

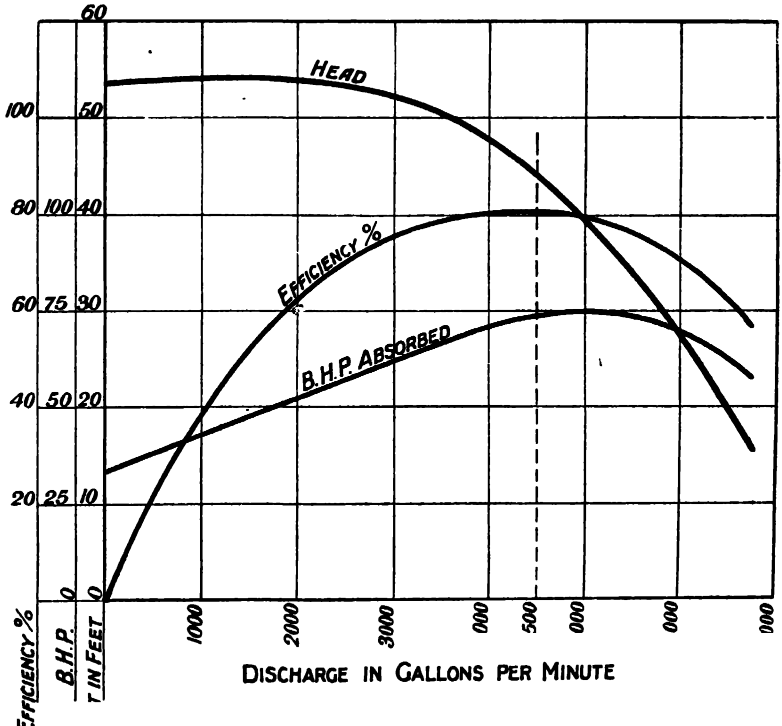
From Equation (2),

$$\frac{Q \text{ for large pump}}{Q \text{ for model}} = \frac{d^2\sqrt{h} \text{ for large pump}}{d^2\sqrt{h} \text{ for model}}$$

$$= \frac{16d^2 \times \sqrt{60}}{d^2 \times \sqrt{20}}$$

$$= 27{\cdot}7$$

132. **Characteristic Curves.** From the results of tests on a centrifugal pump when running at its design speed, curves may be plotted showing the relation between efficiency, brake horse-power, head, and discharge. These are known as characteristic curves, and are plotted for one speed only.



(Worthington-Simpson, Ltd.)

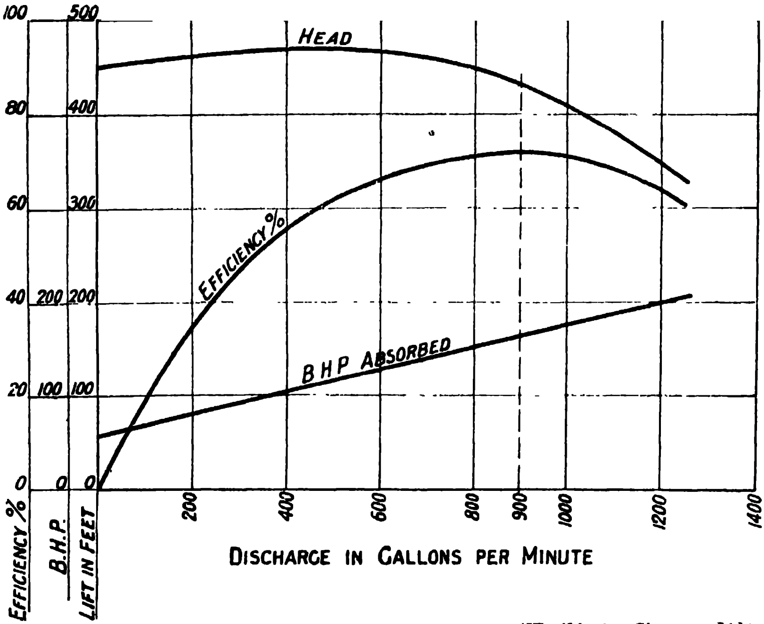
FIG. 162.—PERFORMANCE OF SINGLE IMPELLER LOW-LIFT PUMP
Output, 4,500 gallons per minute; head, 44½ ft. speed, 875 r.p.m.

From these curves it is possible to estimate the performance of the pump under any condition of working at the speed for which it was designed. Characteristic curves of a single impeller low lift pump are shown in Fig. 162; in Fig. 163 are shown the characteristic curves of a multi-stage high lift pump. Both these sets of curves were taken from Worthington pumps,* and were supplied by the makers. The dotted vertical line on each diagram is drawn through the point of maximum

* By courtesy of Messrs. Worthington-Simpson, Ltd.

efficiency ; from the points of intersection of this line with the other curves, the best conditions of running may be read off.

133. **The Least Diameter of Impeller.** It is usual to make the outside diameter of an impeller to be twice the inner diameter. On this assumption, it is possible to obtain an expression for the minimum diameter of an impeller which will enable it to start pumping when running at its normal



(*Worthington-Simpson, Ltd.*)

FIG. 163.—PERFORMANCE OF MULTI-STAGE HIGH-LIFT PUMP

Output, 900 gallons per minute, against a head of 430 ft.; speed, 1,450 r.p.m.

speed. The solution is based on the result of Art. 127 which showed that, for the pump to start pumping, the centrifugal head must equal the actual lift.

From the equation of Art. 127,

$$\text{Actual lift} = h = \frac{v_1^2}{2g} - \frac{v^2}{2g}$$

As $v = \omega r$ and $v_1 = \omega r_1$,

$$h = \frac{\omega^2}{2g}(r_1^2 - r^2)$$

Let d_1 = outer diameter of impeller

d = inner diameter of impeller

Then, $$h = \frac{\omega^2}{8g}(d_1^2 - d^2)$$

But $$d_1 = 2d$$

Hence $$h = \frac{3\omega^2 d_1^2}{32g}$$

From which, $$d_1 = \frac{18{\cdot}54}{\omega}\sqrt{h}\text{ ft.}$$

But, $$\omega = \frac{2\pi n}{60}$$ where n = no. of revs. per min.

Hence, $$d_1 = \frac{18{\cdot}54 \times 60}{2\pi n}\sqrt{h}$$

$$= \frac{177\sqrt{h}}{n}\text{ ft.}$$

$$= \frac{2120\sqrt{h}}{n}\text{ in.}$$

Assuming a manometric efficiency of ·75, the actual lift h will equal ·75 of the theoretical lift H.

Then, $$d_1 = \frac{1840\sqrt{H}}{n}\text{ in.}$$

This equation is used in practice for the design of impellers; the outside diameter should be at least this amount, otherwise the impeller will be unable to start pumping at its normal speed.

134. **The Design of a Turbine Pump.** The following is a rough outline of the design of the piping, impeller, and diffuser of a turbine pump, and is based on the matter already dealt with in this chapter. It is assumed that the required discharge, actual lift and speed are given. The actual lift should not be more than 140 ft. for one impeller, if more than this amount a multi-stage pump should be used consisting of two

or more identical impellers in series. The speed should be between 1,000 and 2,000 revs. per min.

Let Q = required discharge in gals. per min.

h = actual lift in feet per impeller, neglecting pipe losses.

The remaining notation will be the same as given in Art. 113.

First find the specific speed from the equation of Art. 130; if this does not lie below 8,000, more impellers must be used.

THE SUCTION PIPE. Let v_s = velocity in suction pipe in ft. per sec. In practice this velocity varies between 5 and 15 ft. per sec., assume an average value of 10 ft. per sec. for v_s.

$$\text{Then, area of suction pipe} = \frac{Q}{6{\cdot}24 \times 60 \times v_s} \text{ sq. ft.}$$

From this the diameter may be obtained. Use the nearest whole number to this from a list of British Standard Pipes.

THE DELIVERY PIPE. Let v_d = velocity in delivery pipe in ft. per sec. In practice this varies between the same limits as the suction pipe; hence, assume an average value of about 10 ft. per sec.

$$\text{Then, area of delivery pipe} = \frac{Q}{6{\cdot}24 \times 60 \times v_d} \text{ sq. ft.}$$

Hence, find the nearest standard pipe to suit.

THE IMPELLER. Assume the outside diameter to be twice inside diameter. Calculate the outside diameter d_1 from the equation given in Art. 133.

Assuming a manometric efficiency of ·75, the theoretical lift

$$H = \frac{h}{{\cdot}75}$$

Then, $$d_1 = \frac{1840\sqrt{H}}{n} \text{ in.}$$

and $$d = \frac{d_1}{2} \text{ in.}$$

Hence, v and v_1 may now be calculated, as $v = \omega \times$ radius.

In practice, the blade angles θ and ϕ vary between 12° and 30°. Choose values for θ and ϕ between these limits, these may require altering later if the velocities obtained from them do not suit.

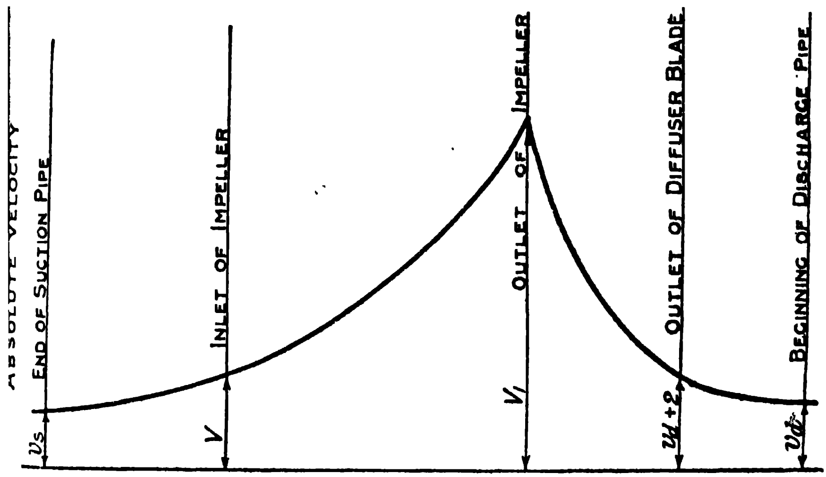
As θ and v are now known, the inlet diagram may be drawn

on the assumption that the water enters radially (Art. 126). Also, from the equation of Art. 126,

$$H = \frac{V_{w_1} v_1}{g}$$

hence V_{w_1} is obtained.*

As ϕ, v_1 and V_{w_1} are now known, the outlet diagram can be drawn (Art. 126). All velocities in the impeller can be obtained from the inlet and outlet triangles.



PATH OF WATER

FIG. 164

To find the width of the impeller at inlet use the equation—

$$\frac{Q}{6{\cdot}24 \times 60} = \pi d b V_f$$

To find the width at outlet use the equation—

$$\frac{Q}{6{\cdot}24 \times 60} = \pi d_1 b_1 V_{f_1}$$

If these widths b and b_1 are not suitable for practical purposes, the assumed angles θ and ϕ must be altered; this can only be done by trial. Another check to prove whether the

* An alternative method is to assume the velocity of flow to be constant throughout the impeller and equal to about 10 ft. per sec. Then θ and ϕ can be obtained from the velocity diagrams at inlet and outlet.

angles θ and ϕ are suitable is to plot the absolute velocity of the water as it passes through the pump, as shown in Fig. 164. The base of this graph represents the centre line path of the water through the pump as a measured distance. The vertical ordinate represents the absolute velocity of the water. The absolute velocities v_s and v_d are known ; the absolute velocities V and V_1 can be obtained from the velocity triangles ; the lengths of the path of the water at various points in the pump may be estimated from an existing pump of a similar design.

v_s is plotted at the point where the suction pipe is attached to the pump casing.

V is plotted at the beginning of the blade.

V_1 is plotted at the tip of the blade.

v_d is plotted at the point where the delivery pipe is attached to the casing.

In a well-designed pump, the absolute velocity of the water should increase smoothly from the suction pipe to the outlet of the impeller blade ; it should then fall smoothly in the diffuser until the delivery pipe is reached, as shown in Fig. 164. Any abrupt change in the absolute velocity will cause a loss of head and should be avoided. If, after plotting these velocities, a suitable curve is not obtained, the assumed values of θ and ϕ must be altered.

The number of blades in the impeller vary with the size ; six would be sufficient for a small pump and twelve for a large pump.

THE DIFFUSER. The diffuser should have about the same number of blades as the impeller, and as these blades are at rest, the relative velocity of the water to the blade will be the absolute velocity of the water. Hence, in order that the water will glide over the diffuser blade without shock, the blade at inlet of diffuser must be parallel to the absolute velocity V_1 of the water leaving the impeller ; that is, to the angle β (Fig. 159). The water will glide over the diffuser blade, and as the area of flow becomes larger with the increased radius of the diffuser, the velocity will become smaller. The water will flow from the diffuser blade into the whirlpool chamber in a direction parallel to the diffuser blade at outlet, and as the flow of the water in the whirlpool chamber is circumferential it follows, therefore, that the diffuser blade angle at outlet

should be as small as possible, which is about 10° to 15°. Hence, assume an angle of 10° for the diffuser blade an outlet.

After leaving the diffuser blade the water will pass through the whirlpool chamber into the discharge pipe. The object of the whirlpool chamber is to collect the water from the diffuser blades and to provide a passage to the discharge pipe; hence, it is spiral in shape owing to its cross-sectional area increasing uniformly up to the discharge pipe diameter; this allows for the increasing volume of water flowing through it.

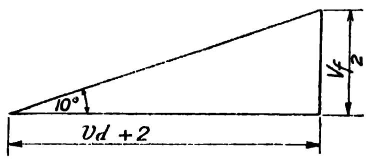


FIG. 165

The full reduction of velocity from V_1 to v_d takes place in the diffuser, but as there is an unavoidable loss of about 2 ft. per sec. in the whirlpool chamber, the water should leave the diffuser blade with a circumferential velocity of $v_d + 2$. Then the outlet velocity triangle for the diffuser blade will be as shown in Fig. 165; it is a 10° right-angled triangle with a base of $v_d + 2$. The hypotenuse represents the actual velocity when leaving the blade, the radial component of this will be the velocity of the flow V_{f_2} which is lost in shock in the whirlpool chamber.

Let b_2 = breadth of diffuser at outlet, in feet.

d_2 = diameter of diffuser at outlet, in feet.

Then, $Q = \pi d_2 b_2 V_{f_2}$

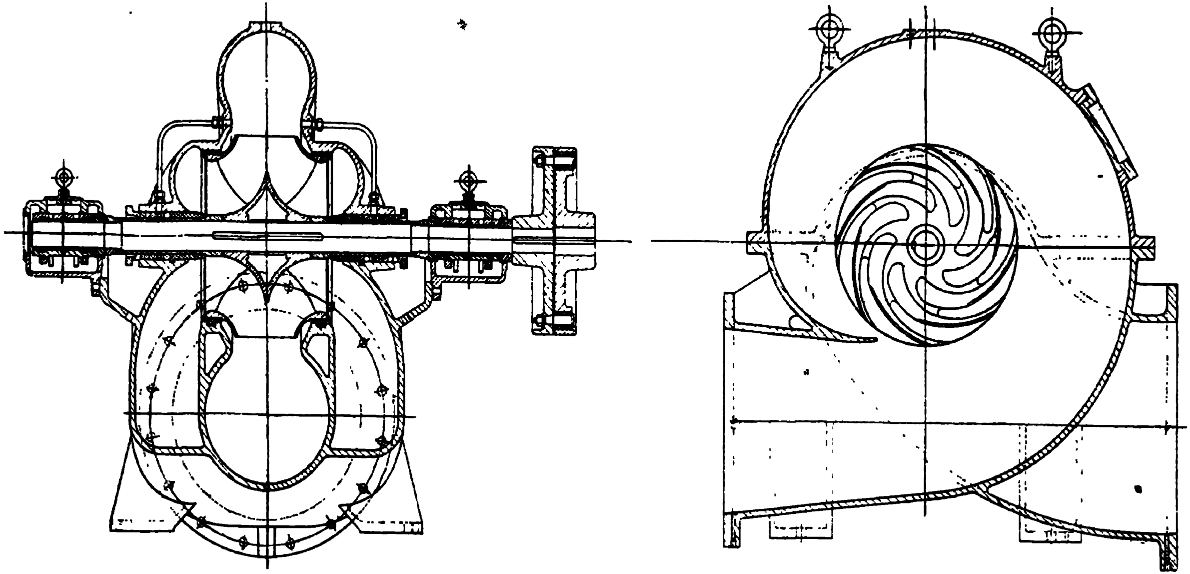
From this equation b_2 can be found if d_2 is first assumed, as V_{f_2} is already known.

As a first attempt, assume d_2 to be 6 in. greater than d_1; if this does not give a suitable value for b_2 another value for d_2 must be chosen.

A view of a single impeller Worthington-Simpson pump* is shown in Fig. 166.

135. **The Multi-stage Pump.** A single impeller will produce a head of not more than 140 ft.; if a larger head than this is

* By courtesy of Messrs. Worthington-Simpson, Ltd.



(*Worthington-Simpson, Ltd.*)

FIG. 166.—HORIZONTAL LOW-LIFT CENTRIFUGAL PUMP

required other impellers are fitted in series, so that the discharge from the first impeller is guided into the inlet of the second impeller. This is repeated with the third impeller, and so on, until the required head is reached ; each impeller will increase the water pressure by the same amount. A pump of this type is called a multi-stage pump, and may be a two-stage, three-stage, etc., according to the number of impellers fitted in the casing. A view of a four-stage pump* is shown in Fig. 167.

All the impellers are keyed to the same shaft, and usually, all impellers and diffusers of one pump are identical ; this has the advantage of reducing the labour in manufacture. The discharge from each diffuser is either circumferential or radial, this is collected by vanes attached to the casing which deflect the water into the eye of the next impeller. The last diffuser will discharge into the delivery pipe.

The design of an impeller and diffuser of a multi-stage pump is the same as for a single stage pump (Art. 134), the head used for design being the head per impeller.

EXAMPLES 11.

(1) A centrifugal pump is required to deliver 6,300 gallons of water per minute, against a head of 20 ft., at a speed of 600 revs. per min. Assuming that all the velocity head is lost, and that the actual head is 75 per cent of the theoretical head, find the diameter and breadth of the impeller at outlet. The velocity of flow, taken as constant, is 10 ft. per sec., and the blades are curved back 30° to the tangent at outlet. Also determine the inlet blade angles, if the inlet diameter is made half the outlet diameter. (London Univ.)

Ans.—$d = 1{\cdot}25$ ft. ; $b = 5{\cdot}15$ in. ; $\theta = 27°$.

(2) A centrifugal pump is employed to pump water from a river into a canal. The pump is fixed with its centre at a height of 20 ft. above the level of the surface of the water in the river, and the mouth of the delivery pipe is 30 ft. above the level of the surface of the water in the river. The velocity of flow through the delivery pipe is 5 ft. per sec. If the angle made by the tangent to the blade with the tangent to the wheel at the discharge edge is 125°, and if the radial velocity of flow through the wheel is 5 ft. per sec., determine (1) the pressure head at the inlet circumference of the wheel, and (2) the pressure at the outlet circumference of the wheel. (London Univ.)

Ans.—(1) 13·61 ft. of water (abs.) ; (2) 30·39 ft. of water (abs.).

(3) A centrifugal pump is placed with the centre of the impeller at a height of 12 ft. above the water in the suction well. The suction pipe is 5 in. in diameter, and the discharge is 350 gallons per min. The total head through which the water is lifted is 75 ft. The vanes of the impeller at exit are set back and make an angle of 150° with the tangent to the wheel. The radial velocity at exit from the wheel is 10 ft. per sec., and the efficiency of the pump is 70 per cent. Determine (*a*) the velocity of the rim of the wheel ;

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* By courtesy of Messrs. Worthington-Simpson, Ltd.

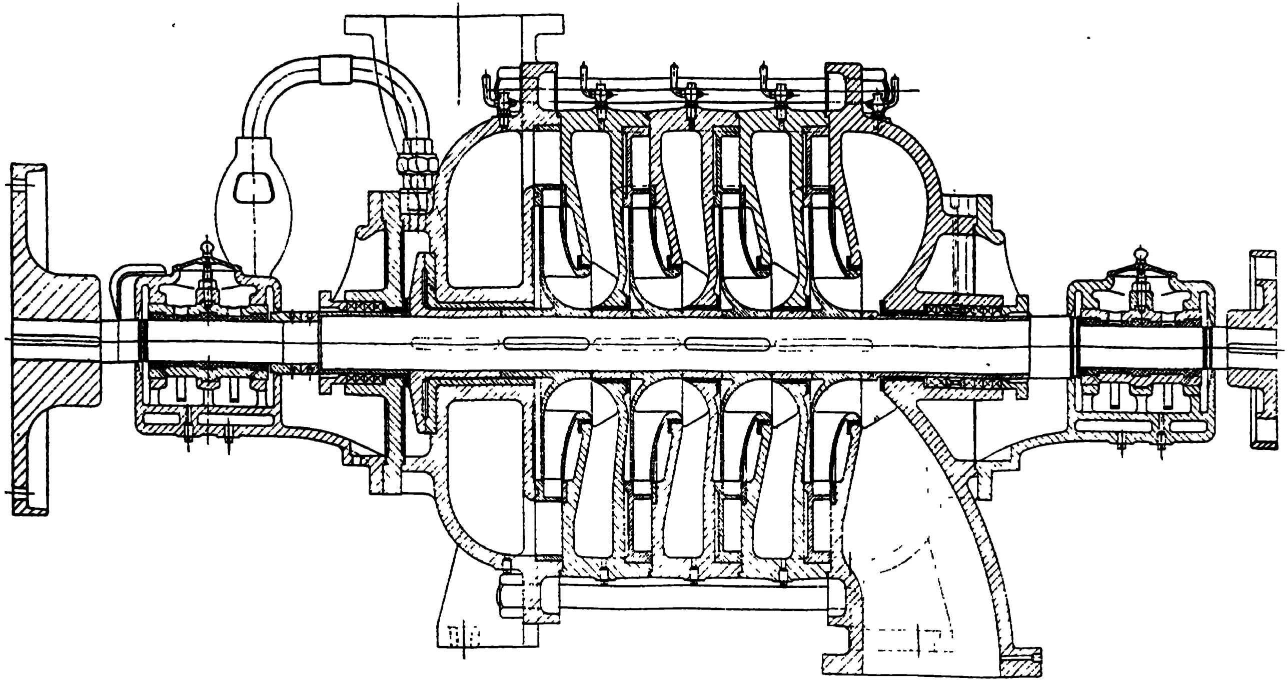

(Worthington-Simpson, Ltd.)

FIG. 167.—FOUR-STAGE HIGH-LIFT TURBINE PUMP

(*b*) the pressures at the inlet and outlet of the wheel, on the assumption that the whole loss of head takes place after the water leaves the wheel. (London Univ.)

Ans.—(*a*) 68·86 ft. per sec.; (*b*) 21·27 and 86·3 ft. of water (abs.).

(4) A centrifugal pump having a wheel 1 ft. outside diameter rotates at 1,000 revs. per min. The vanes are radial at exit and are 3 in. wide. The velocity of radial flow through the wheel is 10 ft. per sec. The velocities in the suction and delivery pipes are 8 and 5 ft. per sec. respectively. Neglecting frictional losses, determine (1) the height through which the pump lifts; (2) the horse-power of the pump. (London Univ.)

Ans.—(1) 84·61 ft.; (2) 75·6 h.p.

(5) A centrifugal pump wheel is 20 in. external and 10 in. internal diameter. It runs at 950 revs. per min. The vanes are set back at an angle of 35° to the outer rim. If the radial velocity of the water through the wheel be maintained constant 6 ft. per sec., find the angle of the vanes at inlet, the velocity and direction of the water at outlet, and the work done by the wheel per pound of water. (London Univ.)

Ans.—$8\frac{1}{2}$°; 74·4 ft. per sec.; $4\frac{1}{2}$°; 191 ft. lb.

(6) A centrifugal pump of 4 ft. diameter runs at 200 revs. per min., and pumps 66·5 cu. ft. per sec., the average lift being 20 ft. The angle which the vanes make at exit with the tangent to the impeller is 26°, and the radial velocity of flow is 8 ft. per sec. Determine the useful horse-power and the efficiency. Find also the lowest speed to start pumping against a head of 20 ft., the inner diameter of the impeller being 2 ft. (London Univ.)

Ans.—151 h.p.; 60·6 per cent; 198 revs. per min.

(7) Give a short account of the various methods which have been adopted to increase the efficiency of a centrifugal pump by altering the shape of the casing or chamber surrounding the impeller. (London Univ.)

(8) A centrifugal pump running at 390 revs. per min. discharges 4 cusecs. The impeller is 10 in. diameter at inlet and 21 in. at outlet; the inlet width is 5 in., and the outlet width $3\frac{1}{2}$ in. Neglecting friction losses and the thickness of the vanes, what is the head pumped against if the vanes at outlet are curved back to give a discharge angle of 28°? (A.M.I. Civil E.)

Ans.—34·4 ft.

(9) A centrifugal pump has an impeller 4 ft. in diameter, whose peripheral speed is 30 ft. per sec. Water enters the eye of the pump radially and is discharged with a velocity whose radial component is 5 ft. per sec. The vanes are curved backward at exit and make an angle of 30° with the periphery. If the pump discharges 120 cu. ft. per min. what will be the turning moment on the shaft? (A.M. Inst. C.E)

Ans.—166 lb. ft.

(10) The impeller of a centrifugal pump has an external diameter of 12 in. and an internal diameter of 6 in. If full of water, with the discharge pipe closed, what would be the difference of pressures at the outer and inner periphery, corresponding to a speed of 300 revs. per min? (A. M. Inst. C. E.)

Ans.—2·87 ft. of water.

(11) A propeller pump 9 in. diameter was found to be most efficient when delivering 2·8 cusecs. at 1,200 r.p.m. against a head of 18 ft. of water. A similar pump is required to deliver 50 cusecs. at 700 r.p.m. Calculate the pump diameter and the head it will develop. (London Univ.)

Ans.—d = 28·2 in.; H = 59·6 ft.

CHAPTER XII

VISCOUS RESISTANCE OF FLUIDS

136. **Viscous Flow.** A fluid flowing steadily in a channel with smooth sides will have a velocity which varies over the cross-section of the channel. The layers of fluid adjoining the sides will be at rest; the layers adjacent to these will have a small velocity; there will then be a gradual increase in the velocity of the layers of fluid as the centre of the cross-section is approached. It follows from this that any two adjacent layers will be moving at different velocities, and there will be a resistance to flow between them. This resistance between two adjacent layers is known as viscosity.

The viscous resistance of a fluid is analogous to the shear resistance of a solid and is probably due to molecular attraction acting on planes inclined to the direction of flow. The frictional resistance of a fluid in a rough pipe is actually due to viscosity, as the rough sides of the pipe will cause cross-currents or eddies, the energy of which are ultimately destroyed by viscous resistance. The resistance of ships or other bodies moving in a fluid is due to viscosity; the movement of the body in the fluid sets up eddies and waves which are ultimately destroyed by viscosity.

The viscous resistance of a fluid will depend on its coefficient of viscosity, on the temperature, its density, its velocity, and on the size of the channel; an equation may be obtained which is applicable to all fluids, whether liquids or gases, which will hold for velocities below and above the critical velocity, and which is suitable for channels of all sizes, from small capillary tubes to large pipes. A viscous resistance is independent of the material of which the pipe is made.

137. **Coefficient of Viscosity.** The phenomenon of viscosity was first investigated by Sir Isaac Newton. Let Fig. 168 represent the section of a channel or pipe through which a fluid is flowing. Consider a longitudinal layer of fluid a distance of y from the side of channel and let the thickness of this layer be dy. Let the velocity of the layer at distance y from the side be v and let the velocity increase by dv over the

layer. Consider a section *abcd* of the layer when the fluid is at rest; then, when the fluid is flowing, the section *abcd* will distort to the shape *aefd* in one second. An enlarged view of the distorted section is shown in the figure. The resistance of the layer to this distortion is known as a viscous resistance, and will cause a stress, known as a viscous stress, to act on the layer. If the fluid in the whole channel section is considered to consist of similar layers, it follows that the

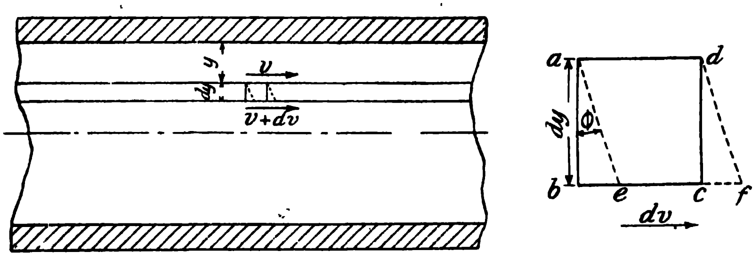


FIG. 168

velocity over the section will increase from the sides to the centre.

Let ϕ = angle of distortion due to viscosity

f = viscous stress

= resistance per unit area

μ = coefficient of viscosity of fluid.

The coefficient of viscosity* of a fluid is defined as the relation between the viscous stress and the angle of distortion.

Or, $$\mu = \frac{f}{\phi}$$

But, from figure 168, $$\phi = \frac{dv}{dy}$$

Hence, $$\mu = \frac{f}{\frac{dv}{dy}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From which, $$f = \mu\frac{dv}{dy} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* For experimental methods of measuring the coefficient of viscosity, see Art. 2I1.

It is interesting to notice the analogy between this viscous distortion of a fluid and the shear distortion of a solid. Consider the solid *abcd* (Fig. 169) under a shear stress q; the solid will distort to the shape *aefd*, ϕ being the angle of shear distortion. If C is the shear modulus or modulus of rigidity,

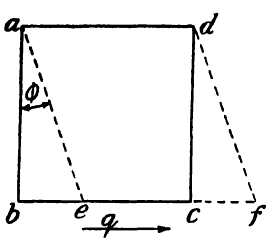


FIG. 169

then $$C = \frac{q}{\phi}$$

This is analogous to $\mu = \frac{f}{\phi}$, the coefficient of viscosity μ corresponding to the modulus of rigidity C, and the viscous stress corresponding to the shear stress q.

If M, L, and T represent the fundamental dimensional units of mass, space, and time, the units of μ may be found by substituting these in Equation (1).

$$\mu = f \div \frac{dv}{dy}$$

$$= \frac{\text{force}}{\text{area}} \div \frac{\text{velocity}}{\text{distance}}$$

$$= \frac{\text{mass} \times \text{acceleration}}{\text{area}} \div \frac{\text{velocity}}{\text{distance}}$$

$$= \left(\frac{M}{L^2} \times \frac{L}{T^2}\right) \div \left(\frac{L}{TL}\right)$$

$$= \frac{M}{TL}$$

138. **Effect of Temperature on Viscosity.** Poiseuille investigated the viscous resistance of water flowing through capillary tubes* and found that the resistance to flow varied inversely with the temperature. This is, of course, well known in the case of oils, which flow more easily when warm.

Let = coefficient of viscosity at any temperature t in degrees centigrade.

μ_0 = coefficient of viscosity at 0° C.

* *Comptes Rendus*, 1840–41.

Then, from Poiseuille's experiments,

$$\mu = \mu_0 \left(\frac{1}{1 + at + bt^2}\right)$$

where a and b are constants.

For water, Poiseuille found that

$$\mu = \mu_0 \left(\frac{1}{1 + \cdot 03368t + \cdot 000221t^2}\right)$$

$$= \frac{\cdot 0179}{(1 + \cdot 03368t + \cdot 000221t^2)} \text{ C.G.S. units*}$$

$$\doteq \frac{\cdot 0179}{1 + \cdot 03368t + \cdot 000221t^2} \times \frac{30\cdot 5}{453\cdot 6 \times 32\cdot 2} \text{ ft. lb. units}$$

$$= \frac{\cdot 00003716}{(1 + \cdot 03368t + \cdot 000221t^2)} \text{ ft. lb. units} \quad . \quad . \quad (1)$$

The relation between the coefficient of viscosity and the density is known as the kinematic viscosity.

Let ν = kinematic viscosity

ρ = density of fluid (absolute units)

$= \frac{w}{g}$ (engineer's units)

Then $\nu = \frac{\mu}{\rho}$

Substituting for μ from Equation (1), for water.

$$\nu = \left(\frac{\cdot 00003716}{1 + \cdot 03368t + \cdot 000221t^2}\right) \times \frac{32\cdot 2}{62\cdot 4}$$

$$= \frac{\cdot 00001926}{(1 + \cdot 03368t + \cdot 000221t^2)} \text{ sq. ft. per sec.} \quad . \quad (2)$$

It will be noticed that the units of the kinematic viscosity ν are $\frac{L^2}{T}$; for,

$$\nu = \mu \div \rho$$

$$= \frac{M}{LT} \div \frac{M}{L^3}$$

$$= \frac{L^2}{T}$$

* This absolute unit is known as a "Poise."

139. Streamline and Turbulent Flow. The experiments of Professor Osborne Reynolds on streamline and turbulent flow of fluids have been described in Art. 68. Reynolds found that

$$v_c = \frac{2000\ \mu}{d\rho}$$

$$= \frac{2000\,\nu}{d}$$

where v_c = lower critical velocity

and d = diameter of pipe.

This may be written

$$\frac{v_c d}{\nu} = 2000$$

and applies to any fluid and to any system of units.

The quantity $\frac{vd}{\nu}$ is called the Reynolds number and is represented by the symbol R_e; it is very important when dealing with problems on the viscous resistance of fluids.

Exhaustive experiments on the flow of fluids in pipes were carried out by Stanton and Pannel.* Their results were plotted with log. $\frac{vd}{\nu}$ as base and $\frac{R}{\rho v^2}$ as ordinate, where R is the viscous resistance per square foot of wetted surface. The curve obtained is shown in Fig. 170. The portion AB of the curve represents their experimental results on streamline flow, the point B being the lower critical velocity. The portion BC represents the results for the state of change from streamline flow to turbulent flow, the point C being the higher critical velocity. The portion CD of the curve represents their experimental results on turbulent flow.

At the lower critical velocity, point B, it was found from the curve that

$$\frac{v\,d}{\nu} = 2000$$

* *Phil. Trans.*, Vol. 214.

At the higher critical velocity, point C, it was found that

$$\frac{vd}{\nu} = 2500$$

Hence it follows, that for any fluid in motion, if the quantity $\frac{vd}{\nu}$ is less than 2,000 the flow is streamline; if the quantity $\frac{vd}{\nu}$ is greater than 2,500 the flow is turbulent. Between these two values the fluid would be in a state of transition from one type of flow to the other. These values hold for all fluids, at all velocities and temperatures.

Stanton and Pannell also plotted on their curve the results of former experimenters on pipe flow and found that these results approximated to their own curve. The complete results plotted by Stanton and Pannell included experiments on the flow of water, air, and oil, through pipes varying in diameter from small capillary tubes to large water supply pipes of 5 ft. diameter. It was found that, excepting for a slight deviation due to the roughness of the inside of the large pipes, they all followed the curve of Fig. 170.*

EXAMPLE.

The density of a fluid A is 0·8, and its coefficient of viscosity is ·01 in C.G.S. units. The density of a second fluid B is 0·6 and its coefficient of viscosity is ·005. Which will have the lower critical velocity under given conditions of flow? What will be the ratio of the critical velocities? (London Univ.)

FLUID A.

$$\nu = \frac{\mu}{\rho} = \frac{\cdot 01}{\cdot 8} = \cdot 0125$$

For critical velocity,

$$\frac{v_c d}{\nu} = 2000$$

Hence, $$v_c = \frac{\cdot 0125 \times 2000}{d}$$

$$= \frac{25}{d} \text{ cm. per sec.}$$

* For latest research on pipe flow, see results of Nikuradse, Prandtl, and Von Kàrmàn, Arts. 213 and 214.

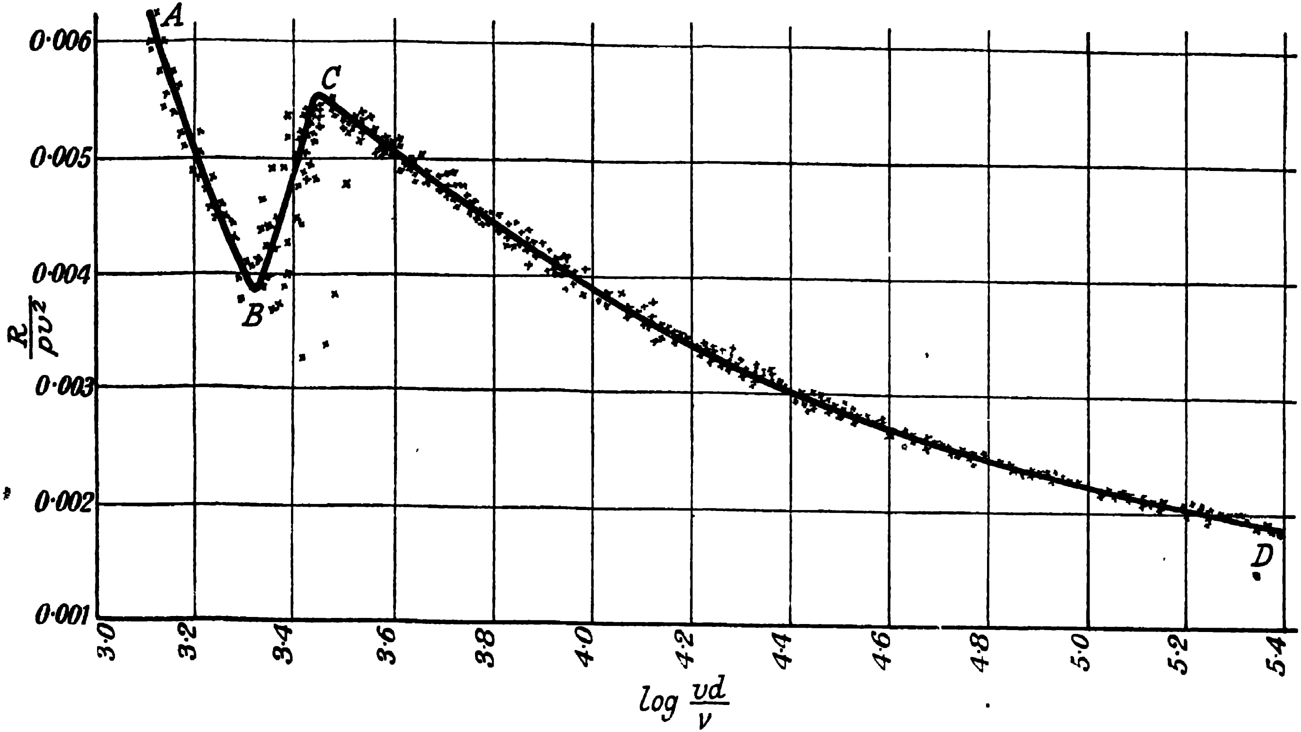

A
B
C
D
0·006
0·005
0·004
0·003
0·002
0·001
R/ρv²
3·0
3·2
3·4
3·6
3·8
4·0
4·2
4·4
4·6
4·8
5·0
5·2
5·4
log vd/ν

FIG. 170

FLUID *B*.

$$\nu = \frac{\mu}{\rho} = \frac{\cdot 005}{\cdot 6} = \cdot 00833$$

Then,

$$v_c = \frac{\nu \times 2000}{d}$$

$$= \frac{\cdot 00833 \times 2000}{d}$$

$$= \frac{16\cdot 66}{d}$$

Hence, fluid *B* has the lower critical velocity

$$\text{Ratio of critical velocities} = \frac{25}{d} \div \frac{16\cdot 66}{d}$$

$$= 1\cdot 5$$

140. **Viscous Flow through Round Pipes.** An equation for the viscous resistance in a round pipe may be obtained by equating the force on the fluid, due to the drop in pressure, to the viscous resistance. Consider a fluid to be flowing along a pipe, the cross-section of which is shown in Fig. 171, and consider a cylinder of fluid of radius y and of unit length.

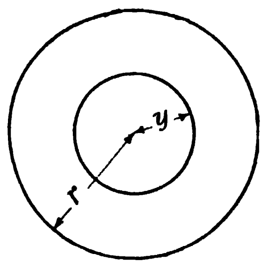


FIG. 171

Let r = radius of pipe
v = velocity of fluid
i = slope of hydraulic gradient
= head lost in resistance per unit length of pipe
p = pressure drop per unit length

$$\left.\begin{array}{l}\text{Viscous resistance of cylinder} \\ \text{of radius } y \text{ and unit length}\end{array}\right\} = \text{surface area} \times \text{viscous stress}$$

$$= 2\pi y \times f$$

$$= -2\pi y\mu \frac{dv}{dy} \qquad . \quad (1)$$

by substituting for f from Equation (2), Art. 137. It should be noted that $\frac{dv}{dy}$ will be negative as y is now measured outwards

from the centre. In Art. 137 y was measured inwards from the sides.

$$\left.\begin{array}{r}\text{Difference of total pressure}\\ \text{on ends of cylinder}\end{array}\right\} = \pi y^2 \times p$$

$$= \pi y^2 \rho g i \quad . \quad . \quad . \quad (2)$$

where $p =$ wt. per cub. ft. $\times$ loss of head per unit length

$= \rho g \times i$

as $\rho = \frac{w}{g}$ in engineer's units

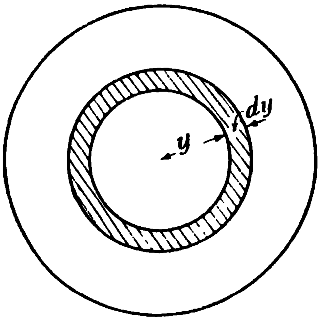


FIG. 172

As the viscous resistance of cylinder must equal the net force on the ends of the cylinder, Equation (1) will equal Equation (2).

Hence, $-2\pi y \mu \frac{dv}{dy} = \pi y^2 \rho g i$

From which, $dv = -\frac{\rho g i y dy}{2\mu}$

Integrating, $v = -\frac{\rho g i y^2}{4\mu} + c_1 \quad . \quad . \quad . \quad . \quad (3)$

where c_1 is the constant of integration.

When $y = r$, $v = 0$, as the fluid is stationary at the sides,

hence, $0 = -\frac{\rho g i r^2}{4\mu} + c_1$

From which, $c_1 = \frac{\rho g i r^2}{4\mu}$

Let $v_y =$ velocity at radius y

Then, from Equation (3), $v_y = \frac{\rho g i}{4\mu}(r^2 - y^2)$* $\quad . \quad . \quad (4)$

Next consider a hollow cylinder of the fluid of radius y and thickness dy (Fig. 172). Let Q be the total quantity flowing through the pipe per second.

* It will be noticed from this equation that maximum velocity occurs when $y = 0$; then maximum velocity $= \frac{\rho g i r^2}{4\mu}$.

$$\left.\begin{array}{l}\text{Then, quantity flowing}\\ \text{through hollow cylinder}\end{array}\right\} = dQ = \text{area} \times \text{velocity}$$

$$= 2\pi y\,dy \times v_y$$

$$= 2\pi y\,dy\,\frac{\rho g i}{4\mu}(r^2 - y^2)$$

from Equation (4)

Integrating between $y = r$ and $y = 0$

$$Q = \frac{\pi \rho g i}{2\mu}\int_0^r (r^2 y - y^3)\,dy$$

$$= \frac{\pi \rho g i}{2\mu}\left[\frac{r^2 y^2}{2} - \frac{y^4}{4}\right]_0^r$$

$$= \frac{\pi \rho g i r^4}{8\mu}$$

Let v = mean velocity of flow in pipe

$$= \frac{Q}{\text{area of cross-section}}$$

$$= \frac{\pi \rho g i r^4}{8\mu} \div \pi r^2$$

$$= \frac{\rho g i r^2}{8\mu}$$

$$= \frac{i g r^2}{8\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

as $\nu = \frac{\mu}{\rho}$ (Art. 138)

For a pipe flowing full, the hydraulic mean depth m is $\frac{r}{2}$, hence, substituting in Equation (5), and calling the diameter of pipe d,

$$m i g = \frac{8\nu v}{d}$$

Dividing both sides by v^2,

$$\frac{m i g}{v^2} = 8\left(\frac{vd}{\nu}\right)^{-1}$$

This may be written,

$$\frac{m i g}{v^2} = C\left(\frac{vd}{\nu}\right)^n \quad . \quad . \quad . \quad (6)$$

where C and n are constants, their values depending on whether the flow is stream line or turbulent.

Let R = viscous resistance per unit area of wetted surface. Then, as resistance at sides is equal to net force on fluid,

$$2\pi r \times R = p \times \pi r^2$$
$$= i\rho g \times \pi r^2 \text{ (as } p = i\rho g)$$

Hence $$R = \frac{\rho i g r}{2}$$

Putting $m = \frac{r}{2}$ and dividing both sides by v^2

$$\frac{R}{\rho v^2} = \frac{mig}{v^2} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Combining Equations (6) and (7),

$$\frac{R}{\rho v^2} = \frac{mig}{v^2} = C\left(\frac{vd}{\nu}\right)^n \quad . \quad . \quad . \quad (8)$$

which is the complete form of the equation for the flow of fluids in pipes.

If the term $\frac{vd}{\nu}$ is less than 2,000, the flow will be streamline* and will belong to the portion AB of the curve in Fig. 170; by plotting log $\frac{R}{\rho v^2}$ and log $\frac{vd}{\nu}$ of this portion of the curve the values of the constants C and n for streamline flow may be obtained.

If the term $\frac{vd}{\nu}$ is more than 2,500, the flow will be turbulent and will be represented by the portion CD of the curve of Fig. 170. Then, by plotting log $\frac{R}{\rho v^2}$ and log $\frac{vd}{\nu}$ for this portion of the curve the values of the constants C and n for turbulent flow are obtained.

For streamline flow,

$$\frac{vd}{\nu} \text{ is less than } 2000$$
$$C = 8$$
$$n = -1$$

* Also termed viscous or laminar flow.

For turbulent flow,

$$\frac{vd}{\nu} \text{ is greater than } 2500$$
$$C = \cdot 032^*$$
$$n = -\cdot 23^*$$

In all problems on pipe flow the valve of $\frac{vd}{\nu}$ must first be worked out; then, if less than 2,000, use the values $C = 8$ and $n = -1$, and solve from Equation (6). If the value of $\frac{vd}{\nu}$ is more than 2,500, use the values $C = \cdot 032$ and $n = -\cdot 23$, and solve from Equation (6).

It is interesting to notice that Equation (6) is another form of the well-known Chezy formula (Art. 67). By substituting in Equation (6), $m = \frac{d}{4}$ and $i = \frac{h}{l}$ the equation becomes—

$$h = \frac{4 f l v^2}{2 g d}$$

where the coefficient $\quad f = 2C\left(\frac{vd}{\nu}\right)^n$

$$= 2\, C R_e^{\,n}$$

It follows from this that the Darcy coefficient f is not a true constant, but varies with R_e; it is a function of the velocity, the diameter, the density, and the temperature.

The viscosity formula of Equation (6) is an alternative method for calculating the flow of fluids in pipes and gives results more accurate than the Chezy formula, although practical engineers still use the latter. The Chezy formula does not allow for the temperature of the fluid, which makes a considerable difference to the flow. For rough approximations the Chezy formula is the simpler to use, and it would be very difficult to apply the viscosity formula to some of the more complicated problems on pipe flow, given in Chapter VI, in which all losses must be expressed in terms of the velocity head.

EXAMPLE 1.

Water flows along a pipe of $\frac{1}{4}$ in. diameter and 100 ft. long; the pipe is running full. Find the loss of head when: (*a*) the temperature is 5° C. and the velocity is 1 ft. per sec.; (*b*) the temperature is 70° C. and the velocity is 10 ft. per sec.

* There is a slight variation in the values of C and n given by different authorities. From the author's plotting, $C = \cdot 048$ and $n = -\cdot 27$.

(*a*) From Equation (2), Art. 138,

$$\nu = \frac{\cdot 00001926}{1 + (\cdot 03368 \times 5) + (\cdot 000221 \times 25)}$$

$$= \cdot 0000164$$

Next find the value of the term $\frac{vd}{\nu}$.

$$\frac{vd}{\nu} = \frac{1}{\cdot 0000164} \times \frac{1}{4 \times 12} = 1270$$

As this is less than 2,000, flow is stream line hence $n = -1$ and $C = 8$.

Applying Equation (6),

$$\frac{m\,i\,g}{v^2} = 8\left(\frac{vd}{\nu}\right)^{-1}$$

$$= 8(1270)^{-1}$$

$$= \cdot 0063$$

From which, $$i = \frac{v^2}{\frac{d}{4}g} \times \cdot 0063$$

$$= \frac{1 \times \cdot 0063 \times 4}{\frac{1}{4} \times \frac{1}{12} \times 32\cdot 2}$$

$$= \cdot 0376 \text{ ft.}$$

But, $$h = i \times l$$

$$= \cdot 0376 \times 100$$

$$= 3\cdot 76 \text{ ft. of water.}$$

(*b*) $$\nu = \frac{\cdot 00001926}{1 + (\cdot 03368 \times 70) + (\cdot 000221 \times 4900)}$$

$$= \cdot 00000435$$

Then, $$\frac{vd}{\nu} = \frac{10}{\cdot 00000435 \times 4 \times 12} = 48{,}000$$

As this is more than 2,500, flow is turbulent, hence $n = -\cdot 23$ and $C = \cdot 032$.

Using Equation (6),

$$\frac{m\,i\,g}{v^2} = \cdot 032(48{,}000)^{-\cdot 23}$$

$$= \cdot 00272$$

From which, $$i = \frac{\cdot 00272\, v^2}{\frac{d}{4} \times g}$$

$$= \frac{\cdot 00272 \times 100 \times 4}{\frac{1}{4} \times \frac{1}{12} \times 32\cdot 2}$$

$$= 1\cdot 62$$

But, $$h = i \times l$$

$$= 1\cdot 62 \times 100$$

$$= 162 \text{ ft. of water.}$$

EXAMPLE 2.

Oil at a temperature of 60° F. has a weight of 57·2 lb. per cu. ft. and a kinematic viscosity of ·0205 ft. sec. units. Find the horse-power required to pump 20 tons of this oil per hour along a pipe line 6 in. diameter and 1,000 ft. long.

$$\text{Quantity per sec.} = \frac{\text{weight per sec.}}{\text{weight per cu. ft.}}$$

$$= \frac{20 \times 2240}{57\cdot 2 \times 3600} = \cdot 218 \text{ cu. ft.}$$

$$v = \frac{Q}{\text{area}} = \frac{\cdot 218}{\frac{\pi}{4} \times \frac{1}{4}} = 1\cdot 11 \text{ ft. per sec.}$$

Then, $$\frac{vd}{\nu} = \frac{1\cdot 11 \times \frac{1}{2}}{\cdot 0205} = 27\cdot 1$$

As this is less than 2,000, flow is stream line, hence $n = -1$ and $c = 8$.

Using Equation (6),

$$\frac{m\,i\,g}{v^2} = 8(27\cdot 1)^{-1}$$

$$= \cdot 296$$

From which, $$i = \frac{\cdot 296 \times (1\cdot 11)^2 \times 4}{\frac{1}{2} \times 32\cdot 2}$$

$$= \cdot 0907$$

But, $h = i \times l$

$= \cdot0907 \times 1000$

$= 90\cdot7$ ft. of oil

Then, horse-power required $= \dfrac{Wh}{550}$

$$= \frac{20 \times 2240}{3600} \times \frac{90\cdot7}{550}$$

$$= 2\cdot05$$

141. **Viscous Flow between Flat Surfaces.** The viscous flow of a fluid between parallel surfaces may be dealt with in a manner similar to the previous article. This type of

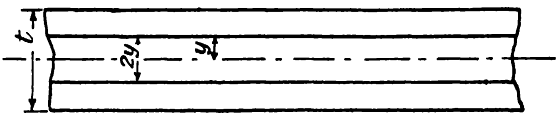


Fig. 173

flow occurs when leakage takes place between two surfaces such as between the piston and the cylinder walls.

Consider two parallel surfaces at a distance of t apart through which a fluid is flowing; let b be the breadth of the surfaces and consider unit length. Fig. 173 represent a cross-sectional view of the plates, perpendicular to the direction of flow. Consider a central layer of the fluid of thickness $2y$; that is, its boundaries are y from the centre. Then, the longitudinal viscous resistance of the layer will equal the longitudinal force on the ends due to the pressure drop.

$$\left.\begin{array}{l}\text{Viscous resistance of}\\ \text{layer per unit length}\end{array}\right\} = \text{stress} \times \text{wetted area}$$

$$= -\mu \frac{dv}{dy} \times 2b$$

This will be negative, as y in the term $\dfrac{dv}{dy}$ was measured from the sides (Art. 137).

$$\left.\begin{array}{l}\text{Longitudinal force on}\\ \text{layer per unit length}\end{array}\right\} = \text{cross-sectional area} \times \text{pressure drop}$$

$$= 2by \times \rho g i$$

as pressure drop $= \rho g h$, and $h = i$ for unit length.

Hence, equating these two equations,

$$-\mu \frac{dv}{dy} 2b = 2 b y \rho g i$$

From which, $$dv = -\frac{\rho g i y \, dy}{\mu}$$

Integrating $$v = -\frac{\rho g i}{\mu} \int y \, dy$$

$$= -\frac{\rho g i y^2}{2\mu} + C_1 \quad . \quad . \quad . \quad . \quad (1)$$

where C_1 is the constant of integration.

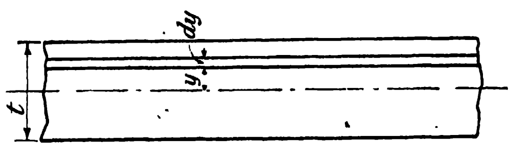


FIG. 174

When $y = \frac{t}{2}$, $v = 0$, as there is no motion at the sides. Putting this limiting condition in Equation (1),

$$0 = -\frac{\rho g i t^2}{8\mu} + C_1$$

From which, $$C_1 = \frac{\rho g i t^2}{8\mu}$$

Substituting this value in Equation (1),

$$v_y = \frac{\rho g i}{2\mu}\left(\frac{t^2}{4} - y^2\right) \quad . \quad . \quad . \quad (2)$$

where v_y is the velocity at any distance y.

The maximum velocity in the whole cross-section is when $y = 0$. Then, from Equation (2),

$$\text{maximum velocity} = \frac{\rho g i t^2}{8\mu} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Next consider a thin layer of the fluid at y from the centre and thickness dy (Fig. 174); let dQ be the quantity flowing through layer per second. Then,

$$dQ = b \, dy \times v_y$$

$$= b \, dy \times \frac{\rho g i}{2\mu}\left(\frac{t^2}{4} - y^2\right)$$

Integrating for total quantity between the surfaces,

$$Q = \frac{b\rho g i}{2\mu}\int_{-\frac{t}{2}}^{\frac{t}{2}}\left(\frac{t^2}{4} - y^2\right)dy$$

$$= \frac{b\rho g i}{2\mu}\left[\frac{t^2 y}{4} - \frac{y^3}{3}\right]_{-\frac{t}{2}}^{\frac{t}{2}}$$

$$= \frac{b\rho g i t^3}{12\mu}.$$

Mean velocity of flow $= v = \dfrac{Q}{\text{cross-sectional area}}$

$$= \frac{b\rho g i t^3}{12\mu} \div bt$$

$$= \frac{\rho g i t^2}{12\mu} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It will be noticed by comparing Equations (3) and (4) that the mean velocity is two-thirds of the maximum velocity.

By substituting the values $m = \dfrac{t}{2}$ and $\nu = \dfrac{\mu}{\rho}$ Equation (4) becomes—

$$6\left(\frac{vt}{\nu}\right)^{-1} = \frac{mig}{v^2} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is a similar form to Equation (6), Art. 140, which was deduced from a round pipe.

Let R = viscous resistance per sq. ft. of wetted surface.

Then, viscous resistance per unit length

= (area of cross-section) × pressure drop per unit length

Or, $R \times 2b = p \times tb$

from which, $R = \rho g i \times \dfrac{t}{2}$ (as $p = \rho g i$)

$$= \rho g i m$$

Dividing throughout by v^2,

$$\frac{R}{\rho v^2} = \frac{mig}{v^2} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Combining Equations (5) and (6),

$$\frac{R}{\rho v^2} = \frac{m i g}{v^2} = C\left(\frac{v t}{\nu}\right)^n \quad . \quad . \quad . \quad (7)$$

where C and n are constants depending on the type of flow and which are determined from experimental results. It will be noticed from Equation (5) that for viscous flow, C = 6 and n = —1

It will also be noticed that Equation (5) reduces to the same form as the Chezy formula, $v = C\sqrt{mi}$, for the flow in open channels (Art. 79), if the Chezy constant C is written

$$\sqrt{\frac{v t g}{6\nu}}$$

but,

$$\sqrt{\frac{v t g}{6\nu}} = \sqrt{\frac{g R_e}{6}}$$

Hence, the Chezy constant C varies with R_e,

then,

$$C = k\sqrt{R_e}$$

EXAMPLE.

The radial clearance between a hydraulic plunger and the cylinder walls is ·004 in.; the length of the plunger is 12 in. and the diameter 4 in. Find the velocity of leakage and the rate of leakage past the plunger at an instant when the difference of pressure between the two ends of the plunger is 30 ft. of water. The temperature of the water is 10° C.

The flow through the clearance area will be the same as the flow between two parallel surfaces.

Assuming the whole of the pressure head is lost,

$$i = \frac{h_f}{l} = \frac{30}{1}$$

From Equation (2), Art. 138,

$$\nu = \frac{\cdot 0000192}{1 + (\cdot 03368 \times 10) + (\cdot 0002221 \times 100)}$$

$$= \frac{\cdot 0000192}{1\cdot 359} = \cdot 0000141$$

From Equation (4),

$$\text{mean velocity} = v = \frac{g i t^2}{12\nu}$$

$$= \frac{32\cdot 2 \times 30 \times \cdot 004^2}{12 \times \cdot 0000141 \times 144}$$

$$= \cdot 634 \text{ ft. per sec.}$$

$$\begin{aligned}\text{Rate of flow} = Q &= v \times \text{area of annular ring}\\ &= v \times \pi D t\\ &= \cdot 634 \times \pi \times \frac{4}{12} \times \frac{\cdot 004}{12}\\ &= \cdot 000221 \text{ cu. ft. per sec.}\end{aligned}$$

141A. **Viscous Flow Through Annular Space.** The viscous flow of a fluid through the annular space between an inner and outer tube can be found in the same manner as in Arts. 140 and 141. Consider the annular space formed by the two tubes of Fig. 174A.

Let R = radius of outer surface

r = radius of inner surface

q = viscous stress at inner surface per unit area

Consider a hollow cylinder of fluid of external radius y, internal radius r, and of unit length.

Let f = viscous stress at radius y

$$= -\mu \frac{dv}{dy}$$

$$\left.\begin{array}{l}\text{Then, viscous resistance of}\\ \quad\text{section considered}\end{array}\right\} = \text{wetted area} \times \text{stress}$$

$$= 2\pi y f + 2\pi r q$$

$$= -2\pi y \frac{dv}{dy} + 2\pi r q$$

Pressure drop on unit length $= \rho g i$ (Art. 140)

$$\left.\begin{array}{l}\text{Longitudinal force on sec-}\\ \quad\text{tion considered}\end{array}\right\} = \begin{pmatrix}\text{cross-sectional}\\ \text{area}\end{pmatrix} \times \begin{pmatrix}\text{pressure}\\ \text{drop}\end{pmatrix}$$

$$= \pi\,(y^2 - r^2) \times \rho g i$$

Equating these equations,

$$-2\pi y \frac{dv}{dy} + 2\pi r q = \pi (y^2 - r^2)\, \rho g i$$

then,
$$dv = \frac{(2rq - \rho g i y^2 + \rho g i r^2)\, dy}{2\mu y}$$

$$= \left(\frac{rq}{\mu y} - \frac{\rho g i y}{2\mu} + \frac{\rho g i r^2}{2\mu y}\right) dy$$

Integrating, $$v = \frac{rq}{\mu}\log_e y - \frac{\rho gi}{4\mu}y^2 + \frac{\rho gir^2}{2\mu}\log_e y + C \qquad (1)$$

where C is the constant of integration.

When $$y = R,\ v = 0$$

hence, $$0 = \frac{rq}{\mu}\log_e R - \frac{\rho giR^2}{4\mu} + \frac{\rho gir^2}{2\mu}\log_e R + C$$

from which, $$C = \frac{\rho giR^2}{4\mu} - \frac{rq}{\mu}\log_e R - \frac{\rho gir^2}{2\mu}\log_e R$$

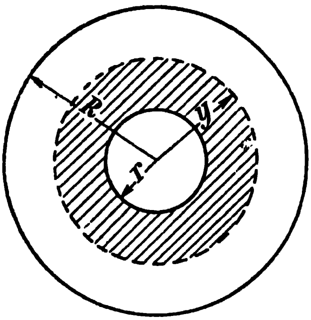


FIG. 174A

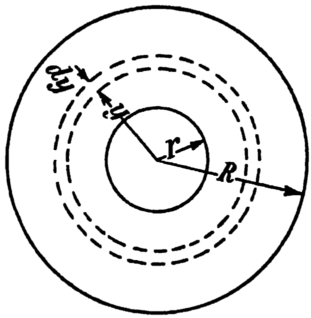


FIG. 174B

Substituting this value of C in Equation (1),

$$v = \frac{rq}{\mu}\log_e\frac{y}{R} + \frac{\rho gir^2}{2\mu}\log_e\frac{y}{R} + \frac{\rho gi}{4\mu}(R^2 - y^2) \quad . \quad . \quad (2)$$

When $y = r$, $v = 0$, hence,

$$0 = \frac{rq}{\mu}\log_e\frac{r}{R} + \frac{\rho gir^2}{2\mu}\log_e\frac{r}{R} + \frac{\rho gi}{4\mu}(R^2 - r^2)$$

from which, $$q = -\frac{\rho gir}{2} - \frac{\rho gi}{4}\frac{(R^2 - r^2)}{r\log_e\frac{r}{R}}$$

Substituting this value of q in Equation (2),

$$v = \frac{\rho gi}{4\mu}\left[(R^2 - y^2) - (R^2 - r^2)\frac{\log_e\frac{y}{R}}{\log_e\frac{r}{R}}\right] \quad . \quad (3)$$

Equation (3) gives the variation of velocity between the radii r and R; differentiating for a maximum,

$$\frac{dv}{dy} = 0 = -2y + \frac{R^2 - r^2}{y \log_e \frac{R}{r}}$$

from which,
$$y = \sqrt{\frac{R^2 - r^2}{2 \log_e \frac{R}{r}}} \qquad . \quad . \quad . \quad . \quad (4)$$

By substituting this value of y in Equation (3), the value of the maximum velocity is obtained.

The quantity of flow through the annular space can be found by considering a thin ring of radius y and thickness dy (Fig. 174B); the velocity through this ring is given by Equation (3). Then,

$$dQ = \text{area of ring} \times \text{velocity}$$

$$= 2\pi y dy \times \frac{\rho g i}{4\mu}\left[(R^2 - y^2) - (R^2 - r^2)\frac{\log_e \frac{y}{R}}{\log_e \frac{r}{R}}\right]$$

Integrating between r and R,

$$Q = \frac{\rho g i \pi}{2\mu}\int_r^R \left\{R^2 y - y^3 - (R^2 - r^2)\, y \frac{\log_e \frac{y}{R}}{\log_e \frac{r}{R}}\right\} dy$$

$$= \frac{\rho g i \pi}{2\mu}\left[\frac{R^2 y^2}{2} - \frac{y^4}{4} - \frac{(R^2 - r^2)}{\log_e \frac{r}{R}}\left(\frac{y^2}{2}\log_e \frac{y}{R} - \frac{y^2}{4}\right)\right]_r^R$$

$$= \frac{\rho g i \pi}{2\mu}\left\{\frac{R^4}{4} + \frac{(R^2 - r^2)}{\log_e \frac{r}{R}} \times \frac{R^2}{4}\right\} - \left\{\frac{R^2 r^2}{2} - \frac{r^4}{4} - \frac{r^2 (R^2 - r^2)}{2 \log_e \frac{r}{R}}\left(\log_e \frac{r}{R} - \frac{1}{2}\right)\right\}$$

$$= \frac{\rho g i \pi}{4\mu}(R^2 - r^2)\left\{\frac{(R^2 + r^2)}{2} + \frac{(R^2 - r^2)}{2 \log_e \frac{r}{R}}\right\}$$

$$\text{Mean velocity} \quad \frac{Q}{\text{area of flow}}$$

$$= \frac{Q}{\pi\,(R^2 - r^2)}$$

$$= \frac{\rho g i}{8\mu}\left\{(R^2 + r^2) + \frac{(R^2 - r^2)}{\log_e \dfrac{r}{R}}\right\} \qquad . \quad . \quad (5)$$

It will be noticed that if $r = 0$, the problem becomes the same as the viscous flow through a round tube and Equation (5) reduces to the same form as Equation (5) of Art. 140.

In the special case when r is nearly of the same value as R, the annular space is very thin, consequently, Equation (5) would have to be calculated to a very fine degree of accuracy in order to obtain a reasonable result. Obviously, if the thickness of the annular ring is very small compared with R, the problem is the same as the flow between two parallel surfaces (Art. 141) and a simpler and more accurate solution is obtained by this treatment; this method was applied to the plunger problem given in the worked-out example at the end of Art. 141.

142. **Resistance of Oiled Bearings.** A shaft revolving in an oiled bearing will be separated from the bearing by a thin film of oil. The layer of this film adjacent to the bearing will be at rest, whilst the layer adjacent to the shaft will be revolving with the same velocity as the shaft. The resistance of the oil film will, therefore, be due to the viscosity of the oil.

Let t = thickness of oil film

D = diam. of shaft

n = speed of shaft in revs. per min.

l = length of bearing.

Then, from Art. 120,

$$f = \mu \frac{dv}{dy}$$

where f is the resistance per unit area. In this case, as the oil film is very thin, dv will be the tangential velocity of the shaft and dy will be the average thickness of the oil film.*

* Actually, the oil film will not be of uniform thickness. For the application of the principle of dimensional similarity to this problem, see Art. 202.

Then, $$f = \mu \frac{v}{t}$$

$$= \mu \times \frac{\pi D n}{60t}$$

Tangential resistance on bearing $= f \pi D l$

$$\text{resisting torque} = f \pi D l \times \frac{D}{2}$$

$$= \frac{\mu \pi^2 D^3 l n}{120 t}$$

Horse-power absorbed in viscosity

$$= \frac{\text{torque in lb. ft.} \times 2\pi n}{33{,}000}$$

Example.

Define the terms "coefficient of viscosity" and "kinematical viscosity." A shaft 4 in. diameter runs in a bearing of length 8 in., the two surfaces being separated by a film of oil ·001 in thick. If the coefficient of viscosity of the oil is 1·53 C.G.S. units, find the torque necessary to rotate the shaft at 30 revs. per min. against the viscous resistance of the oil. (London Univ.)

$$\mu = 1{\cdot}53 \text{ C.G.S. units}$$

$$= 1{\cdot}53 \times \frac{30{\cdot}5}{453{\cdot}6 \times 32{\cdot}2} \text{ ft. lb. units}$$

$$= {\cdot}0032 \text{ ft. lb. units}$$

$$\text{Viscous torque} = \frac{\mu \pi^2 D^3 l n}{120 t}$$

$$= \frac{{\cdot}0032 \times \pi^2 \times (\frac{1}{3})^3 \times \frac{2}{3} \times 30}{120 \times \frac{{\cdot}001}{12}} \text{ lb. ft.}$$

$$= 2{\cdot}34 \text{ lb. ft.}$$

143. **Viscous Resistance of Collar Bearing.** The viscous resistance of a collar bearing can be obtained by assuming the face of the collar to be separated from the bearing surface by a thin film of oil of uniform thickness.

Two views of a collar bearing are shown in Fig. 175. Consider a thin ring of the bearing surface of radius x and thickness dx, and let v be the circumferential velocity at this radius.

Let ω = speed of shaft in radians per sec.

f = viscous stress at radius x

t = thickness of oil film

R_1 and R_2 = external and internal radii of collar respectively.

Using the equation for viscous stress [Equation (2), Art. 137],

$$f = \mu \frac{dv}{dy}$$

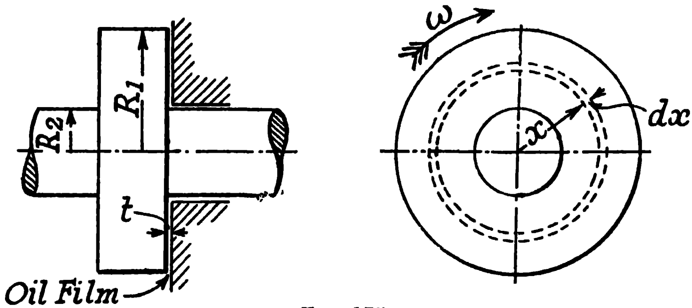


FIG. 175

In this problem, $dy = t$ and $dv = v$, as the oil film is very thin and the velocity changes from 0 to v within the thickness t. Hence, the equation may be written—

$$f = \frac{\mu v}{t}$$

$$= \frac{\mu\omega x}{t}, \text{ as } v = \omega x$$

Tangential viscous force on thin ring $= f \times 2\pi x dx$

$$= \frac{2\pi\mu\omega x^2 dx}{t}$$

Moment of tangential force on ring $= \dfrac{2\pi\mu\omega x^3 dx}{t}$

If the whole bearing surface is assumed to consist of similar concentric rings, the integration of this equation will give the total torque required to overcome the viscous resistance of the bearing.

Then, total torque required $= T = \dfrac{2\pi\mu\omega}{t}\displaystyle\int_{R_2}^{R_1} x^3 dx$

$$= \frac{2\pi\mu\omega}{t}\left[\frac{x^4}{4}\right]_{R_2}^{R_1}$$

$$= \frac{\pi\mu\omega}{2t}(R_1^4 - R_2^4) \quad . \quad . \quad (1)$$

If T is in ft. lb. units, then,

$$\left.\begin{array}{c} HP \text{ required to overcome} \\ \text{viscous resistance} \end{array}\right\} = \frac{T \times 2\pi\,(\text{r.p.m.})}{33{,}000}$$

If the bearing is of the footstep type, Equation (1) may be applied; in this case R_2 is zero and R_1 is the radius of the shaft.

In applying Equation (1), care should be taken to ensure that all dimensions are in ft. lb. units or C.G.S. units.

EXAMPLE.

The thrust of a shaft is taken by a collar bearing provided with a forced lubrication system which maintains a film of oil of uniform thickness between the surface of the collar and the surface of the bearing. The external and internal diameters of the collar are 6 and 4 in. respectively. If the thickness of the film of oil which separates the surfaces is 0·01 in., and the coefficient of viscosity is 0·91 poise, show that the horse-power lost in overcoming friction when the shaft rotates at 300 r.p.m. is 0·02 very nearly. (London Univ.)

In this problem, $\mu = \dfrac{\cdot 91 \times 30{\cdot}5}{32{\cdot}2 \times 453{\cdot}6}$ (Art. 138)

$$= \cdot 0019 \text{ ft. lb. units.}$$

$$\omega = \frac{2\pi \times 300}{60}$$

$$= 10\pi \text{ rads. per sec.}$$

$$t = \frac{\cdot 01}{12} \text{ ft.}$$

R_1 and $R_2 = \cdot 25$ ft. and $\cdot 167$ ft. respectively.

Substituting these values in Equation (1),

$$T = \frac{\pi \times \cdot 0019 \times 10\pi \times 12}{2 \times \cdot 01}(\cdot 25^4 - \cdot 167^4)$$

$$= \cdot 35 \text{ lb. ft.}$$

Then, H.P. absorbed $= \dfrac{\cdot 35 \times 2\pi \times 300}{33{,}000}$

$$= \cdot 02$$

144. Time of Emptying Vessel by Viscous Flow through Pipe. In Art. 73 was shown the method of calculating the time taken to empty a vessel by means of a horizontal pipe, the flow being assumed turbulent. This problem will now be solved for a viscous discharge through the pipe.

From Equation (8), Art. 140, the equation for viscous flow through a round pipe can be expressed in the form:

$$\frac{mig}{v^2} = \frac{8\mu}{\rho v d}$$

where $m = \frac{d}{4}$ and $\rho = \frac{w}{g}$

Let p = drop in pressure over a length l

then, $h_f = \frac{p}{w}$

and $i = \frac{h_f}{l} = \frac{p}{wl}$

Substituting these values of m, ρ and i in the above equation,

$$\frac{dpg}{4lv^2} = \frac{8g\mu}{wvd}$$

From which, $$p = \frac{32\mu v l}{d^2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or, $$h_f = \frac{p}{w} = \frac{32\mu v l}{wd^2} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let the cylindrical vessel of Fig. 176 contain a viscous liquid of height H_1 above the centre of the pipe, and let the vessel discharge through the horizontal pipe shown, the flow being viscous.

Let A = area of surface of liquid in vessel

a = area of pipe

l = length of pipe

C_d = coefficient of discharge at entrance to pipe

It is required find the time taken to lower the level of the liquid from H_1 to H_2. The final velocity head of the liquid leaving the pipe can be neglected as small.

Consider the instant when the surface of the liquid is at a height h above the centre of the pipe and let the surface fall

by $-dh$ in the time dt; this represents a quantity dq flowing from the vessel.

Then, $$dq = -Adh = C_d avdt \quad . \quad . \quad . \quad (3)$$

From Equation (2), and assuming the head is utilized in overcoming the viscous resistance of the pipe,

$$h = h_f = \frac{32\mu vl}{wd^2}$$

from which, $$v = \frac{wd^2h}{32\mu l}$$

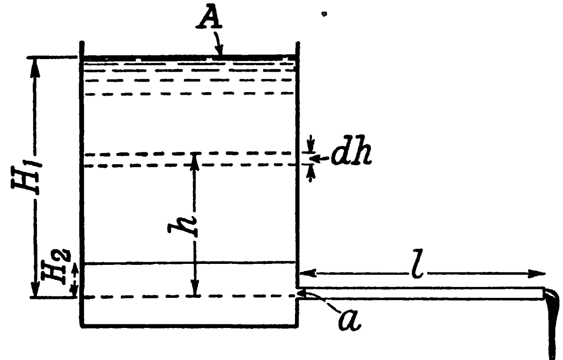


FIG. 176

Substituting this value of v in Equation (3),

$$-Adh = \frac{C_d awd^2hdt}{32\mu l}$$

from which, $$dt = -\frac{32A\mu ldh}{C_d awd^2h}$$

Integrating between the limits H_2 and H_1,

$$\int_0^T dt = -\frac{32A\mu l}{C_d awd^2}\int_{H_1}^{H_2}\frac{dh}{h}$$

then, $$T = -\frac{32A\mu l}{C_d awd^2}\Big[\log_e h\Big]_{H_1}^{H_2}$$

$$= \frac{32A\mu l}{C_d awd^2}\log_e\left(\frac{H_1}{H_2}\right) \quad . \quad . \quad . \quad (4)$$

EXAMPLE.

Prove that when a fluid of viscosity μ flows through a tube of diameter d and mean velocity v, the drop in pressure per unit length of tube is $\frac{32v\mu}{d^2}$ for laminar flow.

The viscosity of a liquid is determined by timing the discharge of 50 c.c. of the liquid from a vessel through a horizontal capillary tube. The vessel is an upright cylinder, open at the top, 5 cm. diameter, and the capillary tube is 1 mm. bore and 10 cm. long. The vessel is at first filled to a height of 5 cm. above the axis of the tube, and it is found that 50 c.c. are discharged in 20 minutes. Find the velocity of the liquid in poises, given that its density is 0·88 grm./cm.³ Neglect the end effects of the tube and the velocity head at discharge. (London Univ.)

The first part of this question is given by Equation (1), and is proved in Art. 140.

$$A = \frac{\pi}{4} \times 5^2 = 19{\cdot}63 \text{ sq. cm.}$$

$$a = \frac{\pi}{4} \times (\cdot 1)^2 = \cdot 00785 \text{ sq. cm.}$$

$$d = \cdot 01 \text{ cm.}, l = 10 \text{ cm.}, w = \cdot 88 \text{ grm. per cub. cm.}, T = 20 \times 60 \text{ sec.}$$

$$Q = 50 = (H_1 - H_2)\,A$$
$$= (5 - H_2) \times 19{\cdot}63$$

From which, $H_2 = 3{\cdot}73$ cm.

Substituting these values in Equation (4),

$$20 \times 60 = \frac{32 \times 19{\cdot}63\mu \times 10}{1 \times \cdot 00785 \times \cdot 88 \times (\cdot 1)^2} \log_e\left(\frac{5}{3{\cdot}73}\right)$$

From which, $$\mu = \frac{1200 \times \cdot 00785 \times \cdot 88 \times \cdot 01}{32 \times 19{\cdot}63 \times 10 \times \cdot 293} \text{ C.G.S. units}$$
$$= \cdot 0000451 \times 981 \text{ poises}$$
$$= \cdot 0442 \text{ poises}$$

145. **Principle of Dimensional Similarity.** Lord Rayleigh has shown* that Reynolds' results on the flow of water through pipes are only one particular case of a general principle applicable to all types of fluid resistance. This general principle is known as the principle of dimensional, or dynamical, similarity, and is obtained by balancing the fundamental units

* *Scientific Papers*, Vol. vi, Art. 392. For the application of the principle of dimensional similarity to other problems see Chapter XVII.

on each side of the equation. The following are the fundamental units for the quantities used—

Mass	$= M$
linear dimension	$= L$
time	$= T$
velocity	$= LT^{-1}$
acceleration	$= LT^{-2}$
force	$=$ mass $\times$ acceleration $= MLT^{-2}$

$$\mu = ML^{-1}T^{-1} \quad \text{(Art. 137)}$$
$$\rho = \text{mass} \div \text{volume} = ML^{-3}$$
$$\nu = L^2T^{-1} \quad \text{(Art. 138)}$$

Case 1.—RESISTANCE DUE TO VISCOSITY ONLY. It is known that the viscous resistance of a fluid depends on the wetted area, the velocity, the density, and the coefficient of viscosity. Assuming the actual law of variation to be unknown, the equation for the resistance may be written

$$R = k\,\rho^a\,\mu^b\,L^c\,v^d \quad . \quad . \quad . \quad . \quad (1)$$

where a, b, c, and d are unknown indices, R is the total resistance, and k is a constant to be determined experimentally. Now the fundamental units for each side of this equation must balance; hence, by putting the total resistance R as a force, and by substituting the fundamental units for ρ, μ, and v, the equation becomes

$$MLT^{-2} = k(ML^{-3})^a\,(ML^{-1}T^{-1})^b\,L^c(LT^{-1})^d$$

That is, $\quad MLT^{-2} = kM^aL^{-3a}M^bL^{-b}T^{-b}L^cL^dT^{-d}$

Now, as the indices of M on each side of the equation are equal,

$$1 = a + b$$

From which, $\quad a = 1 - b \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$

Also, as the indices of L on each side of the equation are equal,

$$1 = -3a - b + c + d \quad . \quad . \quad . \quad . \quad (3)$$

Also, as the indices of T on each side of the equation are equal,

$$-2 = -b - d$$

From which, $\quad d = 2 - b \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$

Substituting the values of Equations (2) and (4) in Equation (3),

$$1 = -3(1-b) - b + c + (2-b)$$

From which, $$c = 2 - b \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting the values of a, c, and d in Equation (1),

$$R = k\,\rho^{1-b}\mu^{b}L^{2-b}v^{2-b}$$

$$= k\,\rho\,L^{2}v^{2}\left(\frac{\mu}{\rho\,L\,v}\right) \quad . \quad . \quad . \quad . \quad (6)$$

But $$\nu = \frac{\mu}{\rho}$$

hence, $$R = k\,\rho\,L^{2}v^{2}\left(\frac{\nu}{L\,v}\right)^{b} \quad . \quad . \quad . \quad . \quad (7)$$

It will be noticed that this is the same equation as that obtained by Froude (Art. 63) if the coefficient f' is written as $\rho\,k\left(\frac{\nu}{L\,v}\right)^{b}$

It will also be noticed, from Equation (7), that true dynamical similarity between two bodies can only be obtained when the term $\frac{\nu}{L\,v}$ is the same for both, as it is necessary that the unknown term $\left(\frac{\nu}{Lv}\right)^{b}$ should cancel when comparing the two resistances.

Equation (7) is usually written

$$R = \rho\,L^{2}v^{2}\phi\left(\frac{\nu}{L\,v}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where ϕ means "a function of."

Case 2.—RESISTANCE DUE TO VISCOSITY AND GRAVITY. This is the case of ships moving in water against a viscous resistance and a wave resistance. The resistance to the formation of surface waves is a gravity resistance, as the wave gains in potential energy. Hence the combined resistance may be written

$$R = k\,\rho^{a}\mu^{b}L^{c}v^{d}g^{e}$$

where k is a constant to be determined experimentally.

Substituting in the above terms their fundamental units, the equation becomes

$$MLT^{-2} = (ML^{-3})^{a}(ML^{-1}T^{-1})^{b}L^{c}(LT^{-1})^{d}(LT^{-2})^{e}$$
$$= M^{a}L^{-3a}M^{b}L^{-b}T^{-b}L^{c}L^{d}T^{-d}L^{e}T^{-2e}$$

Equating the indices of M,

$$1 = a + b$$

From which, $$a = 1 - b \quad . \quad . \quad . \quad . \quad (9)$$

Equating the indices of L,

$$1 = -3a - b + c + d + e \quad . \quad . \quad . \quad . \quad (10)$$

Equating the indices T,

$$-2 = -b - d - 2e$$

From which, $$d = 2 - b - 2e \quad . \quad . \quad . \quad (11)$$

Substituting Equations (9) and (11) in (10),

$$\begin{aligned} 1 &= 3 + 3b - b + c + 2 - b - 2e + e \\ &= -1 + b + c - e \end{aligned}$$

From which, $$c = 2 - b + e$$

Substituting the above values of a, d, and c in the original equation for R,

$$\begin{aligned} R &= k\ \rho^{1-b}\ \mu^{b}\ L^{2-b+e}\ v^{2-b-2e}\ g^{e} \\ &= k\,\rho\,L^2 v^2 \left(\frac{\mu}{\rho\,L\,v}\right)^b \left(\frac{Lg}{v^2}\right)^e \end{aligned}$$

which may be written,

$$R = \rho\,L^2\,v^2\,\phi\left(\frac{\nu}{Lv} \cdot \frac{Lg}{v^2}\right) \quad . \quad . \quad . \quad . \quad (12)$$

where ϕ means " a function of " and where $\nu = \frac{\mu}{\rho}$.

For true dynamical similarity between two bodies the term

$$\phi\left(\frac{\nu}{Lv} \cdot \frac{Lg}{v^2}\right)$$

must be equal in both cases, as the function represented by ϕ is unknown; hence, it is necessary for the whole term to cancel when comparing the resistance of two similar bodies.

That is, $\frac{\nu}{Lv}$ must be equal for both bodies and $\frac{Lg}{v^2}$ must be equal for both bodies.

The ratio $\frac{v}{\sqrt{Lg}}$ is the non-dimensional factor governing the gravity effect and is known as the Froude number.

If the floating body is completely submerged in the fluid to a great depth, there will be no formation of gravity waves;

in this case the resistance is due to viscosity and pressure waves. Such a case occurs with the resistance of submarines and airships.

146. **Applications of Principle of Dynamical Similarity.** The principle of dynamical similarity is used when testing models in order to predict the resistance of large bodies. Before building a new type of ship it is usual to make a model of the same proportions and measure its resistance experimentally. From the results obtained the resistance of the ship can be calculated. The model is propelled at a speed which will give true dynamical similarity; this speed is known as the corresponding speed.

(*a*) RESISTANCE DUE TO VISCOSITY ONLY. This is the case for deeply submerged bodies such as submarines and airships and for the frictional resistance of surface ships.

This type of resistance is given by Equation (8) (Art. 145).

Let suffix m refer to model. Then,

$$R = \rho L^2 v^2 \phi\left(\frac{\nu}{Lv}\right) \text{ for ship}$$

and

$$R_m = \rho_m L^2{}_m v^2{}_m \phi\left(\frac{\nu_m}{L_m v_m}\right) \text{ for model}$$

True dynamical similarity can only be obtained when

$$\left(\frac{\nu}{Lv}\right) = \left(\frac{\nu_m}{L_m v_m}\right)$$

in which case, these terms, which are of an unknown function, will cancel.

If the model is tested in the same fluid as the ship, true dynamical similarity will be obtained when

$$L v = L_m v_m \qquad \text{as } \nu \text{ will then equal } \nu_m.$$

In which case, $\quad v_m = v\dfrac{L}{L_m}$

This is known as the corresponding speed, because if the model is tested at this speed the term $\phi\left(\dfrac{\nu}{Lv}\right)$ is the same for both ship and model and will cancel. Then

$$\frac{R}{R_m} = \frac{L^2 v^2}{L^2{}_m v^2{}_m} = 1 \text{ (by substituting for } v_m\text{)}$$

Thus, the resistance of the model would be the same as that of the ship. The corresponding speed in this case is too large for practical purposes, but by testing the model in a fluid having a kinematic viscosity much less than that of the ship's fluid, a smaller corresponding speed may be obtained. In which case,

$$\frac{\nu}{L v} = \frac{\nu_m}{L_m v_m}$$

hence,
$$v_m = v \times \frac{L}{L_m} \times \frac{\nu_m}{\nu}$$

This type of resistance also occurs in fluids flowing through pipes and channels ; it is possible to predict the resistance to flow in a channel by testing the flow in a similar channel at the corresponding speed. In this case the corresponding speed will be when the term $\frac{\nu}{L v}$ is the same for both channels. Then the corresponding speed will be when

$$\frac{\nu_1}{L_1 v_1} = \frac{\nu_2}{L_2 v_2}$$

where the suffixes 1 and 2 apply to the respective channels. It should be noticed that Equation (8) (Art. 145) is the same form as Equation (8) (Art. 140). This will be seen by putting R in Equation (8) (Art. 145) as the resistance per unit area of wetted surface.

Then,
$$R = \rho v^2 \phi \left(\frac{\nu}{L v}\right)$$

Or
$$\frac{R}{\rho v^2} = \phi \left(\frac{\nu}{L v}\right)$$

which is the same form as Equation (8) (Art. 140).

(*b*) Resistance due to Gravity only. It was shown in Art. 145 that the resistance of a surface ship was partly due to surface friction, or viscosity, and partly due to wave formation, or gravity. The combined resistance was proved to be

$$R = \rho L^2 v^2 \phi \left(\frac{\nu}{L v} \cdot \frac{L g}{v^2}\right)$$

(Equation 12. Art. 145.)

For true dynamical similarity between ship and model both of the following conditions must hold,

$$(1)\qquad \frac{\nu}{Lv} = \frac{\nu_m}{L_m v_m}$$

$$(2)\qquad \frac{Lg}{v^2} = \frac{L_m g}{v^2_m}$$

Case 1 being the condition for viscous resistance and Case 2 the condition for wave resistance. If both model and ship are floating in the same fluid, the corresponding speed for Case 1 is when

$$L\,v = L_m\,v_m$$

Or,
$$v_m = v\frac{L}{L_m}$$

For case 2, the corresponding speed is when

$$\frac{L}{v^2} = \frac{L_m}{v^2_m}$$

Or,
$$v_m = v\sqrt{\frac{L_m}{L}}$$

Hence, the corresponding speed varies in each case; it would, therefore, be impossible to test the model for the total resistance. To overcome this difficulty it is usual to calculate the frictional, or viscous, resistance of the ship and model from the coefficient of friction and surface area. The total resistance of the model is then measured experimentally at the corresponding speed for wave resistance; that is, at a corresponding speed proportional to $\sqrt{\frac{L_m}{L}}$. By subtracting the frictional resistance from the total resistance of model the wave resistance is obtained. The wave resistance of the ship is then obtained by proportion; this, added to the frictional resistance of the ship, will give the total resistance,

Let R_w = wave resistance of ship
R_f = frictional resistance of ship
r_w = wave resistance of model
r_f = frictional resistance of model

Then, $R = R_w + R_f$ . . . . (1)

and, $R_m = r_w + r_f$ . . . . (2)

For wave resistance only,

$$\frac{R_w}{r_w} = \frac{\rho L^2 v^2 \phi\left(\frac{Lg}{v^2}\right)}{\rho_m L^2{}_m v^2{}_m \phi\left(\frac{L_m g}{v^2{}_m}\right)}$$

Then, for corresponding speed, in order that the last term of each will cancel,

$$\frac{v_m}{v} = \sqrt{\frac{L_m}{L}}$$

Then,
$$\frac{R_w}{r_w} = \frac{\rho L^2 v^2}{\rho_m L^2{}_m\left(v\sqrt{\frac{L_m}{L}}\right)^2}$$

$$= \frac{\rho L^3}{\rho_m L^3{}_m}$$

If the same fluid be used, $\rho = \rho_m$

hence,
$$\frac{R_w}{r_w} = \left(\frac{L}{L_m}\right)^3$$

Substituting from Equations (1) and (2),

$$\frac{R - R_f}{R_m - r_f} = \left(\frac{L}{L_m}\right)^3$$

From which the total resistance of the ship is obtained.

EXAMPLE 1.

The resistance of a hydroplane may be assumed to be entirely due to wave formation. The speed of the hydroplane is to be 90 ft. per sec.; calculate its resistance at this speed if the resistance of a model at the corresponding speed was found to be ·5 lb. The linear dimensions of the model were $\frac{1}{20}$ of the hydroplane. What is the speed of the model ?

As wave resistance is a gravity resistance,

$$\text{corresponding speed of model} = 90\sqrt{\frac{L_m}{L}}$$

$$= 90 \times \sqrt{\frac{1}{20}}$$

$$= 20{\cdot}01 \text{ ft. per sec.}$$

Then, $$\frac{}{R_m} = \left(\frac{}{L_m}\right)$$

Hence, $$R = 20^3 \times \cdot 5$$
$$= 4000 \text{ lb.}$$

EXAMPLE 2.

The loss of head in a pipe 1 in. diameter and 100 ft. long through which water is flowing at 10 ft. per sec. was found to be 7 ft. of water. Calculate the loss of head in a 3 in. pipe 60 ft. long through which air is flowing at the corresponding speed. For water, $\rho = 62{\cdot}4$ lb. per cu. ft. and $\mu = {\cdot}01$ C.G.S. units. For air, $\rho = {\cdot}075$ lb. per cu. ft. and $\mu = {\cdot}00015$ C.G.S. units.

As this is a viscous resistance, corresponding speed will be when

$$\frac{\mu}{\rho d v} \text{ for water} = \frac{\mu}{\rho d v} \text{ for air}$$

That is, $$\frac{\cdot 01}{62{\cdot}4 \times \cdot 1 \times 10} = \frac{\cdot 00015}{\cdot 075 \times 3 \times v}$$

from which, $$v = \frac{62{\cdot}4 \times 1 \times 10 \times \cdot 00015}{\cdot 075 \times 3 \times \cdot 01}$$
$$= 416 \text{ ft. per sec.}$$

Now, resistance to flow $= R = \rho L^2 v^2 \phi\left(\frac{\mu}{\rho v d}\right)$

$$= \rho \times \text{wetted area} \times v^2 \phi\left(\frac{\mu}{\rho d v}\right)$$

Hence, $$\frac{R \text{ for air}}{R \text{ for water}} = \frac{(\rho \times \pi d l v^2) \text{ for air}}{(\rho \times \pi d l v^2 \text{ for water}} \quad . \quad (1)$$

as the term $\phi\left(\frac{\mu}{\rho d v}\right)$ will cancel at the corresponding speed

But, $R = \text{pressure} \times \text{cross-sectional area}$

$$= \rho h \times \frac{\pi}{4} d^2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

as $p = \rho h$

Hence, equating Equations (1) and (2),

$$\frac{(\rho h \times \frac{\pi}{4} d^2) \text{ for air}}{(\rho h \times \frac{\pi}{4} d^2) \text{ for water}} = \frac{(\rho \times \pi d l v^2) \text{ for air}}{(\rho \times \pi d l v^2) \text{ for water}}$$

Hence, $$\frac{h \text{ for air}}{h \text{ for water}} = \frac{\frac{1}{d}(l v^2) \text{ for air}}{\frac{1}{d}(l v^2) \text{ for water}}$$

That is, $$\frac{h \text{ for air}}{7} = \frac{60 \times 416^2 \times \cdot 1}{3 \times 100 \times 100}$$

Hence, loss of head for air $= 2420$ ft. of air

$$= \frac{2420 \times \cdot 075}{62 \cdot 4}$$

$$= 2 \cdot 91 \text{ ft. of water.}$$

It is not necessary in a question of this type to bring all terms to foot lb. units, as the factors required to do this would cancel.

EXAMPLES 12.

(1) Find the kinematic viscosity of water at a temperature of 60° C.

Ans.—$5 \cdot 05 \times 10^{-6}$ sq. ft. per sec.

(2) Oil of kinematic viscosity ·000052 sq. ft. per sec. flows through a pipe of 6 in. diameter with a velocity of 1 ft. per sec. Find the value of the term $\frac{vd}{\nu}$ and state whether the flow is streamline or turbulent.

Ans.—9,600 ; turbulent.

(3) Air of kinematic viscosity of $15 \cdot 6 \times 10^{-5}$ sq. ft. per sec. flows through a pipe of 2 in. diameter. What is the maximum velocity for streamline flow ?

Ans.—1·872 ft. per sec.

(4) Water at 20° C. flows through a pipe of 9 in. diameter and of length 2,000 ft. with a velocity of 1·2 ft. per sec. Find the loss of head due to viscosity. *Ans.*—1·13 ft.

(5) Water at a temperature of 20° C. leaks through a horizontal slot ·01 in. deep, 4 in. in breadth, and 6 in. in length. Find the quantity of water leaking through per hour when the difference of pressure between the ends of the slot is 5 lb. per sq. in.

Ans.—3·94 cu. ft. per hour.

(6) Fuel oil at a temperature of 10° C. is pumped through a pipe line of 6 in. diameter and 5,000 ft. in length. Find the horse-power required to pump 10 tons per hour of this oil if the weight of the oil is 57 lb. per cu. ft. and the kinematic viscosity at 10° C. is ·00015 sq. ft. per sec.

Ans.—·0189 h.p.

(7) The resistance of geometrically similar plates when towed edgewise through a fluid are given by $R = \rho l^2 v^2 \phi\left(\frac{\rho l v}{\mu}\right)$ in which ρ is the fluid density and μ the coefficient of viscosity of the fluid, l the linear dimensions, and v the velocity. Determine the torque necessary to rotate a thin disc 24 in. diameter at 3,000 revs. per min. in air for which $\rho = 1 \cdot 2 \times 10^{-3}$ and $\mu = 1 \cdot 86 \times 10^{-4}$ C.G.S. units, given that the torque necessary to rotate a similar disc 9 in. diameter in water at the corresponding speed is ·079 ft. lb. For water $\rho = 1 \cdot 00$, and $\mu = \cdot 0101$ C.G.S. units. (London Univ.)

Ans.—0·0596 lb. ft.

(8) Show that the resistance to motion of a body deeply submerged in a fluid is given by $R = \rho l^2 v^2 \phi\left(\frac{vl}{\nu}\right)$ where l is some one specified dimension of the body and where ρ and ν are respectively the density and kinematic viscosity of the fluid. (London Univ.)

(9) How does the viscous leakage past a long hydraulic plunger having a very small radial clearance depend upon (1) the length of the plunger surrounded by its bush, (2) the radial clearance, (3) the diameter, (4) the difference of pressure ? (London Univ.)

(10) What is meant by "corresponding speeds" in model experiments ? Deduce the law of corresponding speeds (*a*) for viscous resistance, (*b*) for resistances due to the effects of gravity. In the case of a hydroplane the resistance is mainly due to wave formation. If the scale of a model hydroplane is $\frac{1}{25}$ and if its resistance at a speed of 20 ft. per sec. is 0·4 lb., what will be the resistance of the large hydroplane at the corresponding speed ? (London Univ.)

Ans.—6,250 lb.

(11) The resistances to the uniform flow of fluids through similar pipes is given by $\frac{p}{l} = \frac{\rho v^2}{d} \phi\left(\frac{\mu}{\rho d v}\right)$, in which $\frac{p}{l}$ is the pressure drop per unit length, and d is the diameter of the pipe, ρ is the density, μ the viscosity, and v the velocity of the fluid.

Hence, find the pressure drop, expressed in inches of water, in a pipe 8 in. diameter, 1,000 ft. long, in which air is flowing at a velocity of 6·21 ft. per sec., given that the loss of head is 6·1 ft. when water flows through a similar pipe 1 in. diameter 100 ft. long, at the corresponding speed. For water $\rho = 62{\cdot}4$ lb. per cu. ft., and $\mu = 1{\cdot}01 \times 10^{-2}$ C.G.S. units ; for air $\rho = {\cdot}0751$ lb. per cu. ft., and $\mu = 1{\cdot}86 \times 10^{-4}$ C.G.S. units. (London Univ.)

Ans.—·405 in.

(12) Define *Coefficient of Viscosity* and explain how its absolute value can be determined.

A shaft having a diameter of 2 in. rotates centrally in a bush having a diameter of 2·006 in. and length of 4 in. The annular space between the shaft and bush is filled with oil having a viscosity of 0·9 poise. Determine the horsepower required to drive the shaft when the speed of rotation is 600 r.p.m. (London Univ.)

Ans.—·065 h.p.

(13) Define "coefficient of viscosity." The lower end of a vertical shaft rests in a footstep bearing. Assuming that the end of the shaft and the surface of the bearing are both flat and are separated by an oil film 0·05 cm. thick, find the torque required to rotate the shaft at 750 r.p.m. Diameter of shaft, 10 cm.; viscosity of oil, 1·5 C.G.S. units. (I. Mech. E.)

Ans.—2,350 grm. cm.

(14) Find the drag in grammes weight per sq. centimetre on the wall of a pipe 2·5 cm. in diameter, when oil whose viscosity is 2 C.G.S. units flows through with a mean speed of 100 cm. per sec. What Reynolds number would this speed represent ? ρ for oil = 0·8 grm. per cu. cm. (I. Mech. E.)

Ans.—0·652 grm. per sq. cm.; $R_e = 100$.

(15) Name and discuss the factors influencing pipe friction. Find the loss of head when a flow of 33 cu. ft. per sec. takes place through 17,000 ft. of 36-in. pipe, the value of $\tau/\rho v^2$ being 0·003 where τ = drag stress on wall, ρ = density, and v = speed. (I. Mech. E.)

***Ans.*—46 3 ft.**

(16) What is meant by dynamical similarity ?

What are the conditions necessary for dynamical similarity in the case of the flow of two different fluids through similar pipes of different diameter ? (I. Mech. E.)

(17) Find from first principles the head lost due to laminar flow in a pipe of circular section in terms of the length l, the diameter d, the mean velocity v, and the density and viscosity of the fluid. Hence show that if the formula $4flv^2/2gd$ is used for the loss due to friction under these conditions, the value of f is $\frac{16}{R_e}$. Calculate the loss of head in a pipe $\frac{1}{8}$ in. diameter and 20 ft. long when water flows at half the critical velocity, if the critical velocity occurs when Reynolds number is 2,500 and the coefficient of viscosity is 0·0101 poise. (London Univ.)

Ans.—2·63 ft.

(18) Show that the velocity v with which a piston of diameter D and length l moves in a concentric dash-pot is given approximately by $v = \frac{4}{3\pi\mu} \cdot \frac{h^3P}{D^3l}$, where h is the clearance between the piston and the dash-pot, P is the load of the piston, and μ is the coefficient of viscosity of the fluid. State what approximations you make. (London Univ.)

(19) Define coefficient of viscosity. Find the torque to rotate a shaft, diameter 50 mm., at 1,200 r.p.m. concentrically within a sleeve 50·17 mm. in diameter and 90 mm. long, flooded with oil for which $\mu = 0{\cdot}8$ grm.-cm.$^{-1}$ sec.$^{-1}$ (I. Mech. E.)

Ans.—10,600 gm. cm.

(20) Describe the two different types of flow that may occur in a pipe. The difference of pressure at the ends of a 300-ft. pipe line conveying oil is maintained at 10 lb. per sq. in. If the diameter of the pipe line is 4 in. and the viscosity and density of the oil are 3·0 C.G.S. units and 0·9 grm. per cu. cm. respectively, determine the quantity passing. (I. Mech. E.)

Ans.—0·232 cu. ft. per sec.

(21) Oil of specific gravity 0·91 and viscosity 1·24 poises is pumped through a pipe 3 in. diameter at a rate of 15 cu. ft. per min. Show that the motion is streamline, and estimate the horse-power required to pump the oil through a length of 250 ft. of this pipe which rises 10 ft. (London Univ.)

Ans.—$R_e = 868$; H.P. = ·89.

(22) Prove that, for slow flow through a pipe, $\tau d/\mu v = 8$, where τ = sheer stress at pipe wall, d = diameter, μ = viscosity, and v = mean velocity. Find the flow of oil, $\mu = 0{\cdot}5$ grm.-cm^{-1}-sec.$^{-1}$, density 0·8 grm.-cm.$^{-3}$, through a pipe, diameter 0·9 cm. and 12 metres long, when the head lost is 75 cm. (I. Mech. E.)

Ans.—$v = 2{\cdot}49$ cm. per sec.

(23) Deduce an expression for the loss of head h, when a liquid of viscosity ν flows through a pipe of diameter d and length l at a velocity v.

For a liquid flowing along a similar pipe the loss of head was proportional to $v^{1.75}$. Find f in the formula $h = \frac{flv^2}{m2g}$. (London Univ.)

(24) For the flow of a fluid through similar pipes at speeds above the critical velocity prove that the drop in pressure per unit length is given by $p/l = \rho v^2/d \cdot \phi(R)$, in which ρ is the density and v the velocity of the fluid, d is the diameter of the pipe, and R is the Reynolds number.

Hence, show that the frictional coefficient f in the formula $4flv^2/2gd$ for the frictional loss is a function of the Reynolds number. (London Univ.)

CHAPTER XIII

HYDRAULIC MACHINES, METERS, AND VALVES

147. **The Hydraulic Accumulator.** The hydraulic accumulator is a cylinder used for temporary storing the energy of water.

Hydraulic machines such as lifts or cranes are required to do a large amount of work during a small interval of time, which is followed by an idle period. For example, a lift or crane requires the energy to be supplied during the upward motion of the load only, practically no energy being used during the downward motion. But as the pumps are supplying the energy continuously, it may be stored in an accumulator during the idle periods of the machine and given out at an increased rate during the working periods. The uniform supply of energy from the pumps need not, therefore, be as large as that required by the machine when doing its maximum rate of work, as the machine will then draw from the accumulator.

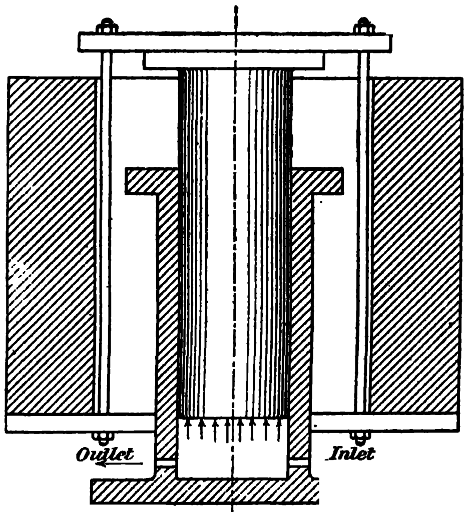


FIG. 177

The accumulator consists of a vertical cylinder containing a sliding ram (Fig. 177). A container, fixed to the ram, is filled with heavy material such as slag, or the ram is loaded with weights. Water is delivered by the pumps into the cylinder when not required by the machine it is working. The pressure of the water lifts up the heavy ram until the cylinder is full. The accumulator has then stored its maximum amount

of energy. During its period of maximum work, the machine will draw from the accumulator and the ram will descend.

The maximum amount of energy the accumulator can store is known as the capacity of the accumulator.

Let A = area of base of ram in square feet,

and H = lift of ram in feet.

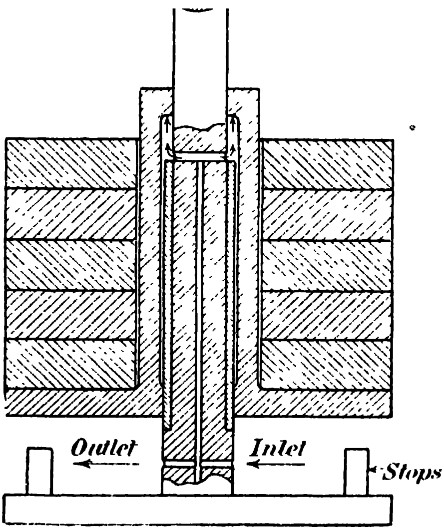


FIG. 178

Then, volume of accumulator $= AH$ cu. ft.

Let p = intensity of pressure of water supplied in pounds per square feet.

Then, weight of ram $= pA$ lb.

Work done in lifting ram $= pAH$ ft. lb.

This equals the energy stored, which is the capacity of the accumulator.

Therefore,

capacity of accumulator

$= pAH$

$= p \times$ volume

Another form of accumulator, known as Tweddell's differential accumulator, is shown in Fig. 178. The advantage of this accumulator is that the water can be stored at a high pressure by a comparatively small load on the ram. It consists of a fixed ram of which the lower portion is made larger than the upper portion by surrounding it with a brass bush. Sliding on the fixed ram is a loaded cylinder, which is forced upwards by the pressure of the water from the main supply. The water enters and leaves the cylinder by a hole through the centre of the fixed ram.

Let a = sectional area of brass bush in square feet

= effective area of ram

Load on cylinder $= pa$.

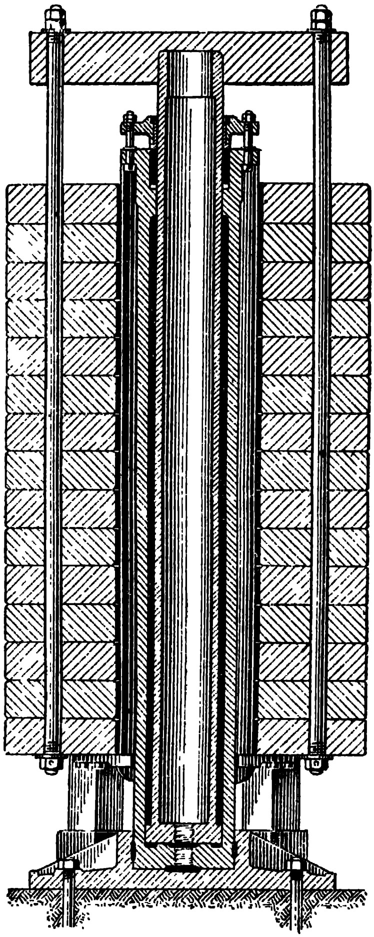
Therefore, by making the area of the bush small, it is possible to store at a high pressure with a small load.

Capacity of accumulator

$$= paH$$
$$= p \times \text{volume}$$

A sectional view of an actual accumulator* is shown in Fig. 179.

If the pipes leading to an accumulator are very long great trouble is experienced owing to surging, which is caused by the inertia effect of the water column. This can be overcome by fitting some form of relief valve (Art. 155) as close to the accumulator as possible.



(*Hydraulic Engineering Co.*)

FIG. 179.—SECTION OF HYDRAULIC ACCUMULATOR

Cast-iron weight type

EXAMPLE 1.

An accumulator has a ram of 6 in. diameter and a lift of 18 ft. Water is supplied at a pressure of 800 lb. per square inch. Find the necessary load on the ram and the capacity in horse-power hours.

$$\text{Load on ram} = p \times a$$
$$= 800 \times \frac{\pi}{4}.(6)^2$$
$$= 22{,}600 \text{ lb.}$$
$$\text{Capacity} = paH$$
$$= 22{,}600 \times 18 \quad 407{,}000 \text{ ft. lb.}$$
$$= \frac{407{,}000}{33{,}000 \times 60} \quad \cdot 206 \text{ h.p. hours}$$

* By courtesy of The Hydraulic Engineering Co., Chester.

EXAMPLE 2.

An accumulator has a 12-in. ram and 15 ft. lift, and is loaded with 80 tons total weight. If packing friction is equivalent to 5 per cent of the load on the ram, determine the horse-power being delivered to the mains if the ram falls steadily through its full range in 1·5 minutes, and if at the same time the pumps are delivering 1 cu. ft. per sec. through the accumulator. (A.M. Inst. C.E.)

First find the pressure of water produced by the falling ram of the accumulator; this will be the pressure of water supplied to mains.

Intensity of pressure when ram is falling

$$= \frac{\text{Weight} \times \frac{95}{100}}{\text{area}}$$

$$= \frac{80 \times 2240 \times 0{\cdot}95}{\frac{\pi}{4}} \text{ lb. per sq. ft.}$$

Head of water due to this pressure,

$$= \frac{80 \times 2240 \times 0{\cdot}95}{\frac{\pi}{4} \times 62{\cdot}4}$$

$$= 3475 \text{ ft. of water}$$

Work supplied by pumps per min.,

$$= WH$$

$$= (62{\cdot}4 \times 60) \times 3475$$

$$= 13{,}000{,}000 \text{ ft. lb.}$$

Work done by accumulator per min.,

$$\text{Weight} \times \text{distance moved}$$

$$= (80 \times 2240 \times {\cdot}95) \times 10$$

$$= 1{,}703{,}000 \text{ ft. lb.}$$

$$\text{Horse-power delivered} = \frac{13{,}000{,}000 + 1{,}703{,}000}{33{,}000}$$

$$= 445$$

148. **The Hydraulic Intensifier.** The hydraulic intensifier is used for increasing the intensity of pressure of water by

means of the energy of a larger quantity of water at low pressure. This is necessary when the pressure of the water supplied to a machine is not of sufficient intensity.

An intensifier consists of a fixed ram (Fig. 180) through which the high pressure water flows to the machine. Mounted externally on the fixed ram is a hollow sliding ram containing the high pressure water. The sliding ram is encased by a fixed cylinder which contains the low pressure water from the main supply. The low pressure water presses on the end of the sliding ram, forcing it downwards on to the fixed ram; this increases the pressure of the water in the sliding ram.

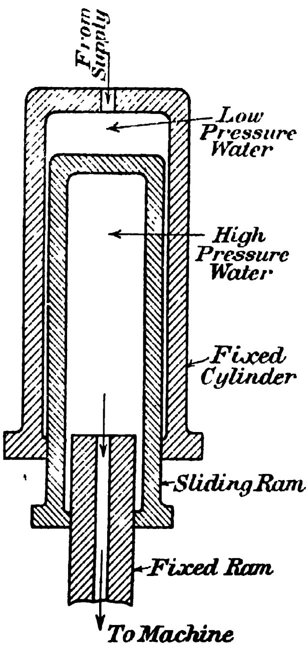


FIG. 180

Let A = external area of end of sliding ram

a = area of end of fixed ram

P = intensity of pressure of low pressure water in fixed cylinder

p = intensity of pressure of high pressure water in sliding ram

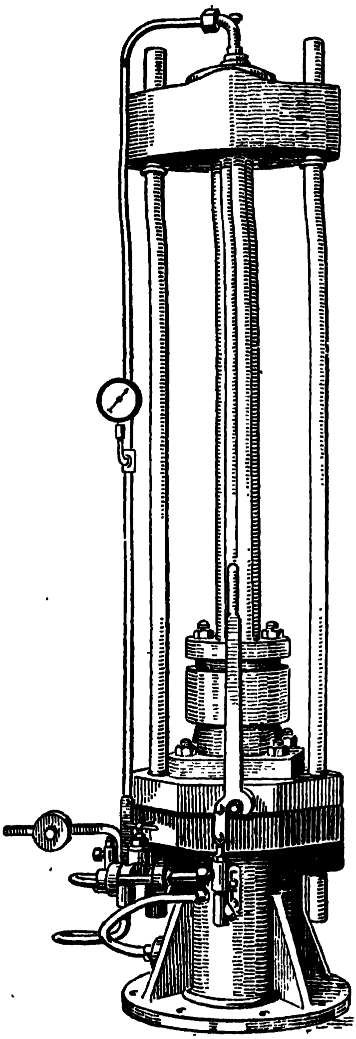
As total upward force = total downward force

$$pa = PA$$

From which
$$p = \frac{PA}{a}$$

When the sliding ram is at the bottom of its stroke the valve admitting the high pressure water to the machine is closed. Low pressure water from the main is then admitted to the inside of sliding ram and the fixed cylinder is open to exhaust; this causes the sliding ram to rise. When it reaches the top of its stroke the valve admitting high pressure water to machine

is opened and the valve admitting low pressure water to inside of sliding cylinder is closed. At the same time the fixed cylinder valve closes to exhaust and opens to the main. Low pressure water then flows into the fixed cylinder and forces the sliding cylinder downwards; this produces the high pressure water in the sliding cylinder which is forced into the machine. The intensifier is thus single acting, supplying high pressure water during the downward stroke only. Double acting intensifiers are made which give a continuous supply of high pressure water.



(Hydraulic Engineering Co.)

FIG. 181
HYDRAULIC INTENSIFIER

It is possible to raise the pressure of water to 10 tons per sq. in. by means of an intensifier.

The view of an actual intensifier is shown in Fig. 181.

EXAMPLE

Water is supplied to an hydraulic intensifier at a pressure of 24 lb. per square inch. The diameters of the sliding and fixed rams of the intensifier are 2 in. and 5 in. respectively. Find the pressure of the water leaving the intensifier.

$$p = \frac{PA}{a} = 24 \times \frac{5^2}{2^2}$$

$$= 150 \text{ lb. per sq. in.}$$

149. **Water Meters.** (*a*) THE KENT VENTURI METER. This consists of the ordinary Venturi meter, which has already been dealt with in Art. 27, on to which is attached a special apparatus for indicating the flow of water. The quantity of

water flowing through the meter is proportional to the square root of the difference of pressure heads at the entrance and throat. The flow is plotted by a pencil on to a drum which is revolved by clockwork, whilst the total flow through the meter is recorded by the small dials shown in Fig. 182.

The instrument, shown in Fig. 182, consists of two cast-iron cylinders, M_1 and M_2, containing mercury on which rests two floats, F_1 and F_2. These cast-iron cylinders form the two limbs of a U-tube and are connected at their bases by the tube m. The water pressure at the entrance and throat of the Venturi meter is transmitted to the cylinders through the pipes P_1 and P_2. The floats are connected to racks D_1 and D_2, which turn the pinions H_1 and H_2 as the floats rise or fall with the pressure difference in the Venturi meter. These pinions transmit the motion to two other racks, J_1 and J_2, which are outside of the cylinders. The rack J_1 operates the pencil G, which plots the pressure difference on the squared paper surrounding the drum D. The latter is rotated by clockwork governed by the pendulum P. The squared paper used on the drum is so divided that the flow may be read direct. As the vertical displacement of the pencil is proportional to the pressure difference, the paper must be divided so as to read the square root of the pressure difference. It should be noted that the float F_1 will rise the same amount as F_2 falls, both displacements being in proportion to the pressure difference of the Venturi meter.

The right-hand external rack J_2 operates the recording dials to register the flow. The vertical displacement of the rack is proportional to the pressure difference; this must be reduced to the square root of the pressure difference in order to register the flow. Inside of the clockwork drum D, and rotating with it, is an integrating drum, the development of which is shown in Fig. 183. The drum is divided by a parabolic curve ABC. The shaded surface above the curve is raised above the surface below. The rack J_2 is connected to a carriage which is in contact with the drum and which gears with the recording dials. When the raised surface of the drum comes in contact with the carriage the latter is put out of gear with the recorder and no flow is registered. Thus, if the carriage is at a height D, the flow will only be registered over the portion of a revolution represented by DB; it will be out of gear during the portion BE. When the float is at the top of the cylinder there will

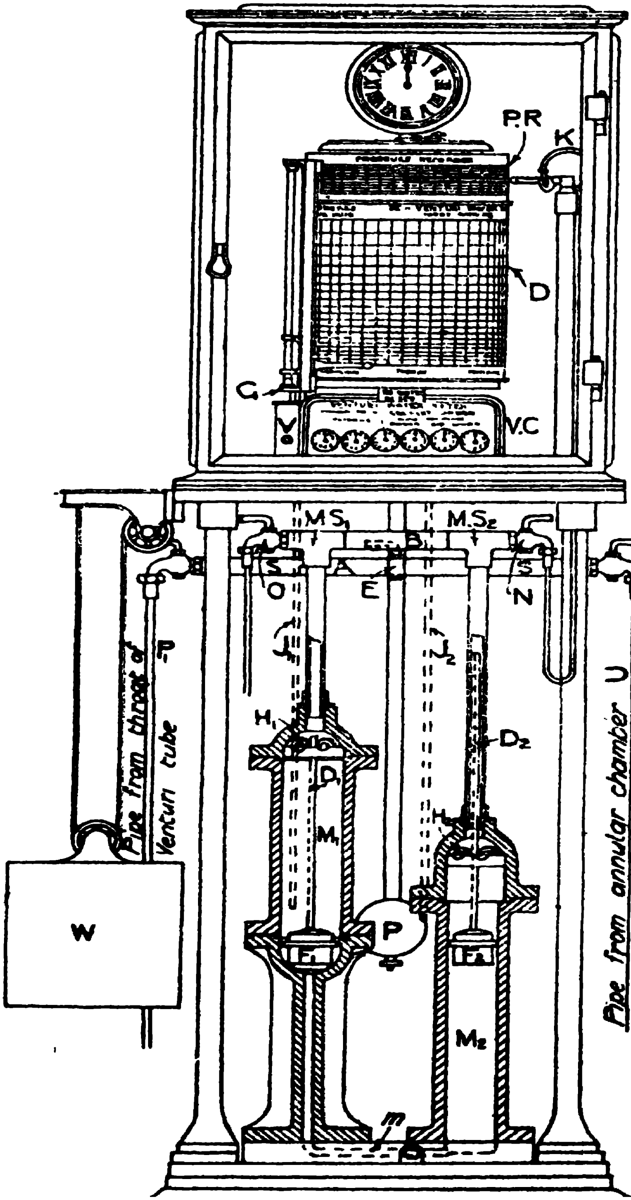


FIG. 182.—KENT VENTURI METER

be no flow taking place; the carriage will then be on the raised surface above A (Fig. 183) during the revolution of the drum, and no flow will be registered by the recorder.

(*b*) THE DEACON METER. This meter is shown in Fig. 184. It consists of a cast-iron body C, into which is fitted a hollow cone A. The water flows into the meter through E, passes through the cone and leaves the meter through F. A disc D, having a diameter equal to the smallest diameter of the cone, is fixed to the rod G, which slides up and down in the boss B. A balance weight Q is attached to a wire W fixed to the top of the rod G, and keeps the disc D at the top of the cone when no water is flowing. When the water flows into the meter it forces down the disc D into a wider part of the cone and passes

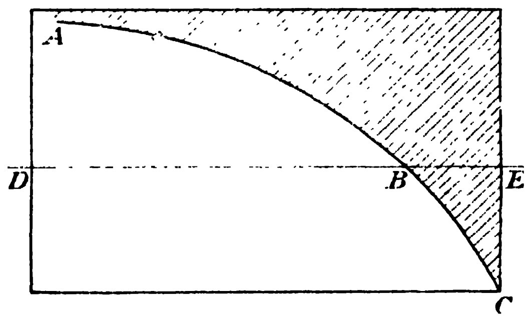


FIG. 183

through the space between D and the cone sides. This space increases as D descends; the vertical drop of D will, therefore, be in proportion to the quantity of water flowing.

The flow through the meter is recorded by means of a pencil connected to the wire from the rod G. The pencil is in contact with the surface of the drum R, which is revolved by clockwork. As the vertical motion of the pencil is proportional to the movement of the disc D, a curve giving the quantity of flow through the meter at any instant will be automatically drawn on suitably graduated squared paper placed around the revolving drum.

This meter is chiefly used for measuring the waste water flow in water mains.

(*c*) THE KENNEDY METER. This is a positive type of meter, the volume of water flowing being actually measured by continually filling a cylinder of known volume.

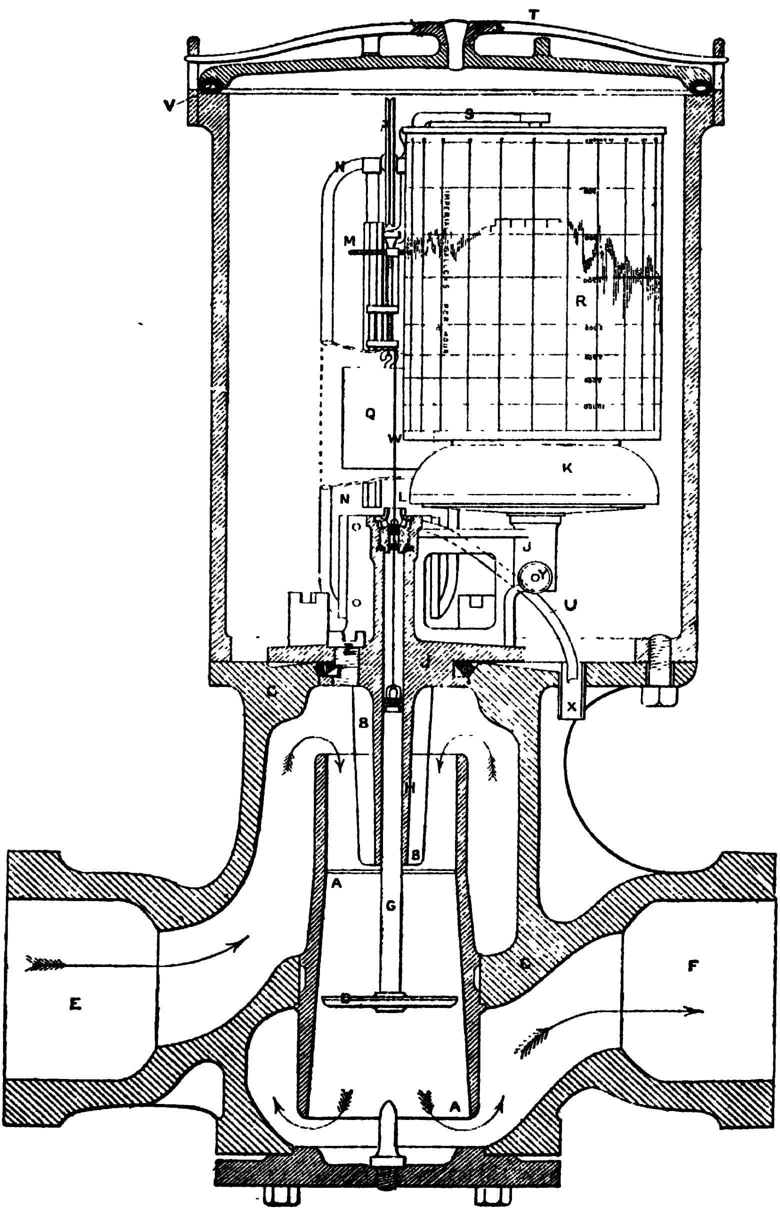


FIG. 184.—DIFFERENTIATING WASTE WATER METER

The meter consists of a cylinder (Fig. 185), in which slides a piston. The piston rod is connected to a rack, which slides up and down with the piston. The rack gears with a pinion, which operates a four-way cock. A diagrammatic view of the passages is shown in Fig. 186. The water

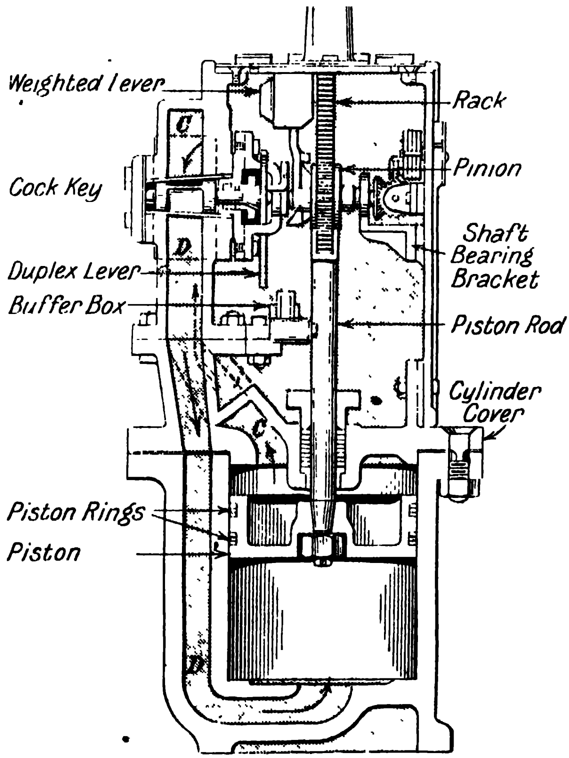


FIG. 185.—THE KENNEDY METER

from the supply pipe flows through *A* into the pipe *D*, through which it enters the lower end of the cylinder and forces up the piston. As the piston rises, the rack turns the pinion. A weight is fixed to the end of a lever which is rotated upwards by a pin fixed to the pinion. When the piston reaches the top of its stroke, the weight is rotated to just beyond the vertical position ; it then falls over suddenly and, by striking a lever, operates the cock into its reverse position. This

new position of the cock cuts off the water supply from the lower end of the cylinder and admits it to the upper end. At the same time, the lower end is open to the outlet pipe *B* of the meter. The piston now moves downwards under the pressure of the incoming water, and forces the water in the

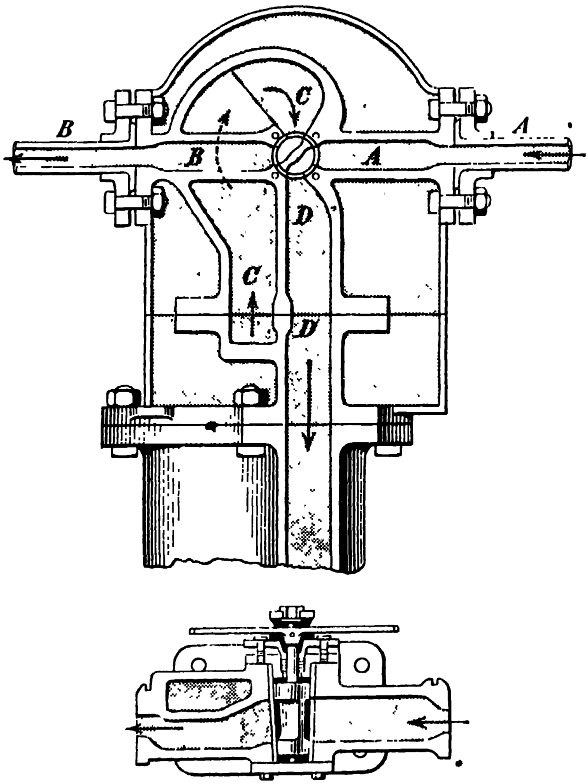


FIG. 186.—KENNEDY METER—SECTION THROUGH PORTS

lower end of the cylinder up the pipe *D* into the outlet pipe. In moving downwards, the rack operates the pinion, which causes the weight to be again raised. When the piston reaches the bottom of its stroke, the weight falls over and turns the cock back to its former position. The upper part of the cylinder is now open to the outlet pipe and the lower part to the supply pipe. The piston will now be forced upwards, driving the water above it through the pipe *C* and into the outlet pipe. The cycle is then repeated.

For each stroke of the piston, a volume of water equal to the volume of the cylinder passes through the meter. The strokes are registered by means of a counter, operated by the pinion, which records the quantity of flow.

150. **The Hydraulic Ram.** The hydraulic ram is an automatic pump by means of which a large quantity of water falling through a small height is utilized in lifting a small quantity of water to a greater height.

A diagrammatic view of a hydraulic ram is shown in Fig. 187. Water from the natural supply A has an available head of H_1;

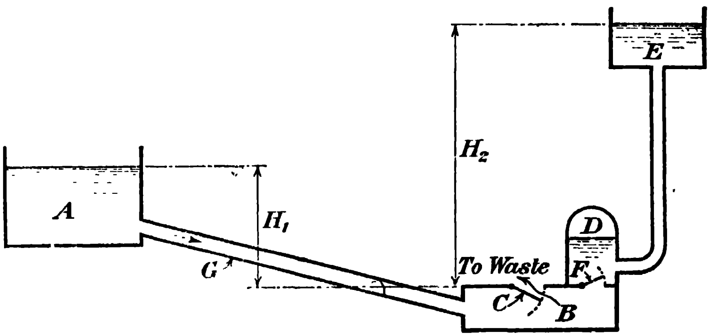


Fig. 187

by means of the ram a small quantity of this water is raised through the height H_2 into the service tank E.

Let W = weight of water flowing per second from A

w = weight of water raised per second to E

Then, as energy supplied by A is theoretically equal to energy supplied to E,

$$WH_1 = wH_2$$

Or

$$w = \frac{WH_1}{H_2}$$

If losses are taken into account,

$$\text{efficiency of ram} = \frac{wH_2}{WH_1}$$

The automatic action of the ram is due to the inertia forces of the water in the pipe G. The water commences to flow down the pipe G into the chamber B. The waste water valve C is open and the water flows through it to waste. As

the speed of the water in *G* increases, the dynamic pressure on the valve *C* increases, until it will ultimately be greater than the weight of the valve lid; the valve will then suddenly close. The closing of the valve *C* brings the water in *G* suddenly to rest, causing an increase of pressure in *B*. This increase of pressure lifts the valve *F* and some of the water will flow into the air vessel *D*, compressing the air in the vessel. This

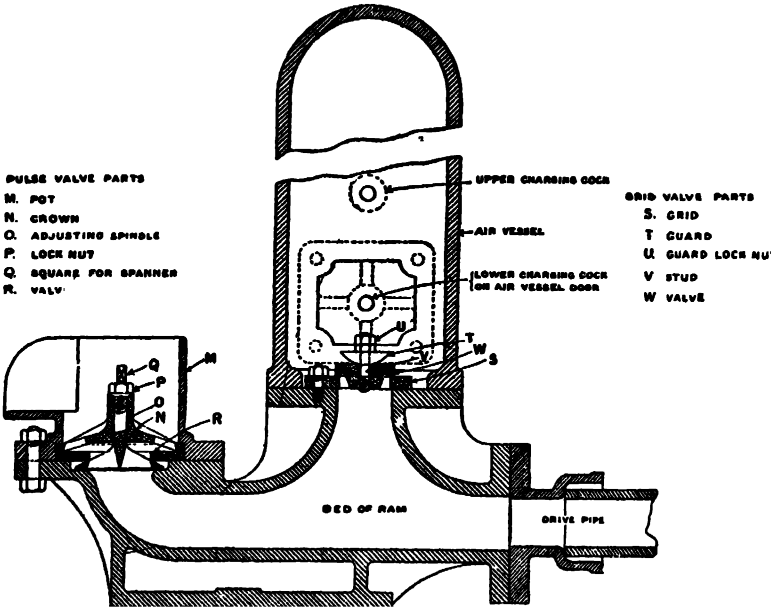


FIG. 188.—HYDRAULIC RAM
The air vessel is drilled so that the delivery may be on either side

increased air pressure forces the water into the tank *E*. When the momentum of the water in *B* is destroyed, the valve *F* closes and the valve *C* opens, causing the flow from *A* to recommence; the cycle is then repeated. The automatic valves *C* and *F* may act by their weight or by a spring.

Hydraulic rams are chiefly used on country estates and farms at which a large quantity of water under a low head is available.

The cross-sectional view of an actual hydraulic ram* is shown in Fig. 188, and a plan and elevation of the complete installation is shown in Fig. 189.

* By courtesy of Messrs. Green & Carter, Ltd., Winchester.

The overall efficiency of the hydraulic ram is as large as 80 per cent, and water can be lifted to a height of fifty times the height of the working fall. Compound hydraulic rams are made which will raise water to any height to which it could be forced by an ordinary pump.

151. **The Hydraulic Press.** Hydraulic presses are used in most branches of industry; in principle they are the same as the Bramah press which was dealt with in Art. 4. They vary greatly in type according to the nature of the work required, but all consists of a ram sliding in a cylinder into which high pressure water is forced. In some large forging presses water

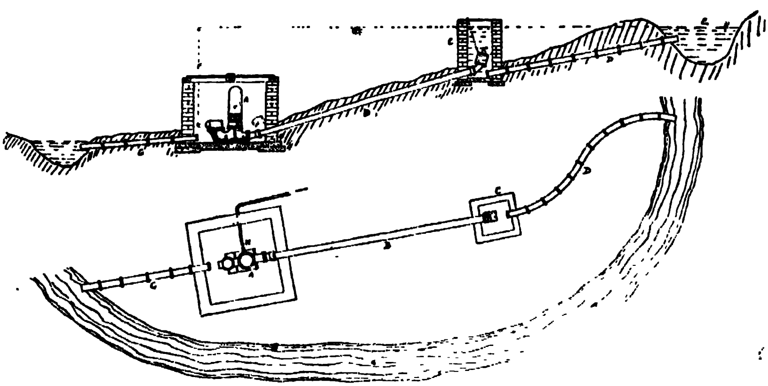


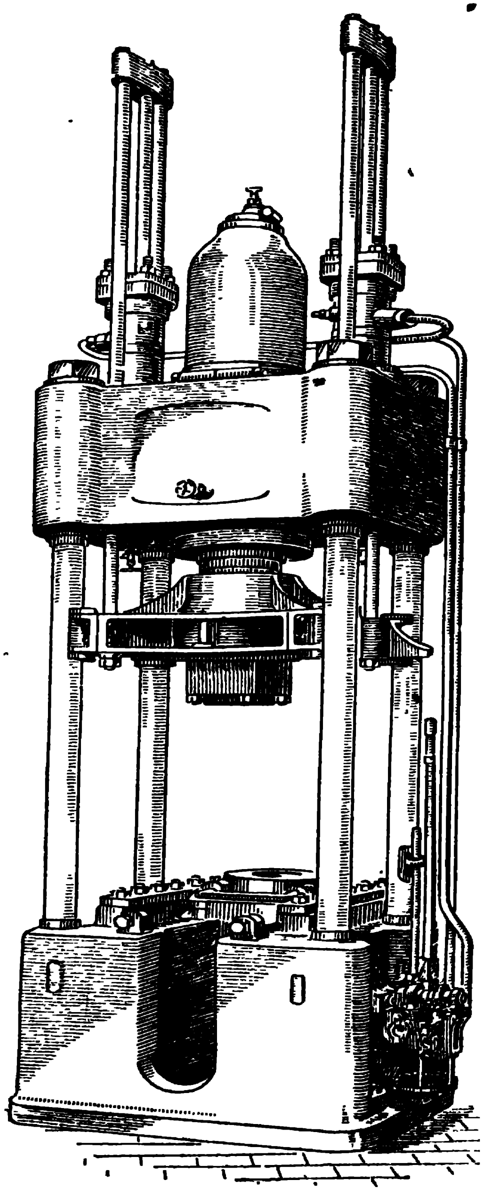
FIG. 189

at a pressure of 5 tons per sq. in. is used in the cylinder and produces a total force of 5,000 tons.

In all heavy presses some means must be adopted to bring about the return stroke of the ram. To do this, small return rams are fitted, their function being to bring the main ram back to the beginning of its stroke. The size of the return ram must be such that the area multiplied by water pressure is sufficient to lift the main ram. In designing the main ram, the area multiplied by water pressure should be large enough to do the work of the press and to overcome the resistance of the return rams.

For balancing purposes it is necessary to have two return rams to one main ram. An alternative method is to have one return ram in tandem with the main ram.

A view of a shell forging press is shown in Fig. 190; in this



(*Hydraulic Engineering Co.*)

FIG. 190.—SHELL FORGING PRESS

view the two return rams can be seen at the sides of the main ram.

EXAMPLE.

The ram of a hydraulic press is 8 in. diameter, and is worked from an intensifier of the piston and ram type which receives its low pressure supply of water from a tank whose surface level is 50 ft. above the level of the intensifier piston, through a pipe 2 in. diameter and 400 ft. long. The intensifier ram is 3 in. diameter and the piston 36 in. diameter. The friction of each of the three packings may be taken to be 3 per cent of the total pressure on the appropriate piston or ram. The frictional coefficient for the low pressure supply pipe is 0·005. Calculate the speed of advance of the press ram in inches per minute when exerting a force of 50 tons. Neglect all other losses. (London Univ.)

$$\text{Water pressure on ram of press} = 50 \times \frac{100}{97} = 51{\cdot}5 \text{ tons}$$

Intensity of pressure on ram of press

$$= \frac{51{\cdot}5}{\frac{\pi}{4}8^2} = 1{\cdot}025 \text{ tons per sq. in.}$$

As this is the same pressure transmitted by the ram of the intensifier, intensity of pressure on intensifier ram

$$= 1{\cdot}025 \times \frac{100}{97} \text{ tons per sq. in.}$$

As load on intensifier ram equals load on intensifier sliding cylinder,

$$1{\cdot}025 \times \frac{100}{97} \times \frac{\pi}{4} \times 3^2 = p \times \frac{\pi}{4} \times 36^2 \times \frac{97}{100}$$

where p = pressure of low pressure water supply

$$\text{Hence,} \quad p = 1{\cdot}025 \times \left(\frac{100}{97}\right)^2 \times \left(\frac{36}{3}\right)^2 \times 2240$$

$$= 16{\cdot}96 \text{ lb. per sq. in.}$$

Hence, pressure head of low pressure water

$$= \frac{16{\cdot}96 \times 144}{62{\cdot}4} = 39{\cdot}2 \text{ ft.}$$

$$\left.\begin{array}{l}\text{Therefore, head lost in} \\ \text{friction in 2-in. pipe}\end{array}\right\} = 50 - 39{\cdot}2 = 10{\cdot}8 \text{ ft.}$$

Let v = velocity of water in 2-in. pipe in ft. per sec.

V = velocity of ram in ft. per sec.

$$\left.\begin{array}{l}\text{Head lost in friction}\\ \quad\text{in 2-in. pipe.}\end{array}\right\} = 10{\cdot}8 = \frac{4 f l v^2}{2 g d}$$

$$= \frac{4 \times {\cdot}005 \times 400 v^2 \times 12}{2 \times 32{\cdot}2 \times 2}$$

From which $v = 3{\cdot}8$ ft. per sec.

As quantity of water flowing along 2-in. pipe per sec. equals the quantity flowing in the press cylinder per sec.,

$$v \times \left(\frac{\pi}{4} \times \frac{2^2}{144}\right) = V \times \left(\frac{\pi}{4} \times \frac{8^2}{144}\right)$$

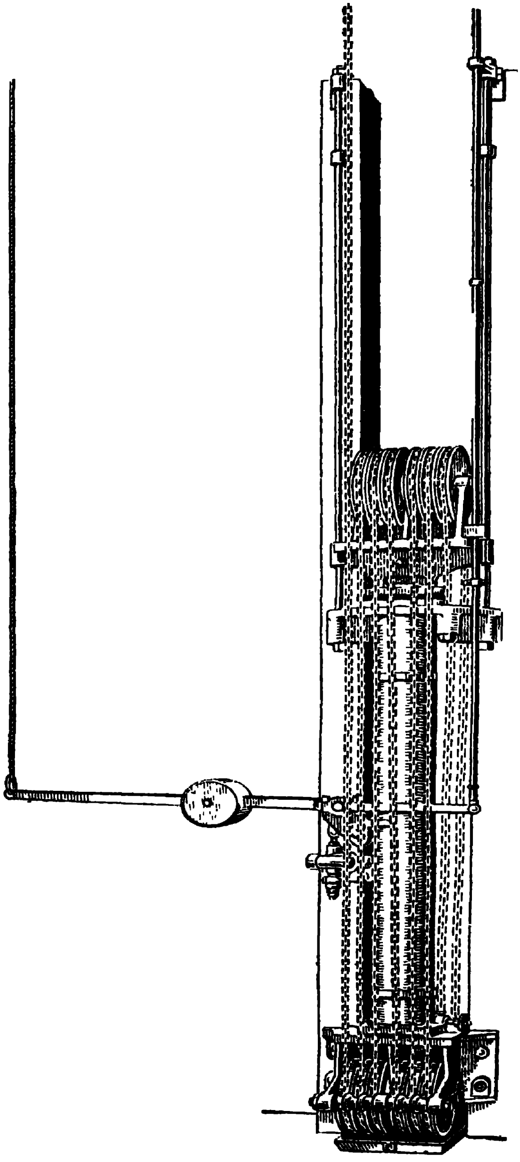
From which $V = 3{\cdot}8 \times \left(\frac{2}{8}\right)^2$ ft. per sec.

$$= 14{\cdot}25 \text{ ft. per min.}$$

$$= 171 \text{ in. per min.}$$

152. **The Hydraulic Crane.** The hydraulic crane is usually found at docks, sidings, and warehouses, and is used for lifting loads up to 250 tons. It consists of a central pedestal supporting a mast from which is suspended a jib, or arm, the latter can be raised or lowered in order to reduce or increase the radius of action. The mast revolves about a vertical axis, the jib swinging with it; thus, by revolving the pedestal and lowering the jib, the suspended load may be moved to any place within the crane's area of action. The principle of the suspended jib enables the load to be lifted over obstacles on the ground.

The load is suspended by a wire rope which passes over pulleys to a hydraulic ram; this ram has an arrangement of pulleys for increasing the velocity ratio and is known as a jigger. The jigger is attached to the mast and consists of a sliding ram and cylinder at the ends of which are pulleys (Fig. 191); it increases the velocity ratio of the ram and cable by acting on the principle of the multi-sheaved pulley blocks. One set of pulleys are fixed to the ram whilst the other set are fixed to the cylinder, the cable being wound over both sets of pulleys. High pressure water is admitted into the cylinder, forcing out the ram; this increases the distance between the two sets of pulleys, thus winding in the cable. A six-sheaf pulley block system will give a velocity ratio of six to one; this means that the suspended load will move at six times the speed of the ram. A modern hydraulic crane may have a lifting speed of 250 ft. per minute.



(*Hydraulic Engineering Co.*)

FIG. 191.—SINGLE POWER JIGGER

Example.

The following particulars refer to a hydraulic crane—

Diameter of ram, 12 in.
Velocity ratio of crane hook to ram, 5 : 1.
Length of supply pipe from accumulator, 500 ft.
Diameter of supply pipe, 2 in.
Pressure at accumulator, 750 lb. per sq. in.
Mechanical friction of ram, pulleys, etc., equivalent to a pressure of 50 lb. per sq. in. on the ram.
Coefficient of friction for the pipe, ·010.
Plot a curve showing the relation between the load lifted and the speed of lifting. (London Univ., 1923.)

Let W = load lifted in pounds

Then load on ram $= 5\,W$

Let v = velocity of water in 2-in. pipe

V = velocity of lifting in ft. per sec.

Then, velocity of ram $= \dfrac{V}{5}$

$$\text{Intensity of pressure on ram} = \frac{5W}{\frac{\pi}{4} \times 1} + (50 \times 144)\ \text{lb. per sq. ft}$$

$$= 6{\cdot}36W + 7200\ \text{lb. per sq. ft.}$$

$$\text{Head of water on ram} = \frac{p}{w} = \frac{6{\cdot}36W + 7200}{62{\cdot}4}\ \text{ft. of water}$$

$$= {\cdot}1W + 115\ \text{ft. of water}$$

$$\text{Head of water in accumulator} = \frac{750 \times 144}{62{\cdot}4} = 1730\ \text{ft. of water}$$

Head lost in friction in pipe = Head in accumulator—head on ram

$$= 1730 - ({\cdot}1W + 115)$$

$$= 1615 - {\cdot}1W\ \text{ft. of water}$$

Hence,
$$1615 - {\cdot}1\,W = \frac{4flv^2}{2gd}$$

$$= \frac{4 \times {\cdot}01 \times 500v^2 \times 12}{62{\cdot}4 \times 2}$$

From which
$$v = \sqrt{840 - {\cdot}052W}\ \text{ft. per sec.}$$

As quantity of flow along pipe per second equals flow per second in ram cylinder,

$$v \times \frac{\pi}{4}\left(\tfrac{1}{6}\right)^2 = \frac{V}{5}\frac{\pi}{4}(1)^2$$

From which
$$V = \frac{5}{36}v$$
$$= \frac{5}{36}\sqrt{840 - \cdot 052\,W}$$
$$= \sqrt{16 \cdot 0 - \cdot 00104\,W}$$

By substituting various values of W in this equation the corresponding values of *V* are obtained.

W	0	2,000	4,000	6,000	8,000	10,000	12,000	14,000
V	4·0	3·72	3·44	3·12	2·77	2·36	1·87	1·2

A curve may now be plotted with these results.

153. **The Hydraulic Lift.** The hydraulic lift obtains its motion from a jigger, in the same way as the crane (Art. 152). The jigger should be fixed with the ram working downwards, so that its weight will be supported by the cables; this prevents any tendency of the ram to move independently of the lift cage. The lift cage runs between guides of hard wood or round steel, and is usually suspended by four lifting ropes, each one being of sufficient strength to support the load. Sliding balance weights are provided to balance the weight of the cage.

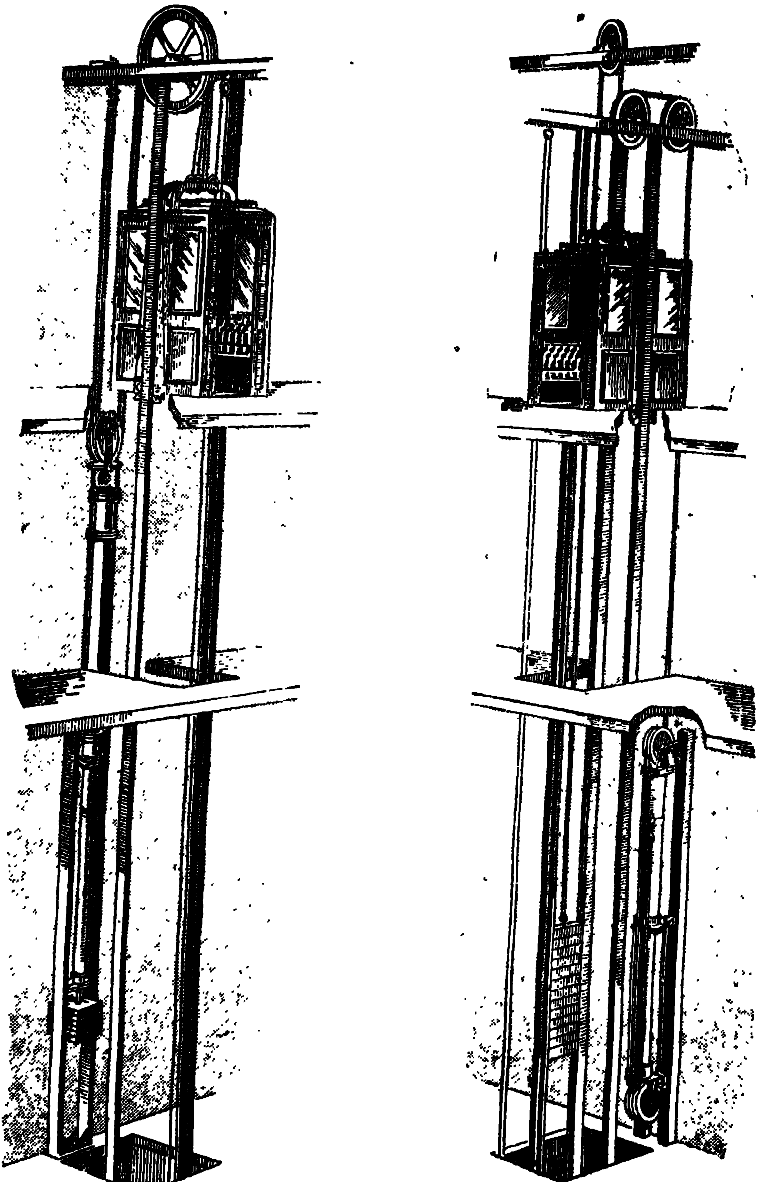
Views of hydraulic suspended lifts are shown in Fig. 192.

Modern hydraulic lifts now have a lifting speed of 350 ft. per minute in this country; in the United States lifting speeds of 400 ft. per minute are in use.

The earlier form of hydraulic lift consisted of a sliding ram and cylinder; the platform or cage was supported on the end of the ram and pushed up by it. Hence, the stroke of the ram was the same as the lift of the platform. This type of lift is known as a direct acting lift.

EXAMPLE.

A hydraulic direct-acting lift has a ram 6 in. diameter. The pipe connecting the valve box to the cylinder is short and is $\frac{3}{4}$ in. diameter. The pressure in the valve box is 800 lb. per sq. in. Neglecting frictional losses



(*Hydraulic Engineering Co.*)

FIG. 192.—HYDRAULIC SUSPENDED LIFTS

and assuming the valve fully open, find the maximum load that can be lifted steadily at a velocity of 2 ft. per sec. Find also the maximum velocity with which the lift with this load could descend steadily with an open exhaust. (A.M.I. Mech. E.)

Let v = velocity of water in $\frac{3}{4}$-in. pipe.

Then, as quantity per second flowing through pipe equals quantity per second flowing in cylinder,

$$v \times \frac{\pi}{4}\left(\tfrac{3}{4}\right)^2 = 2 \times \frac{\pi}{4}(6)^2$$

From which $v = 128$ ft. per sec.

$$\text{Velocity head of water in pipe} = \frac{v^2}{2g} = \frac{128^2}{64 \cdot 4} = 255 \text{ ft. of water}$$

Total intensity of pressure on ram = pressure in valve box + pressure due to velocity in pipe

$$= 800 + \frac{62 \cdot 4 \times 255}{144}$$

$$= 910 \cdot 5 \text{ lb. per sq. in.}$$

$$\text{Load on ram} = 910 \cdot 5 \times \frac{\pi}{4}(6)^2$$

$$= 25{,}700 \text{ lb.}$$

Let V = velocity of descent in feet per second.

$$\text{Then, velocity in } \tfrac{3}{4}\text{-in. pipe} = \left(\frac{6}{\frac{3}{4}}\right)^2 V = 64V$$

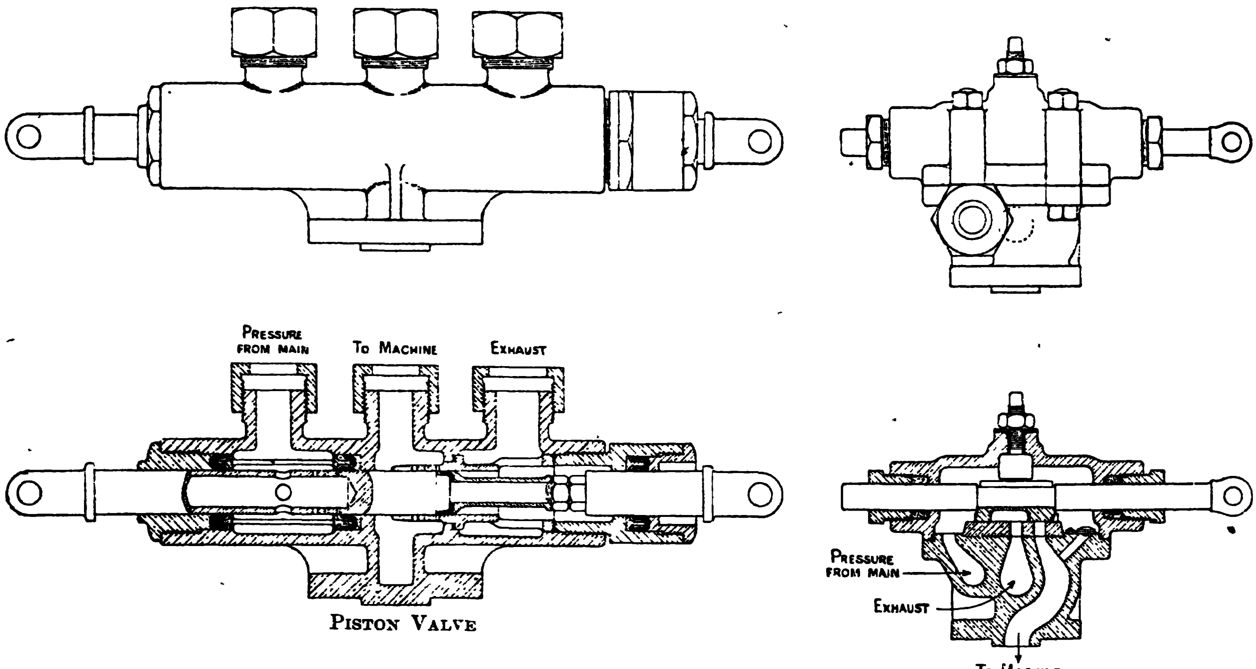
In descending, the ram will give a velocity head to water in the $\frac{3}{4}$-in. pipe; this will be the only resistance. Hence,

pressure head due to ram = velocity head in $\frac{3}{4}$-in pipe

$$\text{That is} \qquad \frac{910 \cdot 5 \times 144}{62 \cdot 4} = \frac{(64V)^2}{2g}$$

From which $V = 5 \cdot 75$ ft. per sec.

154. **The Hydraulic Capstan.** Hydraulic capstans are used for winding a haulage rope and are found in railway goods yards and at docks. They consist of a vertical drum operated by a hydraulic engine. A cable is attached to the wagon or ship which is to be moved, the free end being wrapped round the capstan's drum; the capstan's engine is then started by pressing a lever with the foot; this causes the drum to rotate and wind up the haulage cable.



PISTON VALVE

SLIDE VALVE

(*Hydraulic Engineering Co.*)

FIG. 193.—WORKING VALVES

The hydraulic engine used for capstans is usually of the "Brotherton" type. This engine consists of three fixed radial cylinders at 120°, each containing a piston, with piston rods fixed to the same crank pin. The engine contains one working valve, with three parts, each connected to one of the cylinders. High pressure water is admitted to the head of the cylinder, forcing the piston along the cylinder for the working stroke. During the return stroke the exhaust port is opened and the used water flows out to waste.

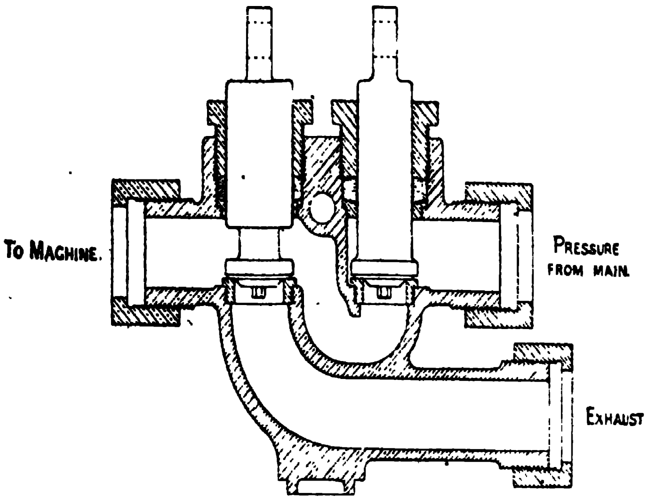
The hauling drum is keyed on to the crank shaft, and is the only part of the machine above the ground. The engine is started by a foot treadle, thus leaving the hands free to manipulate the rope. The foot treadle operates a balanced mitre valve which admits water from the mains to the engine; when the foot is removed from the treadle the valve automatically closes.

155. **Hydraulic Valves.** (*a*) SLIDE VALVES. These valves are for operating hydraulic machinery and consist of the "*D*" slide valve and the piston valve (Fig. 193); they are similar in action to the ordinary steam engine valves. For water pressures up to 1,000 lb. per sq. in. the "*D*" slide valve may be used; but for very high pressures, as in lifts and cranes, the piston valve must be used.

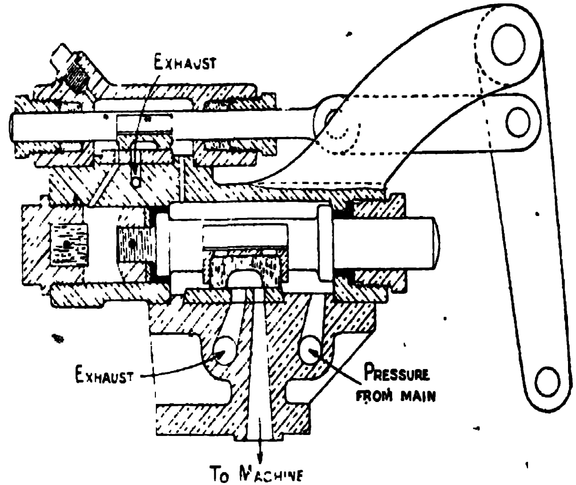
As the valves slide to and fro they uncover or cover the various ports, thus admitting or cutting off the water supply to the machine or to exhaust. The operation of the valve can be clearly seen from Fig. 193.

(*b*) MITRE VALVE. This valve is used on cranes required to lift and lower rapidly; it consists of vertical spindle valves with mitred ends working on seats to suit, and requires very little effort to operate. A view of this valve is shown in Fig. 194. The valve spindles are operated by levers.

(*c*) STOP VALVES. Stop valves are used for shutting off the main water supply. They are spindle valves and are lowered on to the seat by revolving the spindle in a screw thread, a hand wheel being fitted for this purpose. For small valves, an unbalanced valve may be used; but for large valves working under high pressure it would require too large an effort to close the valve by hand. To overcome this, a balanced stop valve is used. The balanced stop valve has the water admitted to both sides of the valve when open, thus relieving the valve spindle of the water pressure.



Mitre Valve



Water-shot Valve

(*Hydraulic Engineering Co.*)

Fig. 194.—Working Valves

Views of an unbalanced and a balanced stop valve are shown in Fig. 195.

(*d*) Relief Valves. One form of relief valve is the safety valve which is arranged to open and reduce the pressure after a certain maximum pressure has been reached. These are fitted to accumulators and to machines with a rising ram of a predetermined stroke. If the ram should rise beyond its proper limit, owing to accidental causes, the pressure of water would become excessive and dangerous; the relief valve will then open and reduce the pressure. Its action, therefore, is the same as the steam safety valve on a boiler. The form of relief valve for this purpose is a lever and weight-loaded valve; a spring-loaded valve may also be used.

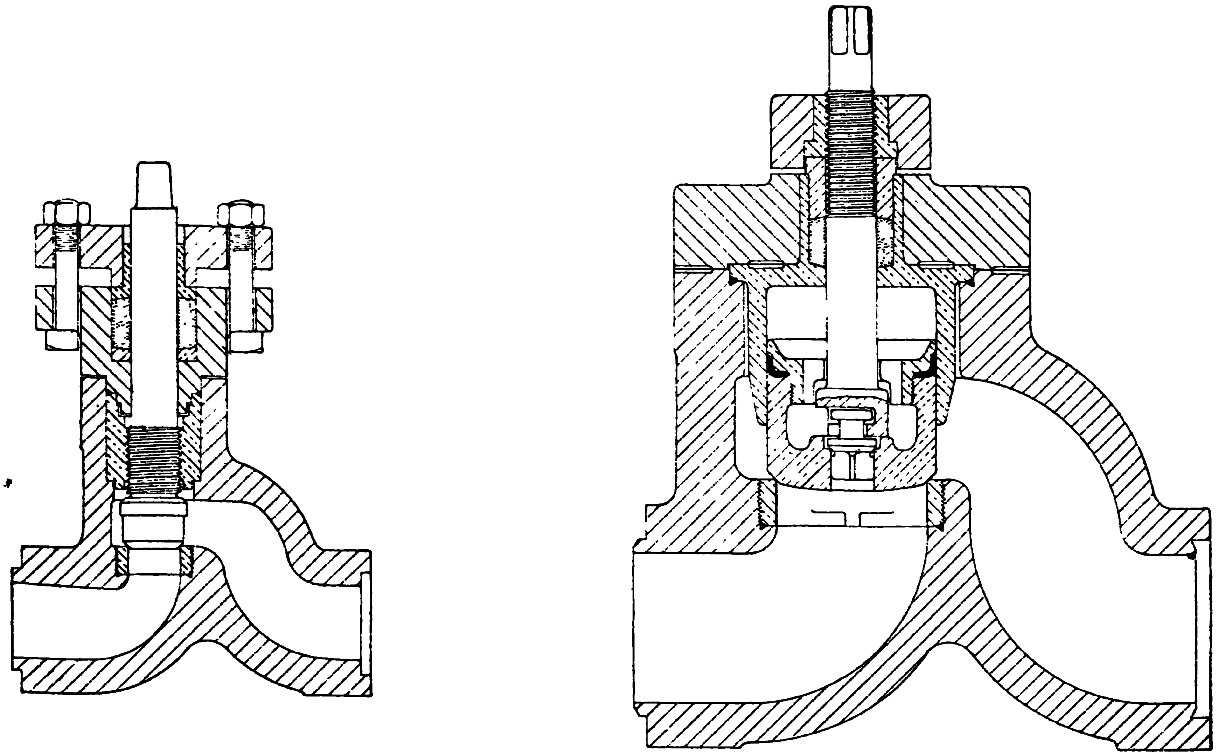
Another use of relief valve is to check the rise in pressure in a long pipe due to the sudden stopping of the flow; such valves are known as momentum valves. They consist of pistons working in a chamber against a spring. These valves are also fitted on machines which receive heavy shocks such as shell forging presses; when used for this purpose they are known as shock absorbers.

156. **Hydraulic Joints and Packing.** Hydraulic pipes of less than 2 in. diameter are usually of wrought iron with screw joints. The ends of each length of pipe are tapped and screwed into a coupling, the thread being first covered with hemp and white lead in order to prevent leakage.

Hydraulic pipes of more than 2 in. diameter are of cast iron with oval or circular flanges cast on the ends. These flanges are bolted together, a strip of packing being placed between them to prevent leakage. The packing consists of a thin sheet of rubber cut to the shape of the flange, or it may consist of one of the many patent hydraulic packing sheets which are on the market; copper rings are also used in place of sheet packing.

Hydraulic glands, pistons, etc., are packed with hemp or yarn soaked in tallow and well pressed into position. Also leather packing rings are used, the leather being first soaked in grease. These leather packing rings are named after the shape of their sections and are known as "U" leathers, "cup" leathers, and "hat" leathers; plain leather washers are also used.

The seams of wrought-iron tanks, ships, etc., are made watertight by "caulking"; the metal is caused to "flow"



UNBALANCED STOP VALVE

DOUBLE BALANCED STOP VALVE
(*Hydraulic Engineering Co.*)

FIG 195.—STOP VALVES

over the seam by blows from a caulking tool. The seams of wood vessels and boats are made watertight by placing a layer of white lead between the planks.

EXAMPLES 13.

(1) A hydraulic accumulator has a ram of 9 in. diameter and a lift of 15 ft. Find the load on the ram and the capacity if supplied with water at 60 lb. per square inch pressure.

Ans.—3,810 lb.; 57,200 ft. lb.

(2) A hydraulic intensifier has ram diameters of 3 in. and 7 in. Find the pressure at which the water is raised when the pressure of the supply is 75 lb. per square inch.

Ans.—408 lbs. per sq. in.

(3) A hydraulic lift has a ram diameter of 6 in. and is supplied with water at a pressure of 400 lb. per square inch. Find the total load the lift will carry if the efficiency is 85 per cent.

If the lift has a velocity of 2 ft. per second, find the horse-power required when lifting.

Ans.—9,600 lb.; 41·2 h.p.

(4) 40 h.p. is to be transmitted from an accumulator through a 4 in. pipe, 5,000 ft. long. If the loss is to be 2 per cent, find the diameter of the ram which is loaded with 120 tons. (Assume coefficient of friction in pipe to be ·01.) (London Univ.)

Ans.—19·9 in.

(5) An accumulator maintains a pressure of 1,200 lb. per square inch in a 3 in. hydraulic main. A hydraulic lift is supplied with pressure water from this main, and the point at which the supply to the lift is drawn off is at a distance of 2,000 ft. from the accumulator. The ram at the lift is 8 in. in diameter, and the load on it, inclusive of its own weight, is 12 tons. The friction of the ram, cage, etc., may be taken as equivalent to an addition of $6\frac{1}{2}$ per cent of the gross load on the ram. Determine the speed at which the lift will ascend, if the value of the coefficient of resistance, f, for the hydraulic main is ·008. Neglect the loss due to shock at entrance to cylinder. (London Univ.)

Ans.—2·69 ft. per second.

(6) Give a careful sketch showing the construction of a hydraulic ram, and explain its action fully by aid of reference letters. In what circumstances would you make use of such a machine and why? (A.M.I. Civil E., 1922.)

(7) Describe with sketches the hydraulic ram, and explain its action. (London Univ.)

(8) An accumulator has a 12 in. ram and 20 ft. lift, is loaded with 100 tons total weight. If packing friction accounts for 2 per cent of the total force on the ram, determine the horse-power being delivered to the mains if the ram falls steadily through its full range in 2 minutes, and if at the same time the pumps are delivering 240 gallons per minute. (A.M.I. Mech. E.)

Ans.—395 h.p.

(9) A hydraulic lift raises a load of 8 tons through a height of 40 ft. once every 2 minutes, the speed of lifting being 2 ft. per second. It is worked from an accumulator which is being continuously charged by a pump. The pressure of the water is 500 lb. per square inch, the efficiency of the lift 75 per cent and the efficiency of the pump 85 per cent. Find the power required to drive the pump, and the minimum capacity of the accumulator. Frictional losses in the pipes may be neglected. (London Univ.)

Ans.—H.P. = 17 ; volume = 11·08 cu. ft.

CHAPTER XIV

THE AEROFOIL AND ITS APPLICATIONS

157. **Modern Theories of Fluid Flow.** The theories on fluid flow given in the earlier chapters were evolved by practical engineers for use in their design problems; in the light of modern science, they are regarded as approximations only, and as having a limited range of application. Although they were deduced from rational laws, it is necessary to introduce a coefficient, found experimentally, in order to make them agree with practical experience. This experimental coefficient is a constant over a very limited range; actually it varies with such factors as the velocity, linear dimensions, temperature and viscosity. Within recent years new theories on fluid flow have been evolved, chiefly due to the development of aeronautics, which take into account these factors. An experimental coefficient is still introduced, but it holds over a much larger range on account of the greater scientific accuracy of the new theories.

In the following chapters an account of some of the more recent theories on fluid flow will be given, and applied to a few practical problems. It should be noted throughout that the effect of fluid flow depends on the relative motion of the body considered and of the fluid. The body considered may be moving in a stationary fluid, or the body may be at rest and surrounded by a moving fluid, or they may both be moving at different velocities. The problem is only affected by their relative velocity.

158. **Fluid Flow Past an Inclined Plate.** In Fig. 196 is shown a flat plate immersed in a fluid stream of velocity V relative to the plate, which is inclined at an angle α to the direction of flow; the fluid may be a liquid or a gas. Some of the fluid streams strike the underside of the plate and are deviated in a direction parallel to the surface by the pressure of the fluid stream beneath them, thus causing a tangential frictional drag on the plate. The impact of the fluid causes a normal pressure P on the plate which is proportional to its surface area and to the kinetic energy of the stream.

Hence,
$$P \propto \frac{\rho V^2}{2} \times \text{area of surface} \quad . \quad (1)$$

Now consider the upper surface of the plate. Local fluid streams will be deflected away from the upper surface of the plate by the action of its front, or leading edge. These are then forced downwards by the pressure of the fluid streams above, thus tending to form a vortex, as shown in the figure. This causes a negative, or vacuum, pressure on the upper surface. The resultant pressure P will, therefore, be partly due to the

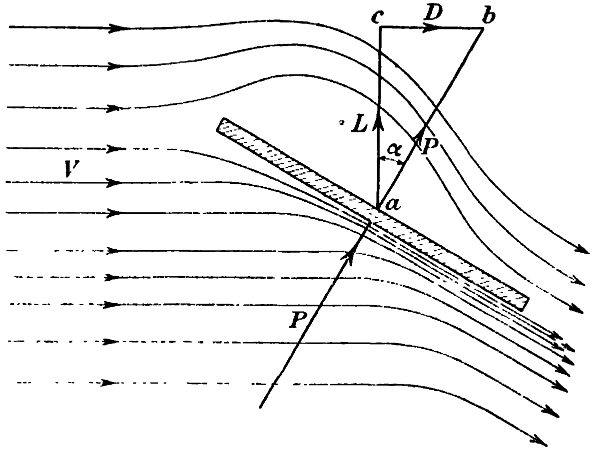


FIG. 196.—FORCES ON FLAT PLATE IN A FLUID STREAM

positive pressure on the lower surface of the plate, and partly due to the effect of the negative pressure on the upper surface.

It will be noticed that the force P will have a component perpendicular to the stream of $P\cos\alpha$; this is sometimes known as the *lift*. There will also be a component in the direction of the stream of $P\sin\alpha$, causing a resistance to motion. The total resistance of the plate in the direction of the stream is known as the *drag*.

Equation (1) may be written

$$L = k_L A \rho V^2 \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where L = force on plate normal to direction of stream

A = area of under surface of plate

ρ = density of fluid (absolute units)

$= \dfrac{w}{g}$

k_L = a coefficient to be determined experimentally for a flat plate inclined at an angle α.

The coefficient k_L is known as the *lift coefficient*; it varies with the type of plate and with the inclination α.

The resistance, or drag, of the plate in a direction parallel to the stream may be expressed in a similar type of equation—

$$D = k_D A \rho V^2 \qquad . \quad . \quad . \quad (3)$$

where D = the drag or resistance of plate in the direction of motion.

The coefficient k_D is known as the *drag coefficient*; it varies with the type of plate and the angle of inclination and is determined experimentally.

The point on the surface through which the resultant force P acts is known as the *centre of pressure*.

Equations (2) and (3) are also used in the form

$$L = C_L \frac{A\rho V^2}{2}$$

and

$$D = C_D \frac{A\rho V^2}{2}$$

Hence, the values of C_L and C_D are twice those of k_L and k_D. If the drag is due to friction only C_D is sometimes written C_f.

This problem of the fluid pressure on an inclined surface or plate frequently occurs in practice. The flat plate is the same in principle as an aeroplane wing, in which case the force L is the lift of the wing and D is its drag or resistance; a large part of the horse-power of the aeroplane is absorbed in overcoming this drag of the wings.

The propelling force on the main sail of a yacht, when tacking, is another example of this problem, the sail corresponding to the flat plate. Other examples of this problem are found in the flying of a kite, the turning force on the rudders of ships and aeroplanes, the dynamic lift on an airship or hydroplane, and the driving force of propeller and fan blades.

EXAMPLE.

A flat plate 4 sq. ft. in area is immersed in a fluid stream and inclined to the direction of motion. Find the force on the plate in a direction normal to that of the stream and also the resistance of the plate in a direction parallel to the stream—

(1) If the fluid is air;
(2) If the fluid is water.

The velocity of the fluid stream is 20 ft. per sec., k_L = ·2, k_D = ·05, weight 1 cu. ft. of air = ·081 lb., weight of 1 cu. ft. water = 62·4 lb.

(1) For air,

Using Equation (2), $L = k_L A \rho V^2$

$$= \cdot 2 \times 4 \times \frac{\cdot 081}{32 \cdot 2} \times 20^2$$

$$= \cdot 805 \text{ lb.}$$

Using Equation (3), $D = k_D A \rho V^2$

$$= \cdot 05 \times 4 \times \frac{\cdot 081}{32 \cdot 2} \times 20^2$$

$$= \cdot 201 \text{ lb.}$$

(2) For water,

$$L = k_L A \rho V^2$$

$$= \cdot 2 \times 4 \times \frac{62 \cdot 4}{32 \cdot 2} \times 20^2$$

$$= 620 \text{ lb.}$$

$$D = k_D A \rho V^2$$

$$= \cdot 05 \times 4 \times \frac{62 \cdot 4}{32 \cdot 2} \times 20^2$$

$$= 155 \text{ lb.}$$

159. **The Aerofoil.** It was shown in Art. 158 that a fluid stream acting on an inclined flat plate causes a force L on the

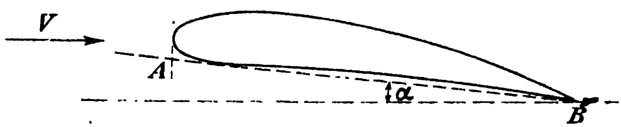


FIG. 197.—CROSS-SECTION OF AN AEROFOIL

plate in a direction normal to the fluid stream, and a drag D on the plate in a direction parallel to the stream. The force L can be increased by substituting for the flat plate a plane having a cross-section of the type shown in Fig. 197; such a section is known as an aerofoil.

Aerofoil sections vary in shape according to the work required of them.* Although the function of an aerofoil is the

* For more advanced work on the aerofoil see *Aerofoil and Airscrew Theory*, by H. Glauert, M.A. (The University Press, Cambridge). Also, see Bairstow's *Applied Aerodynamics* and Piercy's *Applied Aerodynamics*. For elementary work on this subject see *Elementary Applied Aerodynamics*, by Whitlock.

same as that of a flat plate, it is more efficient in its action, because it produces a larger force L and a smaller drag D when acting under similar conditions. Most of the force L is due to the negative pressure on the upper face.

The edge A of the aerofoil (Fig. 197) is known as the *leading edge*; the edge B as the *trailing edge*. The dotted line AB is known as the *chord*; this is the projected length of the aerofoil.

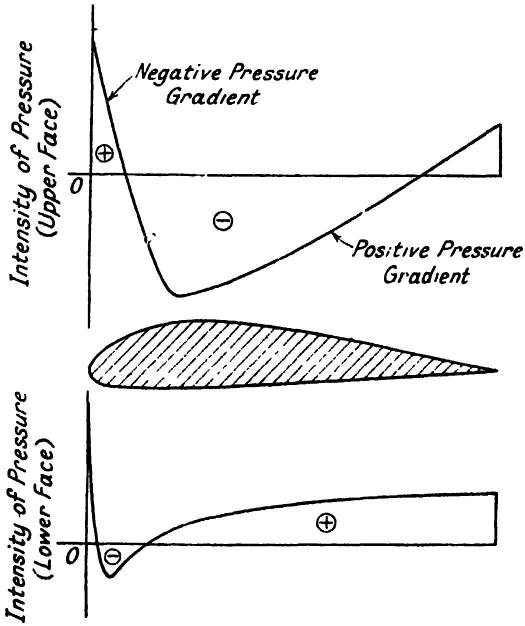


FIG. 198

The inclination α of the chord AB to the direction of motion of the aerofoil, or of the fluid stream, is known as the *angle of incidence* or *angle of attack*; by varying this angle the values of L and D are altered.

The distribution of the intensity of pressure around the surface of a particular aerofoil is shown in Fig. 198; these pressures were obtained* by measurements during a test on an aeroplane wing of this section whilst in flight. It will be noticed that there is a large negative pressure on the upper surface, which accounts for most of the lift.

* These results are due to Stüper.

Let c = length of chord AB in feet

l = longitudinal length of aerofoil in feet (measured in horizontal plane perpendicular to AB)

V = relative velocity of fluid stream to aerofoil in ft. per sec.

A = projected area of aerofoil in sq. ft.

$= c \times l$

The position of the centre of pressure of the aerofoil is also of importance in design. This is the point of application of the resultant force P (Fig. 196) and it varies with the angle of incidence.

The equations of Art. 158, which were derived for a flat plate, can be applied in the same way to an aerofoil. Then, force on aerofoil normal to direction of fluid stream

$$= L = k_L A \rho V^2$$

and resistance, or drag, of aerofoil

$$= D = k_D A \rho V^2$$

Horse-power expended in overcoming resistance

$$= \frac{D \times V}{550} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The total resistance and lift of the wings of an aeroplane can be calculated from these equations; in which case l is the length of both wings. If D_1 is the drag of the fuselage and the remaining parts of the structure other than the wings, then, total horse-power required to overcome drag

$$= \frac{(D + D_1)\, V}{550}$$

160. **Characteristics of the Aerofoil.** The chief characteristics, or properties, of a particular aerofoil are the values of its constants k_L, k_D, and the position of its centre of pressure C_p. These vary with the angle of incidence, and are determined from wind tunnel tests carried out on an aerofoil of the shape considered, the tests being repeated for varying values of α.

The values of these characteristics for a well-known aerofoil section (R.A.F. 31) are shown plotted in Fig. 199 on a base representing the angle of incidence α. The ratio of lift to drag, or L/D ratio, is also shown plotted in this figure. The position of

the centre of pressure is given as a fraction of the chord from the leading edge.

It will be noticed from these curves that the drag coefficient of this aerofoil is a minimum when α has a value of $-6°$. The

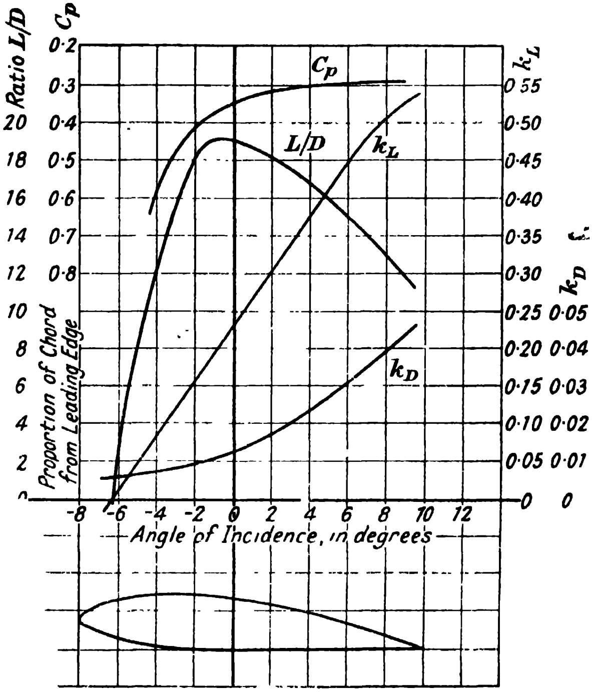


FIG. 199

maximum value of the lift-drag ratio occurs when $\alpha = -1°$; this is the most efficient angle for the aerofoil. It will be noticed from the centre of pressure curve that the position of C_p moves towards the leading edge as the angle of incidence increases.

Similar sets of curves are determined for the hundreds of different aerofoil sections used in practice; these are published in the various aeronautical handbooks,* and the official aeronautical publications of most countries.

Although the curves of Fig. 199 were produced by test in an

* See *Handbook of Aeronautics* (Pitman), *R. & M. Reports* (Air Ministry), *N.A.C.A. Reports* (U.S.A.).

airstream for the purpose of providing data for the design of aeroplane wings, similar results would be obtained for an aerofoil immersed in a stream of other fluids, including water.

The aerofoil is also used to form a cross-section of propeller blades; the thrust of the blade corresponding to the lift of the aerofoil.

EXAMPLE.

An aeroplane wing consists of an aerofoil section of the type given in Fig. 199. It has a length of 20 ft., a chord of 4 ft. and is driven at a speed of 150 m.p.h. Calculate the lift, drag, and horse-power required for this wing, when the angle of incidence is 4°. Find also the position of the centre of pressure at this angle. Weight of 1 cu. ft. of air = ·08 lb.

Using the curves of Fig. 199 and reading the values when $\alpha = 4°$,

$$k_L = \cdot 375$$

$$k_D = \cdot 023$$

$$C_p = \cdot 3 \text{ of chord.}$$

Now,

$$A = c \times l$$

$$= 4 \times 20 = 80 \text{ sq. ft.}$$

$$V = 150 \times \frac{88}{60} = 220 \text{ ft. per sec.}$$

Using the equations of Art. 159,

$$L = k_L A \rho V^2$$

$$= \cdot 375 \times 80 \times \frac{\cdot 08}{32 \cdot 2} \times 220^2$$

$$= 3600 \text{ lb.}$$

$$D = k_D A \rho V^2$$

$$= \cdot 023 \times 80 \times \frac{\cdot 08}{32 \cdot 2} \times 220^2$$

$$= 221 \text{ lb.}$$

$$\text{H.P.} = \frac{221 \times 220}{550} = 88 \cdot 5$$

$$\text{Position of } C_p = \cdot 3 \times 4$$

$$= 1 \cdot 2 \text{ ft. from leading edge.}$$

161. **Aerofoil Blading for Turbines.** In Chapter X the problem of the force on turbine blading was solved from the

consideration of the change of momentum of the water stream when passing over the blades. Another method of solution is obtained by considering the turbine blades acting as aerofoils; the force on the blades and the blade resistance can then be calculated from the equations of Art. 159. In order to apply this method, the characteristic curves for the blades would first have to be obtained from tests.

When based on this theory turbine blades are made of a suitable aerofoil section, similar to those used for aeroplane wings, instead of the concave circular sections at present in use. Such blading is known as aerofoil blading. It is possible that aerofoil blading may prove to be more efficient than circular blading; this can only be proved by test.

The aerofoil will only give its maximum lift if it is clear of any near objects which may interfere with the passing of the fluid stream in its vicinity. Hence, the aerofoil blading of a turbine must be so spaced that there will be no interference of the fluid stream between any two adjacent blades. On the other hand, if the blades are spaced too far apart, some of the fluid stream will flow freely between them without doing any work. Energy will thus be wasted. The exact spacing of the blades to satisfy both of these conditions can only be obtained from tests.

Another factor to consider in the design of aerofoil blading is the effect of the vacuum pressure on the upper face of the aerofoil, shown in Fig. 198. If this negative pressure becomes too large, cavitation will occur, thus interfering with the flow of the fluid stream. This will reduce the force on the blade and, consequently, reduce its efficiency.

162. **Minimum Spacing of Aerofoil Blading.** In order to investigate the minimum spacing of aerofoil blading in turbines, the author tested two model aerofoil sections in a fluid stream by means of the Hele-Shaw apparatus. This apparatus consists of a film of water flowing between two glass plates. At the inlet end of the plates thin streams of coloured liquid are injected into the water stream across the whole width of the film. If there is no obstruction the coloured streams of liquid will flow in straight parallel bands. By placing an object in the fluid stream between the glass plates, the deviation of the streamlines may be observed from the contours of the colour bands.

The two model aerofoil sections were placed between the

glass plates in a parallel position, both having the same angle of incidence. Then, by observing the deviation of the coloured stream bands, it was possible to see if interference was taking place. The aerofoils were then moved further apart until the position was reached when no appreciable interference was obtained.

A photograph of the stream bands is shown in Fig. 200. In the position shown the spacing pitch of the aerofoils was equal

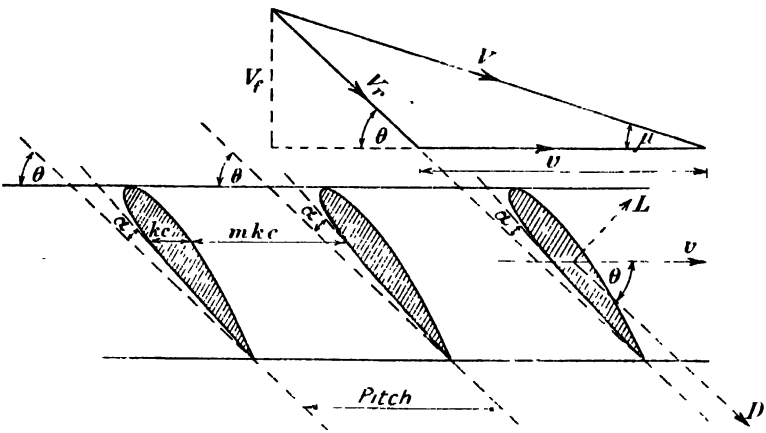


FIG. 204

to 4·3 times the thickness of the aerofoil. It will be noticed from the photograph that a slight deviation of the stream bands still occurs at this pitch. This may cause a slight interference, but it is doubtful whether it would be appreciable enough to affect the lift and drag coefficients.

From these tests it was evident that the pitch of the blades must not be less than 4·3 times their maximum thickness in order to obtain the maximum lift coefficient.

163. **Work Done on Aerofoil Blading.** Consider an axial flow water turbine with aerofoil blades equally spaced around the circumference of the runner of radius r, as shown in Fig. 204. The water is guided on to the blades by a ring of guide blades inclined at an angle μ to the direction of motion. The vector diagram is drawn in the same manner as for ordinary blading (Art. 109), the notation used being also similar. The water will thus pass through the runner in a direction parallel to the

relative velocity; hence V_r represents the magnitude and direction of the fluid stream impinging on the aerofoil. Let the aerofoil be set at angle of incidence α to the direction of V_r.

The fluid stream will cause a lift L and a drag D on the blade, as shown in the figure. The force L causes the runner to rotate, whilst the drag D is equivalent to the blade resistance and causes the hydraulic loss of energy.

Let c = length of chord of aerofoil blade

k = ratio of $\dfrac{\text{maximum blade thickness}}{\text{chord}}$

Then, kc = maximum thickness of blade

Let $m = \dfrac{\text{space between two consecutive blades}}{\text{maximum thickness of blade}}$

Then, space between consecutive blades $= mkc$

pitch of blades $= kc(m + 1)$

number of blades $= n = \dfrac{2\pi r}{\text{pitch}}$

$$= \frac{2\pi r}{kc\,(m+1)}$$

Let b = breadth of blade

A = area of blade

$= b \times c$

θ = direction of relative velocity

Using the equations of Art. 159,

$$L = k_L \rho A V_r^2 \text{ per blade}$$

and

$$D = k_D \rho A V_r^2 \text{ per blade.}$$

The values of k_L and k_D can be obtained from the characteristic curves of the particular aerofoil section chosen for the blades. The value of α should be that which produces the maximum value of k_L. It is assumed that the lift and drag coefficients are the same for water as for air.

For an impulse turbine,

$$V = \sqrt{2gH}$$

It is assumed that H, v, and μ are known; then the velocity diagram of Fig. 204 can be solved and the values of V_r and θ obtained. Resolving parallel to v,

tangential force on runner due to n blades

$$= F = (L \sin\theta + D \cos\theta)n$$
$$= (k_L \rho c b V_r^2 \sin\theta + k_D \rho c b V_r^2 \cos\theta)n$$
$$= \rho c b n V_r^2 (k_L \sin\theta + k_D \cos\theta)$$

Substituting for n and ρ,

$$F = \frac{w}{g} b \frac{2\pi r}{k(m+1)} V_r^2 (k_L \sin\theta + k_D \cos\theta) \quad . \qquad (1)$$

Work done on runner per sec.

$$= F \times v$$
$$= \frac{wb 2\pi r V_r^2 v}{gk(m+1)} (k_L \sin\theta + k_D \cos\theta) \quad . \quad . \qquad (2)$$

Weight of water per sec.

$$= W = w \times \text{area of flow} \times V_f$$
$$= w\, 2\pi r b \left(\frac{m}{m+1}\right) V_f \quad . \quad . \quad . \quad . \qquad (3)$$

Horse-power developed $= \dfrac{\text{work done on runner per sec.}}{550}$

Hydraulic efficiency $= \dfrac{\text{work done on runner per sec.}}{WH}$

Substituting Equations (2) and (3), hydraulic efficiency

$$= \frac{\dfrac{wb\, 2\pi r\, V_r^2\, v}{gk(m+1)} (k_L \sin\theta + k_D \cos\theta)}{w\, 2\pi r b \left(\dfrac{m}{m+1}\right) V_f H}$$
$$= \frac{V_r^2 v (k_L \sin\theta + k_D \cos\theta)}{g k m V_f H}$$

But, from Fig. 204,

$$V_f = V_r \sin\theta$$

Hence, hydraulic efficiency

$$= \frac{V_r v (k_L + k_D \cot\theta)}{g k m H} \quad . \quad . \quad . \qquad (4)$$

From these equations the horse-power, efficiency and quantity of water required can be calculated.

EXAMPLE.

Calculate the horse-power and hydraulic efficiency of an axial flow impulse turbine having aerofoil blading of R.A.F. 31 section (Fig. 199). The mean diameter of blade ring is 4·5 ft.; breadth of blades is 4·5 in.; head of water is 280 ft. The blades are set with an angle of incidence of 6° and have a pitch of 5 times their maximum thickness. The maximum blade thickness is ·135 of chord. Guide blade angle = 40°; speed of runner is 300 r.p.m.

From given data

$$k = \cdot 135$$

$$m = 4$$

From the curves of Fig. 199,

$$k_L = \cdot 45$$

$$k_D = \cdot 03$$

Also $$V = \sqrt{2gH}$$

$$= \sqrt{64 \cdot 4 \times 280} = 134 \text{ ft. per sec.}$$

and $$v = \frac{2\pi r \text{ (r.p.m.)}}{60}$$

$$= \frac{2\pi \times 2 \cdot 25 \times 300}{60}$$

$$= 70 \cdot 7 \text{ ft. per sec.}$$

The velocity diagram may now be drawn as is shown in Fig. 204.

From velocity diagram

$$V_f = 86 \text{ ft. per sec.}$$

$$V_r = 91 \cdot 8 \text{ ft. per sec.}$$

$$\theta = 69 \cdot 5^\circ$$

Applying Equation (2), Art. 163,

Work done on runner per sec.

$$= \frac{wb\, 2\pi r\, V_r^2 v}{gk(m+1)}(k_L \sin\theta + k_D \cos\theta)$$

$$= \frac{62 \cdot 4 \times 4 \cdot 5 \times 2\pi \times 2 \cdot 25 \times 91 \cdot 8^2 \times 70 \cdot 7}{12 \times 32 \cdot 2 \times \cdot 135 \times 5}(\cdot 45 \times \cdot 937 + \cdot 03 \times \cdot 35)$$

= 3,930,000 ft. lb.

$$\textbf{Horse-power} = \frac{3{,}930{,}000}{550} = 7140$$

Using Equation (4), Art. 163,

$$\text{Hydraulic efficiency} = \frac{V_r v\,(k_L + k_D \cot\theta)}{g\,k\,m\,H}$$

$$= \frac{91{\cdot}8 \times 70{\cdot}7\,({\cdot}45 + {\cdot}03 \cot.\ 69{\cdot}5)}{32{\cdot}2 \times {\cdot}135 \times 4 \times 280}$$

$$= 61{\cdot}5 \text{ per cent.}$$

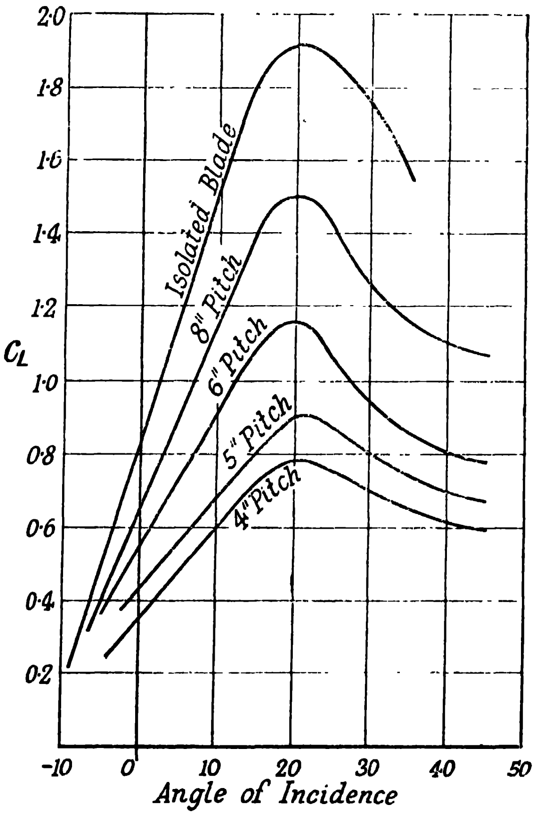


FIG. 205

164. **Effect of Blade Pitch on Lift and Drag.** The lift coefficient curve of Fig. 199 was obtained from a test on an isolated aerofoil and would not hold for a series of similar aerofoils placed close together in parallel, as on the rim of a turbine runner. This is obvious from the position of the

streambands shown in the photograph of Fig. 200. The effect of the interference of adjacent aerofoils on the lift and drag has been investigated experimentally by Youssef.* Five model Parsons reaction steam turbine blades were placed in parallel at a known pitch and were tested in a wind tunnel; the lift and drag of one blade was measured experimentally at various angles of incidence. This was repeated for other pitches. The

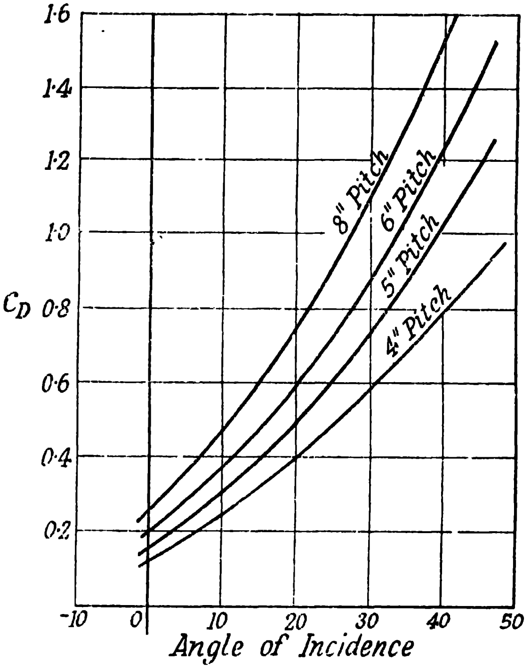


FIG. 205A

lift and drag coefficient curves were plotted for each pitch on a base representing the angle of incidence. These curves are shown in Figs. 205 and 205A. The lift coefficient curve for a single isolated blade was also obtained experimentally and is shown plotted in Fig. 205.

It will be noticed from the curves of Fig. 205 that the lift coefficient decreases considerably as the pitch of the blades is reduced, the maximum value of C_L at 4 in. pitch being less

* See "Wind Tunnel Experiments on Model Reaction Turbine Blades," by Dr. M. R. Youssef, *Engineering*, Vol. 153 (1942), p. 138.

than one-half of the corresponding value for an isolated blade. From the drag coefficient curves of Fig. 205A it will be seen that there is a large decrease in drag as the pitch is reduced.

The model aerofoil used in these experiments had a maximum thickness of 1·4 in. and a chord of 9·2 in., and the pitch of the aerofoils was varied between 4 in. and 8 in. during the tests. If, in order to obtain a large lift, the 8 in. pitch were used, there would be a relatively large space between the aerofoils, in which case a considerable quantity of fluid would pass between the aerofoils without giving up much of its total energy.

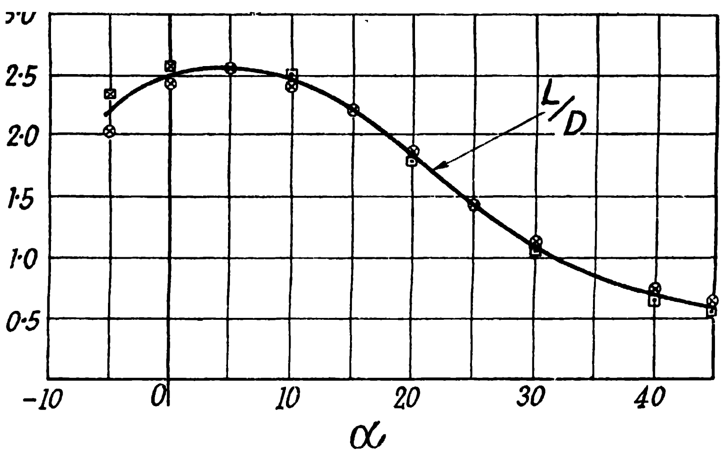


FIG. 205B

Hence, in practice, it is necessary to have several rings of blades arranged in series and keyed on a common shaft in order to absorb the energy of the fluid.

By this method the fluid passes through the first blade ring causing it to rotate, it is then exhausted on to a fixed ring of guide blades which re-direct the fluid at the correct angle on to a second ring of moving blades. This is repeated on each blade ring of the series until almost the whole of the energy of the fluid is absorbed. It will be noticed that in passing over the aerofoil blades the fluid loses velocity and pressure in overcoming the drag. Then,

$$\left.\begin{array}{c}\text{efficiency of one}\\ \text{blade ring}\end{array}\right\} \frac{\text{work done on blades}}{\text{loss of total energy of fluid}}$$

The blades should be set at the angle of incidence which produces the highest lift and the smallest drag; that is, when the lift/drag ratio is a maximum.

The lift/drag curve for these experimental results is shown plotted in Fig. 205B for angles of incidence up to 45°. It was

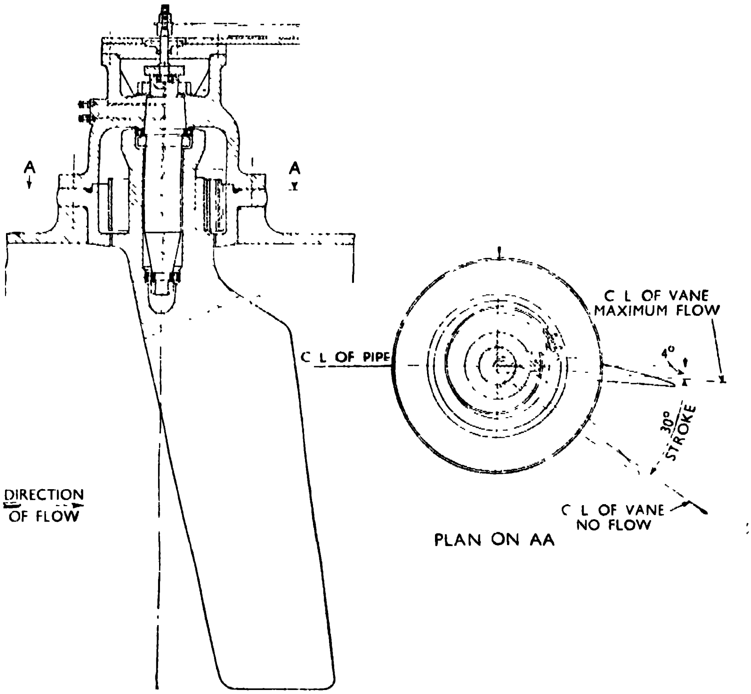


Fig. 206
(*The Institution of Mechanical Engineers*)

found that the lift/drag curve for all of the pitches used coincided. Hence, the lift/drag ratio depended only on the angle of incidence and was independent of the pitch. It will be seen from the curve that the ratio is a maximum when α is 5°; the blades should be set at this angle of incidence for maximum efficiency.

165. **The Aerofoil Flow Recorder.** The aerofoil flow recorder is a very sensitive instrument for measuring the quantity of water flowing through a pipe or channel. It is based on the fact that a freely suspended aerofoil blade automatically

adjusts its angle of incidence to correspond to the velocity of flow of the fluid in which it is immersed. The angle of rotation of the aerofoil is thus proportional to the quantity of flow in a given channel. If the blade is geared to an automatic recorder, such as a drum rotated by clockwork, the angle of incidence may be plotted on a time base. This can be calibrated to give a continuous record of the flow, plotted on a time base.

A sectional view of an aerofoil recorder is shown in Fig. 206. The aerofoil blade rotates about the vertical axis, the angle of rotation being proportional to the velocity of the fluid. The rotation is spring controlled, the action of the spring tending to bring the blade back to its zero position. The aerofoil blade is mounted on ball bearings immersed in grease.

The angular movement of the blade is shown in the plan view (Fig. 206); it will be seen that the maximum angle of rotation is 30 degrees.

The turning moment on the blade is caused by the lift of the fluid acting at the centre of pressure of the aerofoil; both of these quantities will vary with the fluid velocity (Art. 159). The aerofoil blade causes a small amount of resistance due to its drag.

EXAMPLES 14.

The following values of ρ are to be used in these examples—

$$\text{for air} \qquad \rho = \frac{\cdot 075}{g} \text{ ft. lb. units;}$$

$$\text{for water} \qquad \rho = \frac{62\cdot 4}{g} \text{ ft. lb. units.}$$

(1) Using the curves of Fig. 199, calculate the lift and resistance of this aerofoil when inclined at an angle of incidence of 2°. The chord is 6 ft., the length 32 ft., and the air speed 150 ft. per sec.

Ans.—1·33 tons; 160·0 lb.

(2) Using the curves of Fig. 199, calculate the horse-power required by a monoplane, having wings of this section, when travelling at 250 m.p.h The area of each wing is 50 sq. ft. and the angle of incidence 6°. Assume the propeller efficiency to be 70 per cent, and the air resistance of all parts other than the wings to be 30 per cent of the total wing resistance. What is the lift at this speed?

Ans.—1155 h.p.; 6·21 tons.

(3) An axial flow impulse turbine has a mean blade ring diameter of 4 ft. and a speed of 150 r.p.m. The runner blades are of aerofoil section of the shape given in Fig. 199 and are set with an angle of incidence of 8° to the axis of the runner. Assuming the relative velocity of the water impinging on the blade is 80 ft. per sec. and that its direction is axial, find the horse-power developed. There are 30 blades each having a chord of 3·5 in. and a length of 4 in.

Ans.—1050 h.p.

(4) Calculate for the turbine of Question (3), the guide blade angle and the head of water supplied. If the runner blades have a maximum thickness of $\frac{1}{2}$ in., calculate the weight of water supplied per sec. and the hydraulic efficiency of the turbine.

Ans.—68·5°; 114·7 ft.; 9400 lb. per sec.; 53·5 per cent.

(5) An aeroplane is travelling at 180 m.p.h. and its rudder consists of a flat surface of area 2·5 sq. ft. Find the force on the rudder normal to direction of flight when turned through an angle of 30°. Assume k_L for a flat surface at an angle of incidence of 30° to be 0·35.

Ans.—142·2 lb.

CHAPTER XV

THE BOUNDARY LAYER

166. **The Boundary Layer Theory.** When a fluid is flowing past a body, or a surface, it can be noticed that there exists a layer of fluid adjacent to the surface, through which the variation of velocity between the fluid and the surface is transmitted. This layer is known as the boundary layer, and the whole of the viscous or frictional resistance between the fluid and the surface occurs in this layer.

The layer may be imagined to consist of a number of thin parallel stream bands each having a slightly larger velocity than its inner neighbour. The band immediately adjacent to the surface of the body is found to adhere to the surface and has no velocity. Working outwards from the surface, the next band has an extremely small velocity; each successive band beyond will have a slightly higher velocity than its inner neighbour, until, finally, a band is reached which has approximately the full velocity of the fluid. This last band is the outside limit of the boundary layer; no further fluid resistance is transmitted to the surface beyond this outer limit.

The same reasoning holds if the body is moving in a stationary fluid. The boundary layer occurs between any surface and fluid which are in contact, and between which there is a relative velocity.

The existence of the boundary layer was first observed by Hele-Shaw, but the use of the conception of a laminated boundary layer transmitting the fluid resistance to the surface is due to Prandtl.

The flow within the boundary layer may be streamline or turbulent, according to the particular problem or to its distance from the leading edge of the surface. Sometimes, and under certain conditions, the boundary layer will leave the surface and curl up into a vortex or whirlpool; this phenomenon is known as *break-away* or *separation*.

The thickness of the boundary layer increases with its distance from the leading edge in proportion to the square root of the distance; it will depend also on the value of the Reynolds number of the body (Art. 139).

The flow of a liquid past a circular-sectioned body is shown

in the photographs of Figs. 201, 202, and 203. These are due to Prandtl and were obtained by sprinkling small particles of aluminium on the surface of the liquid. The metal particles reflect the light and thus enable the streamlines to be photographed. In Fig. 201 the boundary layer is adhering to the surface throughout. In Fig. 202 the velocity of the stream has been increased. In this photograph the boundary layer can be seen to have left the surface towards the wake, and vortices are commencing to form; this photograph clearly shows a break-away of the boundary layer. In Fig. 203 the speed of the stream has been further increased; an earlier break-away of the boundary layer is noticeable, causing the formation of more pronounced vortices in the wake.

The trail of vortices occurring in the wake of a body, after boundary layer separation has taken place, is known as the *Kàrmàn street*; these are shown forming in the photograph of Fig. 203.

The formation of a boundary layer occurs at the surface of all bodies immersed in a relatively moving fluid. It also occurs on the inner surfaces of short pipes* through which a fluid is being transmitted. It is the deciding factor of the magnitude of fluid resistance in such problems as pipe flow, flow in channels, resistance of ships in water, and resistance of aeroplanes and airships.

167. **Variation of Velocity within Boundary Layer.** Let Fig. 207 represent a surface, shown shaded, past which a fluid is flowing with a velocity V, and let the band adjacent to the surface represent the boundary layer. Consider a vertical section ab through the layer situated at a distance x from the leading edge of the body.

Let δ = thickness of boundary layer at section considered

u = velocity of the fluid within the boundary layer at any distance y from the surface

R_e = Reynolds number for the body immersed in the fluid

$$= \frac{Vl}{\nu}$$

where l represents the linear dimension of the body or surface.

If the Reynolds number is low, say, less than 500,000, the

* In long pipes the boundary layers will intersect at the centre of the pipe and thus interfere with the flow.

flow within the layer is wholly streamline. If the Reynolds number is high the flow is mainly turbulent. Usually the flow is laminar for a short distance from the nose. This is followed by a short length of layer in which the flow changes

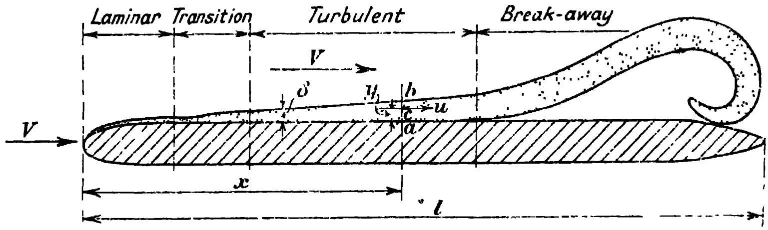


FIG. 207

from laminar to turbulent; in this portion of the layer the flow is unstable. The length of this transition portion of the layer is found to be about the same length as the laminar flow portion. If the Reynolds number is greater than 1,000,000, fully developed turbulence is obtained in the remaining length of the layer. For a short body the flow within the boundary layer may be laminar for its whole length. In a long body the flow may pass through all three stages: laminar, transition, and turbulent, after which break-away may occur as shown in the figure.

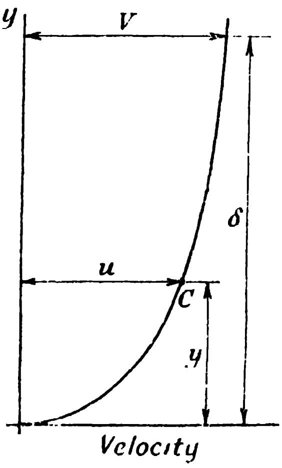


FIG. 208.—VELOCITY VARIATION ACROSS BOUNDARY LAYER

A curve showing the variation of velocity within the layer is shown in Fig. 208. If the velocities on any section *ab* (Fig. 207) are measured with a pitot tube at perpendicular distances y measured along *ab*, they are found to vary as shown by the curve. In this curve the vertical ordinate represents the distance y, and the horizontal ordinate the velocity. The particular value of y at which the velocity approximately reaches the value of the stream velocity V gives the thickness of the boundary layer at this section. Or, the thickness of the layer is the value of y at which the graph becomes vertical.

It is found that the thickness δ varies with $\sqrt{x}$ and increases towards the trailing edge.

The boundary layer theory has been applied to a flat plate surrounded by a fluid flowing longitudinally. For the portion of the boundary layer in which the flow is laminar, the variation of velocity at any section through the layer follows the Prandtl-Blasius law and is closely represented by the equation—

$$\frac{u}{V} = 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4 \quad . \quad . \quad (1)$$

For the portion of the boundary layer in which the flow is turbulent, the velocity distribution on any section is given approximately by the equation—

$$\frac{u}{V} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This may be written

$$\frac{u}{V} = \left(\frac{y}{\delta}\right)^n . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where n is a constant to be determined experimentally. The value of n is found from tests to vary between $\frac{1}{5}$ and $\frac{1}{7}$, the actual value depending on the Reynolds number.

It should be noted that in the testing of small models the Reynolds number is low and the boundary layer flow is usually laminar throughout. The boundary layer surrounding full scale sea-going ships, submarines, aeroplane wings, and airships is mainly turbulent.

168. **Thickness of Boundary Layer.** As stated in Art. 167, the thickness of the boundary layer increases from the leading edge to the trailing edge in proportion to $\sqrt{x}$. Pohlhausen deduced the following approximate value for a laminar flow past a flat plate—

$$\delta = \frac{5{\cdot}83\sqrt{lx}}{\sqrt{R_e}} \quad . \quad . \quad . \quad . \quad (1)$$

Another value for the thickness of the layer was found to be—

$$\delta = 4{\cdot}5\sqrt{\frac{\nu x}{V}} \quad . \quad . \quad . \quad . \quad (2)$$

It will be noticed by substituting for R_e in Equation (1),

that these two equations are of the same type except for the difference of the constants.

The thickness of the boundary layer for turbulent flow past a flat plate has been found experimentally to vary between

$$\delta = \cdot 303\left(\frac{1}{R_e}\right)\sqrt{lx} \tag{3}$$

and

$$\delta = \cdot 18\left(\frac{1}{R_e}\right)\sqrt{lx} \tag{4}$$

The boundary layer theory has been applied to a large American rigid airship* and to a model of the airship of about

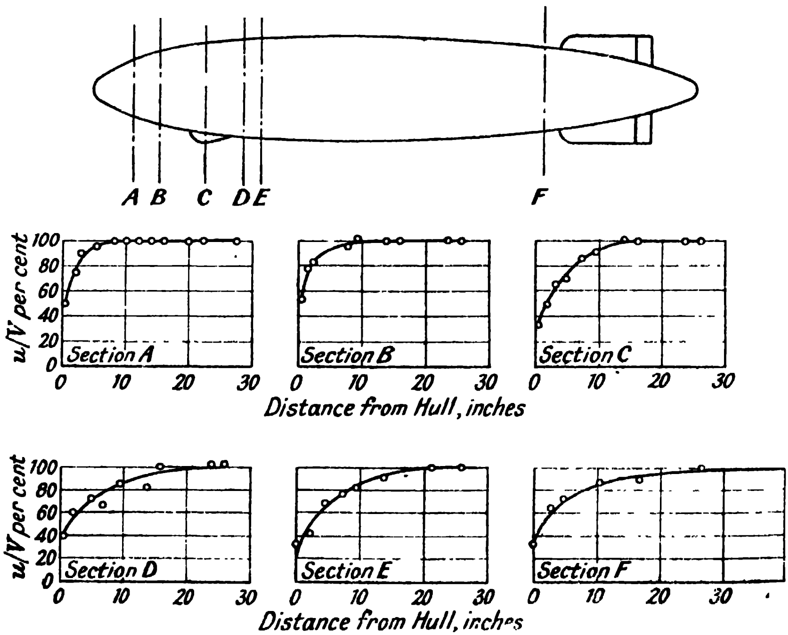


Fig. 209.- -Experimental Velocity Curves for a Rigid Airship

$\frac{1}{300}$ of the size. The velocities within the boundary layer of the airship were measured at several sections by means of pitot tubes, whilst the airship was in flight. By plotting these velocities on each section line, similar velocity curves to Fig. 208 were obtained; it was possible to estimate the thickness of the layer from these velocity curves. The velocity curves

* *Aerodynamic Theory*, Vol. VI. Page 64.

obtained are shown in Fig. 209. These correspond to the sections A, B, C, D, E, and F marked on the airship profile. The Reynolds number for the airship at the speed of this test was 635,000, and its length was 785 ft.

The boundary layer thickness obtained from the curves are shown plotted in Fig. 210 on a base representing the longitudinal position of the section, expressed as a percentage of the length from the nose. It will be noticed that the boundary layer had

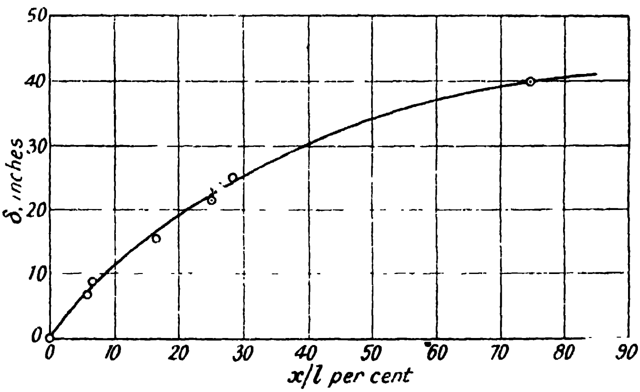


FIG. 210.—BOUNDARY LAYER THICKNESS FOR RIGID AIRSHIP

a thickness of about 40 inches at 75 per cent of the length from the nose.

A model of this airship was similarly tested in a wind tunnel and the boundary layer thicknesses were obtained in the same manner. These are shown plotted in Fig. 211, the thicknesses being expressed as a percentage of the length. The thicknesses obtained from an experiment on a flat plate of the same Reynolds number are also shown plotted in this figure. It will be noticed that the variation is slight between the results for the airship model and those of the flat plate, except at the extreme tail where there is considerable discrepancy.

The boundary layer thicknesses obtained from the model do not correspond with those obtained from the actual airship, on account of the difference in the Reynolds number of the two tests.

If the curve of Fig. 210 is assumed to follow the law of Equation (3), the constant of this equation for an actual

airship can be obtained by plotting the values of δ and $\sqrt{\frac{x}{l}}$ of this curve. This has been done in Fig. 212 and a straight line,

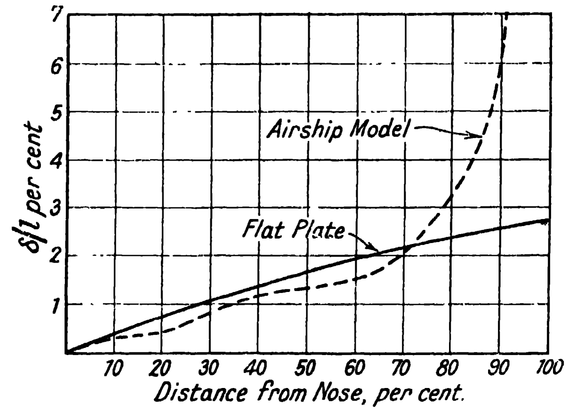


FIG. 211

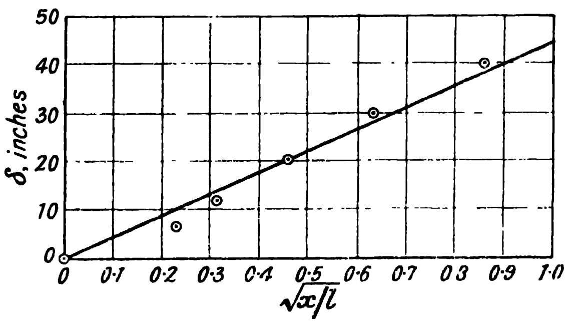


FIG. 212

passing through the origin, has been drawn representing mean of the points obtained.

From Equation (3)

$$\delta = \frac{k\,l}{(R_e)^{\frac{1}{5}}}\sqrt{\frac{x}{l}} \qquad \text{(where } k \text{ is a constant)}$$

From which
$$k = \frac{R_e^{\frac{1}{5}}}{l} \times \frac{\delta}{\sqrt{\frac{x}{l}}}$$

Substituting the values of R_e and l for the airship, and the values of δ and $\sqrt{\frac{x}{l}}$ from Fig. 212,

$$k = \frac{(635{,}000)^{\frac{1}{5}}}{785} \times \frac{43}{12}$$

$$= \cdot 0662$$

Hence, an approximation for the boundary layer thickness for this airship is

$$\delta = \cdot 0662 \left(\frac{1}{R_e}\right)^{\frac{1}{5}} \sqrt{lx}$$

For the value of R_e of this test the boundary layer flow was probably between laminar and turbulent over most of the length, as the results do not agree with either type of flow.

The variation in thickness of the boundary layer of a model

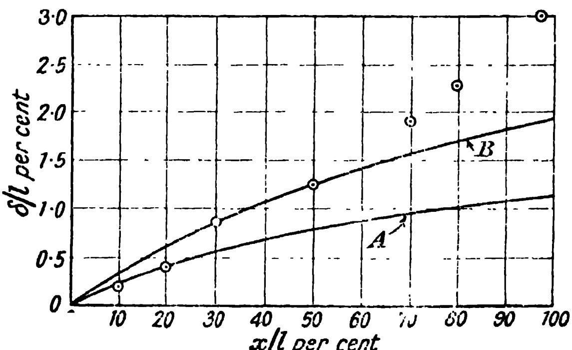


Fig. 213

of a British airship is shown by the curve of Fig. 213. This was obtained from the tests mentioned in Art. 170. The graph at first follows the square root curve A. At about 20 per cent of the length there is a rapid thickening of the boundary layer as the flow changes from laminar to turbulent, after which the graph follows the turbulent curve B. At about 70 per cent of the length the graph becomes steeper on account of a further rapid thickening of the layer, due to irregularities of the flow at the tail.

Millikan found that the thickness of the boundary layer on

a curved surface differs from that of a flat plate. A curved surface, such as a streamline body, produces a thinner layer at the nose and a thicker at the tail. This is demonstrated by the deviation of the points from the straight line in Fig. 212.

In the above, the airship was chosen as an example on the application of the boundary layer theory because of the considerable amount of existing data which is available from airships and airship models.

169. **Drag Coefficients from Boundary Layer Theory.** The resistance of a body based on the boundary layer theory can be obtained from Equation (3), Art. 158, if the drag coefficient is known. Attempts have been made to obtain expressions for the drag coefficients for a flat plate, and to apply them as in an aerofoil problem. This drag coefficient will give the total resistance of the body, and includes the effect of the surfaces of both sides.

It is found that the drag coefficient for laminar flow varies with $\sqrt{\frac{1}{R_e}}$. (Art. 171.)

For laminar flow Blasius found the drag coefficient to be given by the following equation--

$$k_D = 1{\cdot}327\sqrt{\frac{1}{R_e}} \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{then,} \quad D = k_D A \rho V^2$$

For turbulent flow, the drag coefficient for a flat plate was found by Kármán to follow the equation*

$$k_D = {\cdot}072\left(\frac{1}{R_e}\right)^{\frac{1}{5}} \quad . \quad . \quad . \quad . \quad (2)$$

The following equations for k_D for a flat plate have been deduced from experimental results; it is found to vary between—

$$k_D = {\cdot}072\left(\frac{1}{R_e}\right)^{\frac{1}{5}}$$

$$\text{and} \quad k_D = {\cdot}0375\left(\frac{1}{R_e}\right)^{\frac{1}{7}} \quad . \quad . \quad . \quad (3)$$

The following table† gives a summary of the values of k_D for a flat plate compared with its Reynolds number. It should

* See also Art. 172.

† See *Aerofoil and Airscrew Theory*, by GLAUERT.

be noted that the change from laminar flow to turbulent flow causes an increase of drag—

R_e	3×10^5	10^6	7×10^6
Experimental	·0057	·0047	·0035
Kármán	·0058	·0045	·0031
Blasius	·0024	·0013	·0005

Curves showing the complete results of tests on flat plates for a large range of values of R_e are shown in Fig. 214. These curves were obtained by plotting log k_D against log R_e, both for

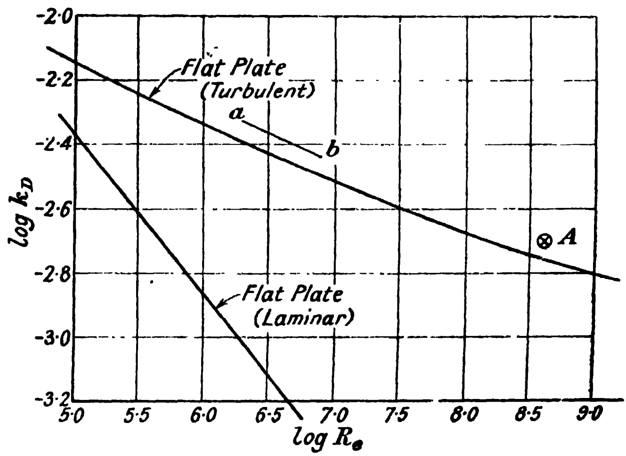


FIG. 214.—RELATION BETWEEN DRAG COEFFICIENT AND REYNOLDS NUMBER

laminar flow and turbulent flow. If these curves are assumed to approximate to straight lines, the value of k_D and the index n can be obtained from the equation—

$$\log k_D = \log k - n \log R_e$$

Having calculated the values of k and n from the graphs, the equation for k_D will be of the form—

$$k_D = k\left(\frac{1}{R_e}\right)^n$$

The equation for the laminar flow curve is found to be—

$$k_D = 1{\cdot}327\left(\frac{1}{R_e}\right)^{\frac{1}{2}}$$

The equation for the turbulent flow curve differs throughout its range as it is not actually a straight line. It is found to vary from

$$k_D = {\cdot}072\left(\frac{1}{R_e}\right)^{\frac{1}{5}} \text{ for the low values of } R_e$$

to

$$k_D = {\cdot}0375\left(\frac{1}{R_e}\right)^{\frac{1}{7}} \text{ for the high values of } R_e.$$

The line *ab* (Fig. 214) was obtained from the results of tests on an airship model. The point *A* represents the result of a test on an actual airship. These results demonstrate the great possibilities of the boundary layer theory for the prediction of the drag of any object. The curves of Fig. 214 hold over a very large range, the size of the airship in question being about 250 times that of the model.

The drag of large objects such as sea-going ships, submarines, and airships can be measured experimentally by means of deceleration tests. The ship is run at a high speed and the engines are then shut off; the time taken to reduce the speed by a measured amount is then noted. As this deceleration is entirely due to the fluid resistance, the value of the drag is the decelerating force. Or,

$$\begin{aligned} D &= \text{mass} \times \text{deceleration} \\ &= \frac{W}{g}\frac{(V_1 - V_2)}{t} \end{aligned}$$

where V_1 and V_2 are the initial and final velocities, and t is the time in seconds in which this reduction in speed took place.

170. **Variation of Pressure within Boundary Layer.** The variation of pressure within the boundary layer is important as a positive pressure gradient* may be the cause of a break-away. The velocities and pressures within the layer have been measured in a wind tunnel† for the smooth streamline model

* See Fig. 198 for definition of positive and negative pressure gradients.

† These results are from *Aerodynamical Research and Hydraulic Practice*, by A. Fage. Proceedings of Inst. of Mechanical Engineers, Vol. 130; 1935.

shown in Fig. 215. The velocity curves for various sections are shown plotted in the figure; from these curves the boundary

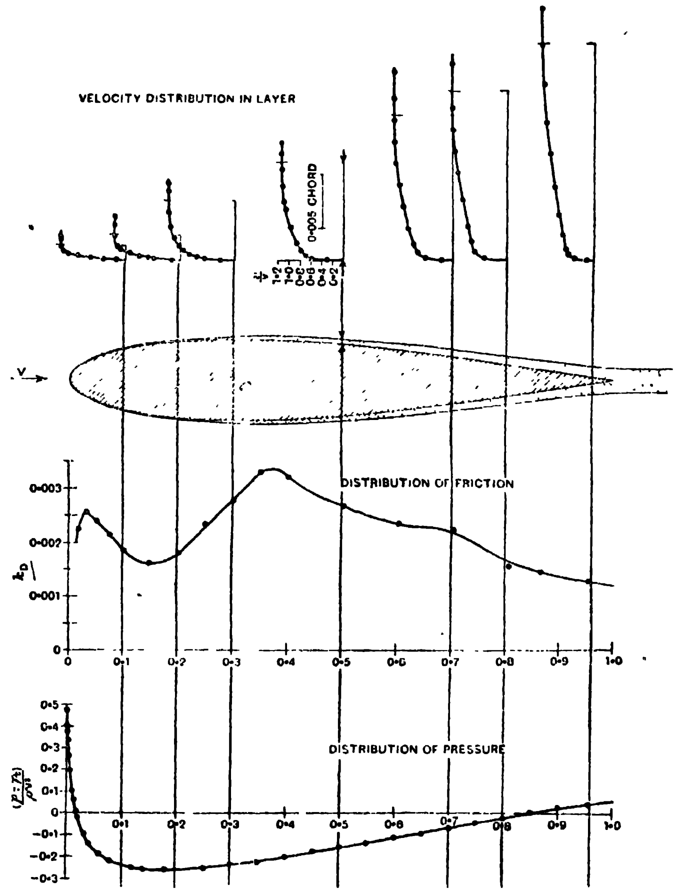


FIG. 215

layer thicknesses were obtained and are shown, drawn to scale, on the profile of the model.

The intensity of pressure within the layer is shown plotted on a base representing the length of the model. It will be noticed that there is a high positive pressure at the nose, but this soon changes to a negative pressure. This causes a local flow of air past the model, from nose to tail, which increases the relative velocity of the air. At the tail the pressure becomes

positive again. It is found that break-away is liable to occur within a portion of the length where a positive pressure gradient exists, and it may occur during a laminated or turbulent type of flow. A turbulent boundary layer can move up a steeper pressure gradient, without break-away occurring, than a laminar layer.

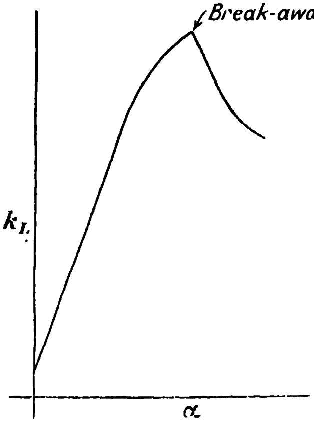


FIG. 216

The variation of k_D throughout the length of the model is also shown in Fig. 215.

Break-away of the boundary layer on an aeroplane wing causes a sudden reduction in the value of the lift coefficient, which may cause the aeroplane to stall. This is illustrated by the shape of the lift coefficient curve shown in Fig. 216. This break-away can be delayed by the use of slotted wings.*

Within a laminar flow layer, the transmission of momentum from the faster moving bands to the slower is carried out by viscosity. In a turbulent layer the transmission of momentum is made by particles of higher velocity moving inwards and giving up their momentum by collisions. Break-away at any section is due to the fact that the energy of the resistance at this section cannot be absorbed fast enough by the layer. Under such conditions break-away will occur, and the energy will then be dissipated in the vortices formed. This is more liable to occur with short bodies.†

Fig. 217 shows the velocity curves within the boundary layer, on several sections, which were obtained during a test on a surface in order to examine the cause of break-away. It will be noticed that break-away has commenced to the right of the fourth section. In the break-away area the velocity diagrams show a reversed current of fluid occurring near the

* See Art. 216 for methods of controlling boundary layer flow.

† For further details of the boundary layer theory the reader is referred to the following works—

Aeronautical Research Committee Reports and Memoranda No. 1664. *Notes on Boundary Layer Flow*, by H. B. Squire, M.A.

Aerodynamic Theory (6 vols.). Published in Berlin by Julius Springer. (Printed in English.) See Vols. III and VI.

surface. This is the commencement of the formation of the vortex, and as the flow is from right to left it denotes the existence of a positive pressure gradient.

171. **Calculation of Drag for Laminar Flow.** The drag on a flat plate due to a laminar boundary layer can be obtained from Equation (7), Art. 141. This equation was obtained for a viscous flow between two parallel surfaces. If the distance

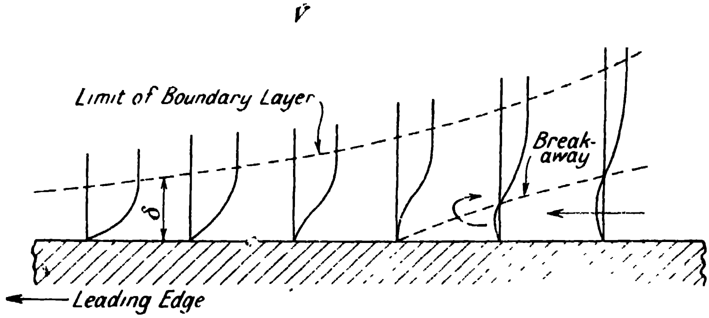


FIG. 217. VELOCITY ACROSS BOUNDARY LAYER

apart of the two surfaces is 2δ, the problem will be identical with that of a flat plate immersed in a fluid, the viscous drag on the surfaces being the same for both problems.

Using Equation (7), Art. 141, and substituting 2δ for t,

$$\frac{\text{drag per sq. ft. of surface}}{\rho V^2} = 6\left(\frac{V\,2\delta}{\nu}\right)^{-1} \quad . \quad . \quad (1)$$

Consider a strip of the plate of length dx, and breadth b, at a distance x from the leading edge (Fig. 218). Then, from Equation (1),

$$\text{drag on both sides of strip} = \frac{3\nu}{V\delta}\rho V^2 b\,dx \ . \qquad (2)$$

But, from Equation (2), Art. 168,

$$\delta = 4\cdot5\sqrt{\frac{\nu x}{V}}$$

Substituting this value of δ in Equation (2),

$$\text{drag of strip} = 3\,b\,\rho V^2\frac{\nu}{V}\frac{1}{4\cdot5}\sqrt{\frac{V}{\nu}}\;x^{-\frac{1}{2}}\,dx$$

$$= \frac{b\,\rho V^2}{1\cdot5}\sqrt{\frac{\nu}{V}}\;x^{-\frac{1}{2}}\,dx$$

Integrating this expression for the whole length l of the plate,

$$\text{total drag on plate} = D = \frac{b\,\rho V^2}{1\cdot5}\sqrt{\frac{\nu}{V}}\int_0^l x^{-\frac{1}{2}}\,dx$$

$$= \frac{b\,\rho V^2}{\cdot75}\sqrt{\frac{\nu}{V}}\;l^{\frac{1}{2}}$$

Substituting for $R_e = \dfrac{Vl}{\nu}$,

$$D = 1\cdot33\,\rho\,b\,l V^2\sqrt{\frac{1}{R_e}} \quad . \quad . \qquad (3)$$

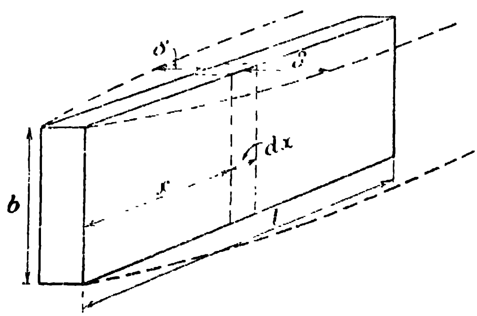


FIG. 218.—DRAG DUE TO LAMINAR FLOW

But, from Equation (3), Art. 158,

$$D = k_D\rho\,A V^2$$

where $\quad A = b\,l =$ projected area on chord

Hence $$k_D = 1\cdot33\sqrt{\frac{1}{R_e}} \quad . \quad . \quad . \qquad (4)$$

which is in close agreement with Blasius's equation (Equation (1), Art. 169).

172. **Drag for Turbulent Flow.** The drag on a flat plate due to a turbulent boundary layer can be calculated from the rate of change of momentum within the layer. It will be assumed that the velocity distribution within the layer is given by Equation (2), Art. 167, and the thickness of the layer is given by Equation (4), Art. 168.

Referring to Fig. 219, consider a section through the layer

at a distance x from the leading edge. The frictional drag occurring between the leading edge and this section will equal the rate of change of momentum of the layer at this section. Consider a thin band of the layer at a distance y from the surface of the plate; let dy be the thickness of this band and u its velocity. Owing to the disturbing effect of the surface, the velocity of this band has been reduced from V to u. This loss of momentum is caused by the drag over the whole length x. By imagining the boundary layer to consist of similar bands,

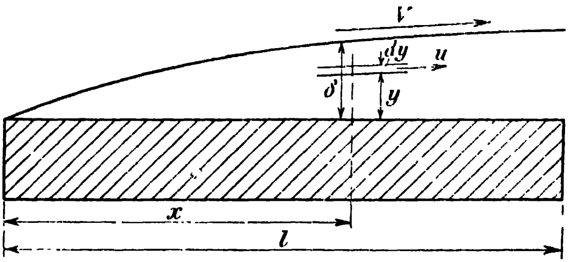


FIG. 219.—DRAG DUE TO TURBULENT FLOW

the total change of momentum throughout the layer can be obtained by integration.

Let b = breadth of plate

l = length of plate

Drag per side over length x due to thin band

= mass per sec. × change of velocity

$= \rho\, b\, dy\, u\, (V - u)$

Total drag on length x due to both sides of plate

$$= D_x = 2\rho b \int_0^\delta u\,(V-u)\,dy$$

$$= 2\rho b V^2 \int_0^\delta \left[\frac{u}{V} - \left(\frac{u}{V}\right)^2\right] dy$$

But, from Equation (2), Art. 167,

$$\frac{u}{V} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$$

Hence, $$D_x = 2\,\rho\, b\, V^2 \int_0^\delta \left[\left(\frac{y}{\delta}\right)^{\frac{1}{7}} - \left(\frac{y}{\delta}\right)^{\frac{2}{7}}\right] dy \quad . \quad . \quad (1)$$

$$= 2\,\rho\, b\, V^2 \left[\frac{7}{8} y^{\frac{8}{7}}\, \delta^{-\frac{1}{7}} - \frac{7}{9} y^{\frac{9}{7}}\, \delta^{-\frac{2}{7}}\right]_0^\delta$$

$$\doteq \frac{7}{36}\,\rho\, b\, V^2\, \delta \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But, from Equation (4), Art. 168,

$$\delta = \cdot 18 \left(\frac{1}{R_e}\right)^{\frac{1}{7}} \sqrt{l x}$$

substituting this value of δ in Equation (2),

$$D_x = \cdot 035 \left(\frac{1}{R_e}\right)^{\frac{1}{7}} \rho\, b\, V^2 \sqrt{l x}$$

The drag for the whole plate is when $x = l$;
then, total drag on whole plate

$$= \cdot 035 \left(\frac{1}{R_e}\right)^{\frac{1}{7}} \rho\, l\, b\, V^2 \quad . \quad . \quad . \quad (3)$$

But, from Equation (3), Art. 158,

$$D = k_D \rho\, A V^2$$

where $A = b\,l$ = projected area on chord

Hence, from Equation (3),

$$k_D = \cdot 035 \left(\frac{1}{R_e}\right)^{\frac{1}{7}} \quad . \quad . \quad . \quad (4)$$

It will be noticed that Equation (3) is of the same form as Equation (3), Art. 169, and agrees with the experimental results given in the table of Art. 169.

If the velocity distribution within the layer had been assumed to follow the "$\frac{1}{6}$ power law" instead of $\frac{1}{7}$ power, an equation similar to Equation (2), Art. 169, would have been obtained.

EXAMPLE.

What is meant by "boundary layer"?

A roughened thin board 1 ft. wide and 8 ft. long moves at 10 ft. per sec. in water. Each boundary layer is 3 in. thick at the rear end of the board, and $\frac{u}{V} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$.

Find the resistance and express it—(*a*) in lb., (*b*) as a pure number independent of δ. (London Univ.)

(*a*) Using Equation (1) and substituting the power of $\frac{1}{4}$ in place of $\frac{1}{7}$,

$$\text{Drag} = 2\,\rho\, b\, V^2 \int_0^\delta \left\{ \left(\frac{y}{\delta}\right)^{\frac{1}{4}} - \left(\frac{y}{\delta}\right)^{\frac{1}{2}} \right\} dy$$

$$= 2\,\rho\, b\, V^2 \left[\frac{4}{5} y^{\frac{5}{4}}\, \delta^{-\frac{1}{4}} - \frac{2}{3} y^{\frac{3}{2}}\, \delta^{-\frac{1}{2}} \right]_0^\delta$$

$$= \frac{4}{15}\, \rho\, b\, V^2 \delta$$

$$= \frac{4}{15} \times \frac{62{\cdot}4}{32{\cdot}2} \times 1 \times (10)^2 \times \frac{1}{4}$$

$$= 12{\cdot}93 \text{ lb.}$$

(*b*) Using the equation

$$\text{Drag} = k_D \rho\, A\, V^2$$

where $\quad A = bl$

hence, $\quad k_D = \dfrac{\text{drag}}{\rho\, b\, l V^2}$

$$= \frac{12{\cdot}93 \times 32{\cdot}2}{62{\cdot}4 \times 1 \times 8 \times (10)^2}$$

$$= {\cdot}00834$$

173. **The Laminar Sub-layer.** In a turbulent boundary layer a narrow portion of the layer, adjacent to the surface of the

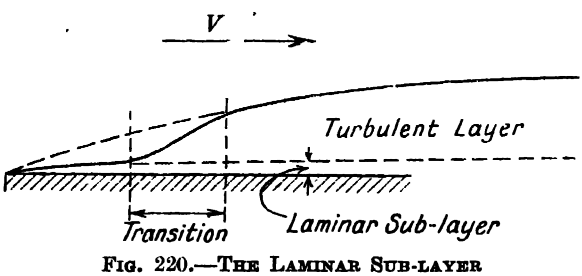


FIG. 220.—THE LAMINAR SUB-LAYER

body, is found to be of a laminar type of flow and is a continuation of the laminar boundary layer commencing at the leading edge (Fig. 220). This thin laminar layer occurring within a turbulent layer is known as a *laminar sub-layer*. It has an important effect in problems dealing with the roughness of the surface, and with the flow of heat between the surface and a fluid in contact with it.

A surface is said to be *aerodynamically smooth* when the projections due to its roughness do not penetrate beyond this laminar sub-layer. If the surface is sufficiently rough that its projections extend beyond the laminar sub-layer into the turbulent layer, there is caused a thickening of the turbulent layer and a corresponding increase in the frictional resistance.

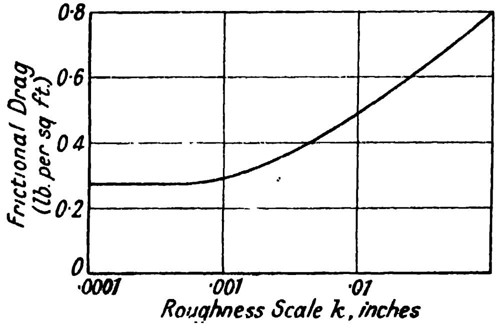


FIG. 221.—EFFECT OF ROUGHNESS ON FRICTIONAL DRAG

The drag due to a turbulent layer is, therefore, unaffected by the roughness of the surface, providing projections due to roughness do not protrude above the laminar sub-layer. The permissible roughness of surface is thus limited by the thickness of the laminar sub-layer.

The effect of the surface roughness on the frictional drag has been investigated experimentally by Nikuradse and Prandtl.* They have deduced the following empirical law—

$$\frac{1}{k} = 200V\sqrt{C_f}$$

where k represents a scale of roughness in inches, V is the fluid velocity in m.p.h., and C_f is the frictional drag coefficient. The term C_f is the same coefficient as used in the continental drag formula†—

$$\text{Frictional drag} = C_f \frac{A\rho V^2}{2}$$

and is consequently equal to twice the value of k_D.

The variation of the frictional drag of a flat plate, with the roughness of the surface is shown by the experimental curve of

* See paper, *Profile Drag*, by Melville Jones in the Proceedings of the Royal Aeronautical Society, 1937; and *Aerodynamic Theory*, Vol. III.

† See Art. 158.

Fig. 221. In this curve the drag per square foot has been plotted on a base representing the roughness scale k in inches. The fluid velocity of this test was 200 m.p.h. and the plate had a length of 10 ft. The surface of the plate was varied during the tests to conform with different degrees of roughness.

In this test the limiting value for aerodynamic smoothness was when $k = \cdot 0005$ in. It will be noticed from the curve that the rapid rise in the value of the drag did not commence until $k = \cdot 0007$ in.

For low values of k the frictional drag remained constant, as denoted by the horizontal portion of the curve; evidently the surface was aerodynamically smooth over the range represented by this horizontal portion.

174. **Methods of Obtaining True Dynamical Similarity.** It was shown in Arts. 145 and 146 that it is possible to predict the resistance of a large body by testing a small model of the same shape. This can only be done if there is dynamical similarity between the body and the model. It was shown that true dynamical similarity occurs when such factors as $\frac{\nu}{Lv}$ or $\frac{Lg}{v^2}$ are the same for both body and model.

In the case of surface ships, true dynamical similarity for wave resistance can be attained between the ship and the model, because the required corresponding speed for the model is within practical possibilities. For this type of resistance it was shown that

$$v_m = v\sqrt{\frac{L_m}{L}} \qquad \text{(Art. 146.)}$$

Hence, the required model speed is less than that of the large craft under consideration.

In a test for frictional or viscous resistance, the corresponding speed of the model is given by the equation

$$v_m = v \times \frac{L}{L_m} \times \frac{\nu_m}{\nu} \qquad \text{(Art. 146).}$$

Hence, if the model is tested in the same fluid as used for the large craft, the corresponding speed is very high, and, consequently, unattainable.* This difficulty can be overcome by testing the model in a fluid of lower viscosity or of a higher density, thus reducing the corresponding speed.

* A wind tunnel is being constructed in U.S.A. having an air speed of 800 m.p.h., and a jet diameter of 6 ft.

It is the latest practice to test models of aeroplanes and aeroplane wings in wind tunnels containing air compressed to 25 or 30 atmospheres. This causes alterations in the density and coefficient of viscosity of such magnitudes that the ratio $\frac{\nu_m}{\nu}$ is sufficiently reduced to make the corresponding speed a practical amount. In this way tests for frictional resistance with true dynamical similarity are made possible. Compressed air wind tunnels, of 6 ft. diameter, have been constructed for this purpose.

When compressed air wind tunnels are not available for model testing, or when speeds up to 100 m.p.h. only are possible, dynamical similarity cannot be attained, as the air speed is too low; hence, some other device must be used for predicting the resistance of the craft. The equation for the resistance of the full-sized craft and of the model will be of the form

$$D = k_D \rho A V^2$$

If the model is tested at any convenient speed and the value of its k_D found from the test, this value cannot be used to predict the resistance of the full-sized craft because the speed of the test was not the corresponding speed to give dynamical similarity. In order to find the value of k_D for the large craft it is necessary to investigate the variation of k_D with the Reynolds number R_e. In order to do this, the model is tested with various values of R_e and each corresponding value of k_D calculated. A curve with k_D and R_e as ordinates is then plotted. Then, by extrapolating, the value of k_D for the same Reynolds number as the large craft can be obtained.

This variation of k_D with Reynolds number is known as the *scale effect.*

175. **Comparison of Fluid Flow Theories.** A comparison may now be made of the fluid flow theories which have been dealt with in previous chapters. This involves a comparison of the following equations. Let the symbol V represent velocity.

(1) FROUDE.

Resistance $= f'AV^2$ (Art. 63).

(2) CHEZY.

$$h_f = \frac{4flV^2}{2gd} \text{ [Equation (3), Art. 67.]}$$

(3) VISCOSITY FORMULA.

$$\frac{R}{\rho V^2} = C\left(\frac{Vd}{\nu}\right)^n \quad \text{[Equation (8), Art. 140.]}$$

(4) BOUNDARY LAYER THEORY combined with DRAG FORMULA.

$$D = k_D \rho A V^2 \quad \text{[Equation (3), Art. 158.]}$$

where $$k_D = k\left(\frac{1}{R_e}\right)^n \quad \text{[Equations (1), (2) and (3), Art. 169.]}$$

k being an experimental constant.

It will be noticed that the drag formula of (4) is the same equation as Froude's if

$$f' = k_D \rho.$$

The Chezy formula was built up from Froude's assumptions and is merely another form of Froude's equation.

It was shown in Art. 158 that the viscosity formula of (3) is the same as the Chezy formula if the Chezy constant f is equal to

$$2C\left(\frac{Vd}{\nu}\right)^n$$

Substituting for $R_e = \left(\frac{Vd}{\nu}\right)$ in the above equation,

Then, $$f = 2C\,(R_e)^n$$

which is of the same form as the equation for k_D in the boundary layer theory.

The viscosity formula (3) may be written

$$\frac{\text{Resistance per sq. ft.}}{\rho V^2} = C(R_e)^n \quad . \quad . \quad (1)$$

But, $$D = k_D \rho A V^2$$

hence, $$\text{Resistance per sq. ft.} = \frac{D}{A} = k_D \rho V^2$$

Substituting this in Equation (1),

$$\frac{k_D \rho V^2}{\rho V^2} = C(R_e)^n$$

from which $$k_D = C(R_e)^n \quad . \quad . \quad (2)$$

Hence, it follows that the Viscosity Equation of (3) is of the

same form as the boundary layer equation, except for the values of the constants C and n. The constants of the viscosity equation are given in Art. 158. These were obtained from analytical results and experiments on pipe flow respectively, and were found to be—

For laminar flow, $C = 8, \; n = -1$
For turbulent flow, $C = \cdot 032, \; n = \cdot 23$

Substituting these values in Equation (2),

(*a*) for laminar flow, $k_D = 8\left(\frac{1}{R_e}\right)$,

but the boundary layer theory gives this value for a flat plate as

$$k_D = 1\cdot 327\left(\frac{1}{R_e}\right)^{\frac{1}{2}} \quad \text{[Equation (1), Art. 169.]}$$

which is not of the same form.

(*b*) For turbulent flow, substituting the values of C and n in Equation (2),

$$k_D = \cdot 032\left(\frac{1}{R_e}\right)^{\cdot 23}$$

which compares favourably with the Boundary Layer Equations for a flat plate, namely—

$$k_D = \cdot 072\left(\frac{1}{R_e}\right)^{\frac{1}{5}} \quad \text{[Equation (2), Art. 169.]}$$

and $$k_D = \cdot 0375\left(\frac{1}{R_e}\right)^{\frac{1}{7}} \quad \text{[Equation (3), Art. 169.]}$$

A summary of these comparisons is given in the following table—

Type of Flow	Equivalent Values of k_D in Terms of R_e		
	Froude and Chezy	Viscosity Equation	Boundary Layer Theory
Laminar	a constant	$8\left(\frac{1}{R_e}\right)$	$1\cdot 327\left(\frac{1}{R_e}\right)^{\frac{1}{2}}$
Turbulent	a constant	$\cdot 032\left(\frac{1}{R_e}\right)^{\cdot 23}$	$\cdot 072\left(\frac{1}{R_e}\right)^{\frac{1}{5}}$ to $\cdot 0375\left(\frac{1}{R_e}\right)^{\frac{1}{7}}$

From this comparison it will be seen that the correct fluid resistance, which depends on the Reynolds number and on the thickness of the boundary layer, is given only by the Boundary Layer theory. The Froude and Chezy formulae use the same constant for all types of flow and for all values of R_e; consequently, they hold over a very limited range. The Viscosity equation takes into account the variation of R_e and the type of flow, but does not allow for the varying thickness of the boundary layer; hence, the range of its constants are also limited. The Boundary Layer theory, as it allows for the variation of R_e and of the thickness of the layer, gives accurate results over a large range.

For the practical design problems of civil, mechanical and shipbuilding engineering, the results obtained from the Froude and Chezy equations are of sufficient accuracy. It is impossible to calculate the exact resistance of water pipes, when the lining is of varying roughness when new, contains irregularities at the joints, and is usually covered after use with encrustation and fungus of varying amounts. The resistance of a ship will also vary; its surfaces may be newly painted or they may be covered with variable amounts of fungus or other organic growths. These unavoidable changes make it impossible to apply any equation with close accuracy.

In aeronautical and other more scientific design problems the surface conditions are more constant; also, it is important that more exact calculations be made on account of the high degree of accuracy required.

EXAMPLES 15.

The following values of ρ and ν are to be used in these examples—

$$\text{for air,} \quad \rho = \frac{\cdot 075}{g} \text{ ft. lb. units;} \quad \nu = \cdot 000167 \text{ ft. lb. units}$$

$$\text{for water,} \quad \rho = \frac{62 \cdot 4}{g} \text{ ft. lb. units;} \quad \nu = \cdot 0000108 \text{ ft. lb. units}$$

(1) A fluid is flowing past a flat surface with a velocity of 30 ft. per sec. If the thickness of the boundary layer at a certain section is 2·5 in., find the velocity within the layer at a distance of 1·5 in. from the surface. Assume the flow is turbulent and that the velocity variation within the layer follows the "$\frac{1}{7}$ power law." (Equation (2), Art. 167.)

Ans.—27·9 ft. per sec.

(2) Calculate the Reynolds number and the thickness of the boundary layer of a flat plate, moving through water with a speed of 20 ft. per sec., at a section 15 ft. from the leading edge. The length of the plate is 60 ft. Assume the "$\frac{1}{7}$ power" law. (Equation (4), Art. 168.)

Ans.—$1 \cdot 11 \times 10^8$; 4·6 in.

(3) Calculate, from the Boundary Layer theory, the Reynolds number and the drag coefficient for a flat plate 3 ft. long immersed in an air stream of 60 m.p.h. Assume the "$\frac{1}{7}$ power" law to hold. (Equation (2), Art. 169.) If the plate is 6 in. broad, find the total resistance at this speed.

Ans.—$R_e = 1{\cdot}58 \times 10^6$; $k_D = {\cdot}00417$; $D = {\cdot}133$ lb.

(4) Using the Boundary Layer theory, calculate the Reynolds number and the drag of the hull of a large airship when travelling at a speed of 60 m.p.h. The length is 812·7 ft. and the diameter 135·4 ft., giving an area on chord of 91,000 sq. ft. Use the "$\frac{1}{7}$ power" law for the value of k_D. (Equation (3), Art. 169.) Find also the horse-power required to overcome the hull drag at this speed.

Ans.—$R_e = 4{\cdot}28 \times 10^8$; $k_D = {\cdot}00219$; 577 h.p.

(5) During a series of tests on a model in a wind tunnel the following values of k_D were obtained for the given values of R_e, in order to obtain the scale effect.

R_e	126,000	224,000	631,000	1,728,000
k_D	·00398	·00251	·00158	·00100

By plotting these values and their logarithms, find by extrapolating, the value of k_D for the full-size object, the value of its Reynolds number being $4{\cdot}467 \times 10^6$.

Ans.—·000631.

(6) Define "boundary layer." A thin plate 6 ft. long and 1 ft. wide moves at 10 ft. per sec. through water. The boundary layer is 2 in. thick at the rear end of the plate and the velocity distribution is approximately

$$\frac{u}{V} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$$

Find the resistance of the plate and express it (*a*) in pounds, (*b*) as a pure number independent of δ. (London Univ.)

Ans.—8·62 lb.; ·0074.

(7) What is meant by "boundary layer"?

Water flows at 15 ft. per sec. past a thin rough board 1 ft. 6 in. wide and 12 ft. long. The boundary layer is 4 in. thick at the downstream end of the board. Neglecting pressure variations and assuming

$$\frac{u}{V} = \left(\frac{y}{\delta}\right)^{0\cdot2}$$

within the boundary layer, find the drag and express it (*a*) in lb.; (*b*) as a pure number independent of δ. (London Univ.)

Ans.—51·8 lb.; ·00662

(8) A ship-shaped body is immersed in a current. Sketch the general outline of the lines of flow past the body (*a*) at very low speeds, and (*b*) at high speeds. State briefly how the resistance depends upon the velocity, density, and viscosity of the fluid in each case. (I. Mech. E.)

CHAPTER XVI

COMPRESSIBLE FLUIDS

176. **Properties of Gases.** Problems dealing with the pressure and flow of gases are more complex than those applied to liquids. As a gas is easily compressible, in comparison with a liquid, the variation in its density is considerable. It is this variation in density which complicates the calculations on gas flow.

The behaviour of gases under changes of temperature, pressure, and volume is governed by several well-known laws, all of which have been verified experimentally.

Let p = pressure of gas in lb. per sq. ft.

V = volume in cu. ft.

t = temperature as measured by a thermometer

T = absolute temperature

$= t + 273$ degrees C.

$= t + 460$ degrees F.

ρ = density of gas

w = weight per cu. ft. of gas

$= \rho g$

Then, volume 1 lb. of gas $= \dfrac{1}{w}$ cu. ft.

Let K_p and K_v = specific heats at constant pressure and constant volume in ft. lb. units

γ = ratio of specific heats

$= \dfrac{K_p}{K_v}$

J = Joule's equivalent of heat

= 1400 ft. lb. per degree centigrade

= 778 ft. lb. per degree Fahrenheit

E = internal energy of 1 lb. of gas reckoned above 0° C. or 32° F.

Let suffix 1 represent the initial condition of the gas and suffix 2 the final condition.

Boyle's Law. This law states that if a gas is expanded or compressed at constant temperature the product of its pressure and volume at any instant is a constant. Or,

$$p \times V = \text{a constant}$$

then,
$$p_1 V_1 = p_2 V_2$$

Charles' Law. If a gas is expanded or compressed at constant pressure, then

$$\frac{V}{T} = \text{a constant}$$

If the change takes place at constant volume, then

$$\frac{p}{T} = \text{a constant}$$

Characteristic Equation of a Gas. By combining the laws of Boyle and Charles the characteristic equation of a gas is obtained. That is,

$$pV = RT$$

where V is the volume of 1 lb. of the gas at pressure p and absolute temperature T. R is known as the gas constant.

For atmospheric air, $R = 96$ ft. lb. centigrade units
$= 53{\cdot}3$ ft. lb. Fahrenheit units

177. **The Internal Energy of a Gas.** The internal energy stored in a gas is the amount of energy in the form of heat; that is, the energy due to the vibration of its molecules. In gases such as oxygen, nitrogen and hydrogen, the internal energy is registered by the temperature. It can be proved from Joule's experiment* that the internal energy of 1 lb. of gas at an absolute temperature T, and reckoned above the freezing point of water, is given by the equation,

$$E = K_v\,(T - 273) \text{ ft. lb., when } T \text{ is in centigrade units}$$
$$= K_v\,(T - 460) \text{ ft. lb., when } T \text{ is in Fahrenheit units}$$

If the temperature of a gas changes from T_1 to T_2, then increase of internal energy per lb. of gas is,

$$E_2 - E_1 = K_v\,(T_2 - T_1) \text{ ft. lb. units} \quad . \quad . \quad . \quad (1)$$

178. **Application of Law of Conservation of Energy.** If a gas is expanded in such a manner that it performs some form of external work, it follows that the heat absorbed, the work done, and the change of internal energy, must balance in

* See author's textbook *Thermodynamics Applied to Heat Engines* (Pitman).

accordance with the principle of conservation of energy. This may be written in the form of an equation—

$$\left\{\begin{array}{c}\text{Heat absorbed}\\ \text{by 1 lb. of gas in}\\ \text{ft. lb. units}\end{array}\right\} = \left\{\begin{array}{c}\text{Work done}\\ \text{by 1 lb. of}\\ \text{gas}\end{array}\right\} + \left\{\begin{array}{c}\text{Increase of in-}\\ \text{ternal energy per}\\ \text{lb. of gas}\end{array}\right\}$$

or, $$H = W + (E_2 - E_1)$$

It should be noted that if the gas is rejecting heat, then H is negative; if external work is being done on the gas in compressing it, then W is negative. If the term $(E_2 - E_1)$ is negative, it then represents a decrease in internal energy.

179. **Isothermal Process.** If a gas is expanded or compressed whilst its temperature is maintained constant the process is called isothermal. External work of some description is performed by the gas. As the temperature is constant the expansion follows Boyle's law, then

$$pV = \text{constant}$$

and $$p_1V_1 = p_2V_2$$

As there is no change of temperature there cannot be any change of internal energy; this will be seen from Equation (1) (Art. 177).

Applying the law of conservation of energy to the isothermal process,

$$\begin{aligned} H &= W + (E_2 - E_1) \\ &= W + 0 \end{aligned}$$

Hence, it follows, that the heat absorbed by the gas during this process is equal to the work done by the gas.

The work done during an isothermal process can be calculated by finding the area under the isothermal curve of Fig. 222. Let this curve represent the expansion pV = constant, and consider a thin strip at pressure p and volume V. Let the thickness of this strip be dV.

$$\text{Work done} = W = \text{area under curve}$$

$$= \int_{V}^{V_2} pdV \quad . \quad . \quad . \quad . \quad (1)$$

But, $pV = p_1V_1$

hence, $p = \dfrac{p_1V_1}{V}$

Substituting this value in Equation (1),

$$W = p_1 V_1 \int_{V}^{V_2} dV$$

$$= p_1 V_1 \Big[\log_e V\Big]_{V}^{V_2}$$

$$= p_1 V_1 \log_e \frac{V_2}{V_1}$$

Let $\quad r =$ ratio of expansion or pressure ratio

$$= \frac{V_2}{V_1}$$

then, $\quad W = p_1 V_1 \log_e r$ (ft. lb.) . . . . . (2)

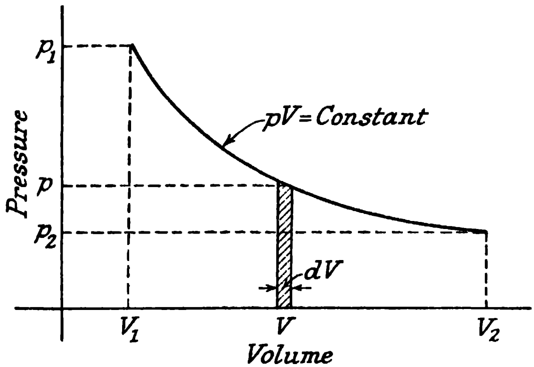


FIG. 222

From the characteristic equation of a gas,

$$p_1 V_1 = RT$$

then, for 1 lb. of gas, $\quad W = RT \log_e r$ (ft. lb.) (3)

This equation represents the heat energy absorbed during the expansion. If the process is a compression, r is less than unity and $\log r$ becomes negative; then the work done by the gas is negative. This means that work is done on the gas and heat is consequently rejected.

180. **Adiabatic Process.** If a gas is expanded or compressed in such a manner that no interchange of heat takes place during the process, the process is called adiabatic. External

work of some description is performed by the gas. It can be proved* that the equation representing an adiabatic process is

$$pV^{\gamma} = \text{constant}$$

then, $$p_1V_1^{\gamma} = p_2V_2^{\gamma}$$

where γ is the ratio of the specific heats.

Applying the law of conservation of energy to this process, and putting the heat supplied as 0,

$$H = W + (E_2 - E_1)$$

then, $$O = W + (E_2 - E_1)$$

or $$W = (E_1 - E_2) \quad . \qquad (1)$$

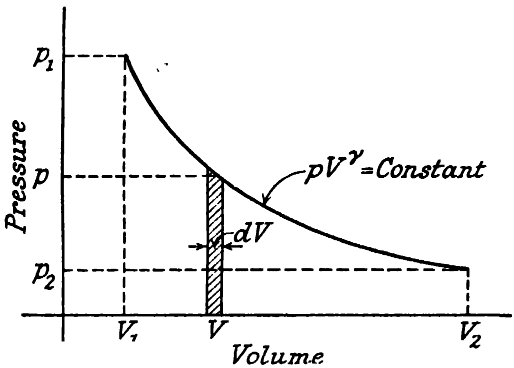


FIG. 223

Thus, the work done by the gas is performed at the expense of its own internal energy.

An equation for the work done during this process can be obtained from the adiabatic curve of Fig. 223. Consider the thin strip of pressure p, volume V and thickness dV.

$$\text{Work done} = W = \int_{V_1}^{V_2} p\,dV$$

but, $$pV^{\gamma} = p_1V_1^{\gamma}$$

from which $$p = p_1V_1^{\gamma}V^{-\gamma}$$

* See author's textbook *Thermodynamics Applied to Heat Engines* (Pitman)

then, substituting for p, we have $W = p_1 V_1^{\gamma} \int_{V_1}^{V_2} V^{-\gamma} dV$

$$= \frac{p_1 V_1^{\gamma}}{1-\gamma}\Big[V^{1-\gamma}\Big]_{V_1}^{V_2}$$

$$= \frac{p_1 V_1^{\gamma}}{1-\gamma}\,(V_2^{1-\gamma} - V^{1-\gamma})$$

but, $\quad p_1 V_1^{\gamma} = p_2 V_2^{\gamma}$

hence, $\quad W = \frac{(p_2 V_2 - p_1 V_1)}{1-\gamma}$

$$= \frac{(p_1 V_1 - p_2 V_2)}{\gamma - 1} \text{ ft. lb.} \qquad (2)$$

As $p_1 V_1 = RT_1$ and $p_2 V_2 = RT_2$, this equation may be written,

$$W = \frac{R(T_1 - T_2)}{\gamma - 1} \text{ ft. lb.} \qquad (3)$$

It will be seen from Equation (1) that Equations (2) and (3) represent the decrease of internal energy per lb. of gas during an adiabatic expansion, and the increase of internal energy during an adiabatic compression.

Or, $\quad (E_1 - E_2) = \frac{p_1 V_1 - p_2 V_2}{\gamma - 1} \qquad (4)$

As $p_1 V_1^{\gamma} = p_2 V_2^{\gamma}$, and as $\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}$ it follows that

$$\frac{p_1}{p_2} = \left(\frac{V_2}{V_1}\right)^{\gamma} \qquad (5)$$

also, $\quad \frac{T_1}{T_2} = \left(\frac{V_2}{V_1}\right)^{\gamma-1} \qquad (6)$

and, $\quad \frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} \qquad (7)$

From Equations (5), (6), and (7) the relation between pressure, volume and temperature during an adiabatic change can be found.

181. **Bulk Modulus of a Fluid.** The bulk elastic modulus of a fluid is the ratio between the increase of pressure and the volumetric strain caused by this pressure increase. This applies to liquids and gases.

Let a quantity of fluid at pressure p and volume V be subjected to an increase of pressure dp. Let this increase of pressure cause a change of volume $-dV$.

Then, volumetric strain $= \dfrac{-dV}{V}$

$$\text{Bulk modulus} = K = \frac{\text{increase of pressure}}{\text{volumetric strain}}$$

$$= \frac{dp}{\dfrac{-dV}{V}}$$

$$= -V\frac{dp}{dV} \quad . \quad . \quad . \quad (1)$$

The bulk modulus for a liquid is large on account of the small amount of compressibility. For a gas, the value of K is relatively small as dV is large.

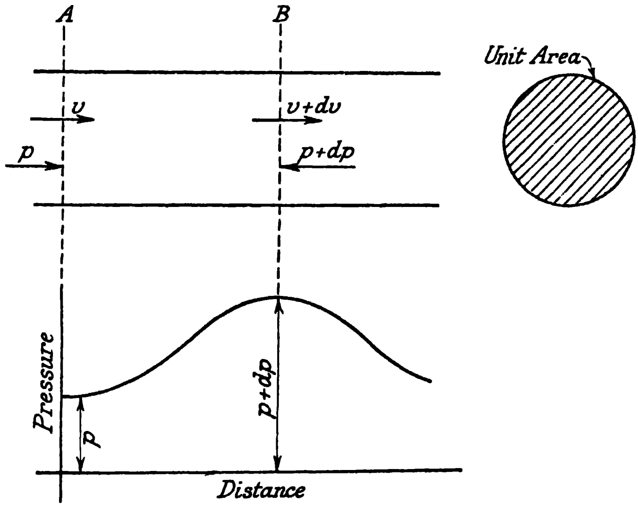


FIG. 224

182. **Velocity of Pressure Wave in a Fluid.** The velocity of a pressure wave transmitted through a fluid depends on the bulk modulus and density of the fluid.

Consider a tube of fluid of unit cross-sectional area (Fig. 224) through which a pressure wave is being transmitted from right to left, having a velocity of v. Now bring the wave to rest by

imagining the fluid to have a velocity of v in the opposite direction. Consider a section A at which the intensity of pressure is p and the velocity v; let this section represent that portion of the fluid which is at the least pressure. Let the section B represent the adjacent portion of the fluid at which the pressure is a maximum. Let the pressure between the sections A and B increase by dp and let the velocity change by dv; this is due to a local circulation caused by the wave. The pressures are shown plotted on the pressure-distance diagram projected vertically below the tube (Fig. 224). It will be noticed that section A corresponds to the trough of the pressure wave and section B to the crest. Then,

pressure of fluid at $B = p + dp$

and, velocity of fluid at $B = v + dv$

Let V = volume of fluid at A compressed per sec. by wave

and ρ = density of fluid at A

Let volume compressed per sec. increase by dV between sections A and B

Then, at section A, $V = a \times v = v$ (as a is unity)

At section B, volume compressed per sec.

$$= V + dV$$
$$= a(v + dv)$$

But, $a = 1$ and $V = v$,

hence, $$dV = dv \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, force on fluid between sections A and B

= change of momentum per sec.

= mass per sec. $\times$ change of velocity

That is, $$\{p - (p + dp)\}a = \rho a V \times dv$$

or $$dp = -\rho V dv$$

But, from Equation (1) $$dv = dV$$

hence, $$dp = -\rho V dV$$

or $$\frac{dp}{dV} = -\rho V \quad . \quad . \quad . \quad (2)$$

But it was proved in Art. 181, Equation (1), that

$$K = -V\frac{dp}{dV}$$

or $$\frac{dp}{dV} = -\frac{K}{V} \qquad (3)$$

From Equations (2) and (3), $\rho V = \frac{K}{V}$

But as $V = v$, then, $\rho v = \frac{K}{v}$

or $$v = \sqrt{\frac{K}{\rho}} \qquad (4)$$

Thus, the velocity of a pressure wave in a fluid depends on the values of its bulk modulus and density. As sound is propagated by means of a pressure wave transmitted in a fluid, Equation (4) gives the velocity of sound in a fluid.

EXAMPLE.

Calculate the velocity of sound in water of weight 62·4 lb. per cu. ft. if the value of its bulk modulus is 300,000 lb. per sq. in.

Using Equation (4),

$$v = \sqrt{\frac{K}{\rho}}$$

Now, $$\rho = \frac{w}{g} = \frac{62{\cdot}4}{32{\cdot}2}$$

and, $$K = 300{,}000 \times 144 \text{ lb. per sq. ft.}$$

Hence, $$v = \sqrt{\frac{300{,}000 \times 144 \times 32{\cdot}2}{62{\cdot}4}}$$

$$= 4720 \text{ ft. per sec.}$$

183. **Wave Velocity for Isothermal Process.** The velocity of a pressure wave in a gas depends on whether the transmission is an isothermal process or an adiabatic. If the process is assumed to be isothermal, then

$$pV = C, \text{ where } C \text{ is a constant}$$

differentiating, $$pdV + Vdp = 0$$

then, $$pdV = -Vdp$$

or, $$\frac{dp}{dV} = -\frac{p}{V} \qquad (1)$$

But from Equation (3), Art. 182,

$$\frac{dp}{dV} = -\frac{K}{V}$$

Equating these two values of $\frac{dp}{dV}$,

$$\frac{K}{V} = \frac{p}{V}$$

hence, $$K = p$$

Substituting this value of K in Equation (4), Art. 182,

$$\text{Velocity of wave} = v = \sqrt{\frac{p}{\rho}} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equation (2) was deduced by Newton for the velocity of sound in a gas, but it is found from tests that this equation does not give good results. An adiabatic process should be assumed for the velocity of sound in a gas, as the temperature does not remain constant during the pressure variation of the wave.

Equation (2) is found to give good results for a liquid; this is because the temperature change in a liquid due to a variation of pressure is extremely small and, consequently, the process approximates to an isothermal.

184. **Wave Velocity for Adiabatic Process.** The transmission of a pressure wave in a gas is approximately adiabatic; this is because the pressure variation takes place very suddenly and, consequently, there is no time for any appreciable interchange of heat. Applying the law for an adiabatic process (Art. 180),

$$pV^\gamma = C, \text{ where } C \text{ is a constant}$$

differentiating,

$$\gamma p V^{\gamma-1}\, dV + V^\gamma dp = 0$$

then, $$\gamma p V^{\gamma-1}\, dV = -V^\gamma dp$$

hence, $$\frac{dp}{dV} = -\frac{\gamma p}{V} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But, from Equation (3), Art. 182,

$$\frac{dp}{dV} = -\frac{K}{V}$$

Equating these two values of $\frac{dp}{dV}$,

$$-\frac{K}{V} = -\frac{\gamma p}{V}$$

hence, $$K = \gamma p$$

Substituting this value of K in Equation (4), Art. 182,

$$\text{Velocity of wave} \quad = v = \sqrt{\frac{\gamma p}{\rho}} \quad . \quad . \quad . \quad . \quad (2)$$

This equation was deduced by Laplace for the velocity of sound in a gas. Results of tests are found to be in agreement with this equation. Equation (2) is also used for obtaining the value of γ for a gas; the velocity of sound at a known pressure being measured experimentally.

EXAMPLE.

Calculate the velocity of sound in air at a pressure of 14·7 lb. per sq. in. and at a temperature of 0° C. γ for air $= 1{\cdot}41$.

Using the characteristic equation of a gas,

$$pV = RT$$

$$V = \frac{1}{w}, R = 96 \text{ and } T = 273° \text{ C. abs.}$$

hence,

$$p \times \frac{1}{w} = 96 \times 273$$

or $$w = \frac{14{\cdot}7 \times 144}{96 \times 273}$$

$$= {\cdot}081 \text{ lb. per cu. ft.}$$

Now, $$\rho = \frac{w}{g} = \frac{{\cdot}081}{32{\cdot}2}$$

Using Equation (2),

$$v = \sqrt{\frac{\gamma p}{\rho}}$$

$$= \sqrt{\frac{1{\cdot}41 \times 14{\cdot}7 \times 144 \times 32{\cdot}2}{{\cdot}081}}$$

$$= 1090 \text{ ft. per sec.}$$

185. **Bernoulli's Equation Applied to Gases.** Bernoulli's equation of Art. 26 can be extended to apply to problems dealing with the flow of gases. When applying this equation to gases it is necessary to allow for the following additional factors—

(1) The change of density with pressure and temperature changes.

(2) The change of internal energy, due to temperature changes.

(3) Any absorption or rejection of heat.

Let E_1 and E_2 represent the initial and final internal energies of the gas in ft. lb. units per lb., reckoned above freezing point of water, and let H be the heat absorbed by the gas in ft. lb. units during the process. Then, Bernoulli's equation may be written

$$Z_1 + \frac{p_1}{w_1} + \frac{v_1^2}{2g} + E_1 = Z_2 + \frac{p_2}{w_2} + \frac{v_2^2}{2g} + E_2 - H$$

But, $$\frac{1}{w_1} = V_1$$

and $$\frac{1}{w_2} = V_2$$

where V_1 and V_2 represent the volumes of 1 lb. gas at pressures p_1 and p_2 respectively.
Hence, for 1 lb. of gas,

$$Z_1 + p_1V_1 + \frac{v_1^2}{2g} + E_1 = Z_2 + p_2V_2 + \frac{v_2^2}{2g} + E_2 - H \quad . \qquad (1)$$

(1) For Adiabatic Change. If the process is adiabatic, $H = 0$, and from Equation (4), Art. 180,

$$E_1 - E_2 = \frac{p_1V_1 - p_2V_2}{\gamma - 1}$$

Substituting these values in Equation (1),

$$(Z_1 - Z_2) + (p_1V_1 - p_2V_2) + \frac{p_1V_1 - p_2V_2}{\gamma - 1} = \frac{v_2^2 - v_1^2}{2g}$$

that is, $$(Z_1 - Z_2) + \frac{\gamma}{\gamma - 1}(p_1V_1 - p_2V_2) = \frac{v_2^2 - v_1^2}{2g} \quad . \qquad (2)$$

This equation can be further simplified. Let n be the ratio of the initial and final pressures; then,

$$n = \frac{p_2}{p_1}.$$

Then $$\frac{p_2 V_2}{p_1 V_1} = n\frac{V_2}{V_1}$$

$$= n\left(\frac{p_2}{p_1}\right)^{-\frac{1}{\gamma}} \text{[from Equation (5), Art. 180]}$$

$$= n^{\frac{\gamma-1}{\gamma}}$$

Taking $p_1 V_1$ outside of brackets of Equation (2) and substituting for n, Equation (2) becomes

$$Z_1 - Z_2 + \left(\frac{\gamma}{\gamma - 1}\right) p_1 V_1 \left[1 - n^{\frac{\gamma-1}{\gamma}}\right] = \frac{v_2^2 - v_1^2}{2g} \quad . \quad (3)$$

This equation represents any adiabatic flow of a gas, and may be applied to a pipe of varying section, a Venturi meter, an orifice, or a nozzle.

(2) FOR ISOTHERMAL CHANGE. If the process is an isothermal change, heat must be absorbed or rejected by the gas in order to maintain a constant temperature.

Then, from Art. 179,

$$E_1 = E_2$$

$$p_1 V_1 = p_2 V_2$$

and $$H = RT \log_e r$$

Substituting these values in Equation (1),

$$(Z_1 - Z_2) + RT \log_e r = \frac{v_2^2 - v_1^2}{2g} \quad . \quad . \quad . \quad . \quad (4)$$

186. **Isothermal Flow in Pipe.** When gas flows isothermally along a pipe of uniform bore there is a loss of head due to the frictional resistance. This can be calculated from the equation

$$h_f = \frac{4flv^2}{2gd} \text{ ft. of gas}$$

There is a continuous fall in pressure as the gas flows along the pipe, due to the overcoming of the frictional resistance. This fall of pressure causes an expansion in volume and a corresponding reduction in density. The weight of gas per sec. flowing past any section must be constant; hence, an expansion in volume causes an increase in velocity. It will be seen from this that the frictional resistance causes the gas

to expand; the temperature thus tends to fall. But as the flow is isothermal, the fall in temperature is prevented by heat being absorbed from the surroundings. The effect of friction also tends to maintain a constant temperature, because the energy lost in friction is converted into heat, some of which reheats the gas.

It follows from this that the pressure drop during an isothermal flow is less than that of an adiabatic flow for the same frictional resistance. Also, it can be shown that the change of velocity along the pipe, due to frictional causes, is small for moderate lengths of piping; hence, the mean velocity can be used in the frictional formula.

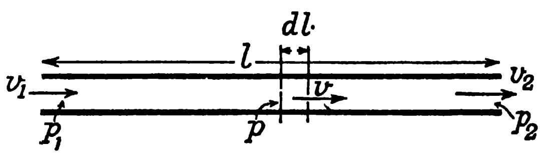


FIG. 224A

Referring to the pipe of Fig. 224A, let suffix 1 apply to the inlet conditions and suffix 2 to the outlet.

Let H = amount of heat absorbed by gas in maintaining a constant temperature, in ft. lb. units

Applying Equation (1) (Art. 185) to the entrance and exit ends of the pipe,

$$Z_1 + p_1V_1 + \frac{v_1^2}{2g} + E_1 = Z_2 + p_2V_2 + \frac{v_2^2}{2g} + E_2 + h_f - H$$

For a horizontal pipe $\qquad Z_1 = Z_2$

As the flow is isothermal, $p_1V_1 = p_2V_2$

and $\qquad E_1 = E_2$

hence, the equation becomes

$$\frac{v_1^2}{2g} = \frac{v_2^2}{2g} + h_f - H \qquad . \qquad . \qquad (1)$$

The work done against friction is converted to heat, which helps to reheat the gas to its former temperature. This reduces the amount H which flows in through the pipe walls. If it is assumed that the whole of the frictional resistance is utilized

in reheating the gas, the term h_f is not a loss of energy. Then Equation (1) becomes—

$$\frac{v_1^2}{2g} = \frac{v_2^2}{2g} - H \quad . \quad . \quad . \quad (2)$$

In this case, H is not the heat absorbed during a pure isothermal expansion as given by Equation (2) (Art. 179), the expansion being affected by the frictional resistance. A flow at constant temperature which is frictionally resisted cannot be regarded as a true isothermal expansion, as defined in Art. 179.

For short pipes it is usually of sufficient accuracy to assume a constant density and velocity and to treat the problem as was done with liquid flow (Art. 74). An approximate solution for pressure drop during isothermal flow is given in Art. 187.

187. **Approximate Solution for Isothermal Flow in Pipes.** The following approximate solution is satisfactory for most problems on isothermal flow through pipes. In this approximation the gain of kinetic energy of the gas is neglected as relatively small compared with the energy absorbed by the frictional resistance.

Consider a short length of the pipe dl (Fig. 224A) at a point where the velocity is v, the pressure p, the density w, and the specific volume V. Then,

Weight of gas flowing $= wav = w_1av_1$

also, for isothermal flow, $pV = p_1V_1$

or $pv = p_1v_1$, as $\frac{v}{v_1} = \frac{V}{V_1}$

then,
$$v^2 = \frac{p_1^2v_1^2}{p^2} \quad . \quad . \quad . \quad (1)$$

As,
$$\frac{p}{w} = \frac{p_1}{w_1}$$

then,
$$w = \frac{pw_1}{p_1} \quad . \quad . \quad . \quad . \quad (2)$$

Combining Equations (1) and (2),

$$wv^2 = \frac{pw_1}{p_1} \times \frac{p_1^2v_1^2}{p^2}$$

$$= \frac{w_1p_1v_1^2}{p} \quad . \quad . \quad . \quad (3)$$

From the characteristic equation for a gas,

$$p_1V_1 = RT$$

then,

$$\frac{p_1}{w_1} = RT$$

or

$$w_1 = \frac{p_1}{RT} \quad . \quad . \quad . \quad . \quad (1)$$

As

$$h_f = \frac{4flv^2}{2gd}$$

then for a short length of pipe dl,

$$d(h_f) = \frac{dp}{w} = -\frac{4fv^2dl}{2gd}$$

where dp is the pressure change on length dl and is negative,

hence,

$$dp = -\frac{4fwv^2dl}{2gd}$$

Substituting for wv^2 from Equation (3),

$$pdp = -\frac{4fw_1p_1v_1^2dl}{2gd}$$

Integrating between the two ends of the pipe,

$$\int_{p_1}^{p_2} pdp = -\frac{4fw_1p_1v_1^2}{2gd}\int_o^l dl$$

that is,

$$\frac{p_2^2 - p_1^2}{2} = -\frac{4fw_1p_1v_1^2l}{2gd}$$

Substituting for w_1 from Equation (4),

$$\frac{p_2^2 - p_1^2}{2} = -\frac{4fp_1^2v_1^2l}{2RTgd}$$

from which

$$p_2^2 = p_1^2\left(1 - \frac{8flv_1^2}{2gdRT}\right)$$

then,

$$p_2 = p_1\sqrt{1 - \frac{8flv_1^2}{2gdRT}} \quad . \quad . \quad . \quad (5)$$

EXAMPLE.

Compressed air is transmitted through 300 ft. of 2 in. pipe. The supply pressure is 100 lb. per sq. in., and the flow is 80 cub. ft. per min. at the supply end. Calculate the delivery pressure assuming the temperature remains at 15° C. throughout, and that $PV = 96T$ for 1 lb. of air. Prove any formula used. Take $f = \cdot005$. (London Univ.)

$$T = 15 + 273 = 288° \text{ C. abs.}$$

$$a = \frac{\pi}{4}\left(\frac{1}{6}\right)^2$$

$$= \cdot 0218 \text{ sq. ft.}$$

$$v_2 = \frac{Q}{a}$$

$$= \frac{80}{60 \times \cdot 0218} = 61 \cdot 15 \text{ ft. per sec.}$$

Applying Equation (5),

$$p_2 = p_1 \sqrt{1 - \frac{8flv_1^2}{2gdRT}}$$

$$= 100 \sqrt{1 - \frac{8 \times \cdot 005 \times 300 \times (61 \cdot 15)^2}{64 \cdot 4 \times \frac{1}{6} \times 96 \times 288}} \text{ lb. per sq. in.}$$

$$= 100 \sqrt{1 - \cdot 146}$$

$$= 92 \cdot 5 \text{ lb. per sq. in.}$$

188. **Adiabatic Flow through Pipe.** This type of flow occurs when the pipe is efficiently lagged so that no heat flows into the moving gas. As the gas flows along the pipe it is subjected to a frictional resistance causing a pressure drop. This pressure drop causes the gas to expand and thus increases its kinetic energy.

In practice, the energy lost in friction will partly reheat the gas, causing an increase in internal energy at the outlet end of the pipe which is theoretically equal to the loss of energy due to friction.

(*a*) Adiabatic Flow, Neglecting Reheating.

Let v = mean velocity in pipe

then, $h_f = \dfrac{4fvl^2}{2gd}$ (approximately)

Applying Equation (3) (Art. 185) to a horizontal pipe and allowing for the frictional loss h_f,

$$\left(\frac{\gamma}{\gamma - 1}\right) p_1 V_1 \left[1 - n^{\frac{\gamma-1}{\gamma}}\right] = \frac{v_2^2 - v_1^2}{2g} + h_f$$

that is, $$\left(\frac{\gamma}{\gamma - 1}\right) p_1 V_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right] = \frac{v_2^2 - v_1^2}{2g} + \frac{4flv^2}{2gd} \quad . \quad (1)$$

but from Equation (5) (Art. 180),

$$\frac{v_2}{v_1} = \frac{V_2}{V_1} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}}$$

hence,

$$v_2 = v_1 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}} \quad . \quad . \quad . \quad (2)$$

Substituting this value of v_2 in Equation (1),

$$\left(\frac{\gamma}{\gamma - 1}\right) p_1 V_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right] = \frac{v_1^2}{2g}\left[\left(\frac{p_1}{p_2}\right)^{\frac{2}{\gamma}} - 1\right] + \frac{4flv^2}{2gd} \quad . \quad (3)$$

The value of p_2 can be obtained from this equation by successive approximations. First assume the mean velocity v to be equal to v_1 and solve for p_2 by trial or by plotting. Repeat the process using the mean of v_1 and v_2 for v. No further adjustment will be necessary.

(*b*) Adiabatic Flow, with Reheating.

In this case it is assumed that the energy lost due to friction is absorbed by the gas in the form of heat. Hence, the total energy at the outlet end of the pipe is equal to the total energy at the inlet end. Owing to the effect of reheating the expansion is not a reversible adiabatic*; Equation (3) (Art. 185) can be applied, but the law of expansion is of the form $pv^n =$ constant.

Then, for horizontal pipe,

$$\left(\frac{n}{n - 1}\right) p_1 V_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}\right] = \frac{v_2^2 - v_1^2}{2g} \quad . \quad . \quad (4)$$

Substituting the value of v_2 from Equation (2),

$$\left(\frac{n}{n - 1}\right) p_1 V_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}\right] = \frac{v_1^2}{2g}\left[\left(\frac{p_1}{p_2}\right)^{\frac{2}{n}} - 1\right] \quad . \quad (5)$$

If n, p_1, V_1, and p_2 are known, the value of v_1 can be calculated from this equation.

Then, as $p_1 V_1^n = p_2 V_2^n$, V_2 can be obtained.

Also, as $\dfrac{v_2}{v_1} = \dfrac{V_2}{V_1}$, the velocity v_2 can be calculated.

189. **Variation of Atmospheric Pressure with Altitude.** The

* Some authorities apply the terms "adiabatic" and "isothermal" only to the reversible processes described in Arts. 179 and 180. Other authorities apply the term "adiabatic" to all insulated processes, and the term "isothermal" to all constant temperature processes; the more general application of these terms may cause considerable confusion.

pressure of the atmosphere at the earth's surface is due to the weight of the column of air above. This will not be directly proportional to the head, as in liquids, on account of the varying density of the air.

If the temperature of the column of air is assumed constant, the variation in pressure will follow an isothermal process and obeys Boyle's law. This is not the case in practice; it is found that the atmosphere gets colder as the altitude increases. On this account, it is more accurate to consider the pressure variation to follow the adiabatic law; that is, to assume there is no interchange of heat between the layers. The ratio between the temperature drop and the altitude is known as the *temperature gradient*.

In the solution of problems on the pressure of the atmosphere, the atmosphere is assumed to be *quiet* and to contain a constant amount of moisture; it is then said to be in *convective equilibrium*.

It is found from measurements that the true condition of the atmosphere lies between the isothermal and adiabatic processes, and it appears to follow a law

$$pv^n = \text{constant}$$

In aeronautical problems, a condition called the *International Standard Atmosphere* is assumed. This assumes the above law of variation, the value of n being 1·235. The solution of problems based on this law can be obtained from the equations for the adiabatic process by substituting the value of n for γ.

The variation of atmospheric pressure can also be obtained direct from the temperature gradient, if the latter is known.

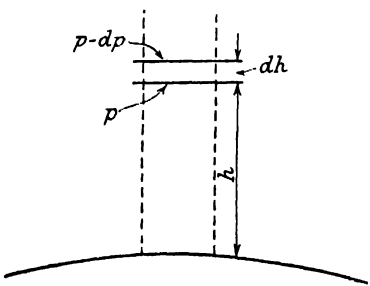


Fig. 225

Consider the column of atmospheric air of unit cross-sectional area (Fig. 225). Consider a section of the column at a height h above the earth's surface, and let the pressure at this height be p lb. per sq. ft. Now consider another section at a height of dh above the first section and let the pressure at this section be $p - dp$.

The difference of pressure between the two sections is equal to the weight of air between them. Or

$$-dp = w(dh \times 1)$$

that is, $$dp = -wdh \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let V = volume of 1 lb. of air between the two sections

then, $$w = \frac{1}{V}$$

Substituting this value of w in Equation (1),

$$dh = -V\,dp \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But, from the characteristic equation of a gas,

$$pV = RT$$

hence, substituting in Equation (2),

$$dh = -\frac{RT}{p}\,dp \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equations for the variation of atmospheric pressure with altitude will now be obtained for each of the four methods previously defined.

(1) Assuming Isothermal Process. If an isothermal process is assumed for the pressure variation, the temperature of the column remains constant. Then Equation (3) becomes

$$dh = -RT\frac{dp}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Let p_1 and p_2 be the pressures at two heights from the ground h_1 and h_2.

Integrating Equation (4),

$$\int_{h_1}^{h_2} dh = -RT\int_{p_1}^{p_2}\frac{dp}{p}$$

or $$h_2 - h_1 = -RT\log_e\frac{p_2}{p_1} \quad . \quad . \quad . \quad (5)$$

Equation (5) gives the pressure ratio between any two heights h_2 and h_1. If h_1 is zero it corresponds to the ground level, then p_1 is 14·7 lb. per sq. in. Equation (5) then becomes

$$h_2 = -RT\log_e\frac{p_2}{14{\cdot}7} \quad . \quad . \quad . \quad . \quad (6)$$

This equation gives the pressure p_2 in lb. per sq. in. at any altitude h_2 ft.

This assumption of an isothermal process does not give good results in practice for a large variation in altitude, on account of the alteration in temperature. For a large change in altitude the adiabatic law would be more accurate. It should be noticed that Equation (5) could also have been obtained by the application of the extended Bernoulli's equation [Equation (4), Art. 185] to the atmosphere at the two given altitudes, the velocity at each altitude being zero.

(2) ASSUMING ADIABATIC PROCESS. The law for an adiabatic change is

$$pV^\gamma = p_1 V_1^\gamma$$

where p and V apply to any height h above the earth's surface, and p_1 and V_1 apply to a particular height h_1.

For an adiabatic process (Art. 180)

$$p = p_1 \left(\frac{T_1}{T}\right)^{\frac{\gamma}{1-\gamma}} \qquad (7)$$

where T and T_1 are the absolute temperatures at altitudes h and h_1. Differentiating in terms of T,

$$dp = -\left(\frac{\gamma}{1-\gamma}\right) p_1 T_1^{\frac{\gamma}{1-\gamma}} T^{-\left(\frac{1}{1-\gamma}\right)} dT$$

Substituting the above values of p and dp in Equation (3),

$$dh = \frac{-RT\left[-\left(\frac{\gamma}{1-\gamma}\right) p_1 T_1^{\left(\frac{\gamma}{1-\gamma}\right)} T^{-\left(\frac{1}{1-\gamma}\right)}\right] dT}{p_1 T_1^{\left(\frac{\gamma}{1-\gamma}\right)} T^{-\left(\frac{1}{1-\gamma}\right)}}$$

$$= \left(\frac{\gamma}{1-\gamma}\right) R dT \qquad (8)$$

hence, $$\frac{dT}{dh} = -\left(\frac{\gamma-1}{\gamma}\right)\frac{1}{R} \qquad (9)$$

This is known as the temperature gradient.

Integrating Equation (8) between altitudes h_2 and h_1 and between the corresponding temperatures T_2 and T_1,

$$\int_{h_1}^{h_2} dh = \left(\frac{\gamma}{1-\gamma}\right) R \int_{T_1}^{T_2} dT$$

or $$(h_2 - h_1) = \left(\frac{\gamma}{1-\gamma}\right) R(T_2 - T_1)$$

$$= \left(\frac{\gamma}{1-\gamma}\right) T_1 R \left(\frac{T_2}{T_1} - 1\right) \quad . \quad . \quad . \quad (10)$$

But, from Equation (7), Art. 180,

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

Substituting this value of $\frac{T_2}{T_1}$ in Equation (10),

$$h_2 - h_1 = \left(\frac{\gamma}{1-\gamma}\right) RT_1 \left[\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] \quad . \quad . \quad . \quad (11)$$

If Equation (11) is applied to find the pressure at any height h_2 from the ground, h_1, T_2, and p_1 will apply to the ground-level; then $h_1 = 0$ and $p_1 = 14{\cdot}7$ lb. per sq. in.

Then, $$h_2 = \left(\frac{\gamma}{1-\gamma}\right) RT_1 \left[\left(\frac{p_2}{14{\cdot}7}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] \quad . \quad . \quad (12)$$

From this equation the value of p_2 in lb. per sq. in. can be calculated.

It should be noticed that Equation (11) could have been obtained more simply by the application of the extended Bernoulli's equation [Equation (3), Art. 185] to the atmosphere at the two given altitudes, the velocity at each being zero.

(3) ASSUMING THE TEMPERATURE GRADIENT IS KNOWN. If the temperature gradient is known, the variation of atmospheric pressure with altitude can be calculated from Equation (3).

Let T_0 = absolute temperature at ground-level

p_0 = atmospheric pressure at ground-level

t = temperature drop per ft. increase of altitude

T = temperature at any altitude h ft.

p = pressure at altitude h ft.

p_1 = pressure at any given altitude h_1 ft.

then, $T = T_0 - th$

Using Equation (3),

$$dh = -\frac{RTdp}{p}$$

$$= -\frac{B(T_0 - th)dp}{p}$$

hence,
$$\frac{dp}{p} = -\frac{dh}{R(T_0 - th)}$$

Integrating between the pressure limits p_0 and p_1 and between the altitude limits of 0 and h_1,

$$\log_e \left[p\right]_{p_0}^{p_1} = \frac{1}{Rt}\log_e \left[T_0 - th\right]_0^{h_1}$$

that is,
$$\log_e\left(\frac{p_1}{p_0}\right) = \frac{1}{Rt}\log_e\left(\frac{T_0 - th_1}{T_0 - 0}\right)$$

$$= \frac{1}{Rt}\log_e\left(1 - \frac{th_1}{T_0}\right) \quad . \quad . \quad . \quad (13)$$

This equation gives the pressure p_1 at any given altitude h_1.

(4) ASSUMING PROCESS FOLLOWS LAW pv^n = CONSTANT. The solution of this process is the same as for an adiabatic process except that n should be substituted for γ. Equation (11) then becomes

$$h_2 - h_1 = \left(\frac{n}{1 - n}\right) RT_1 \left[\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}} - 1\right] \quad . \quad . \quad (14)$$

For the international standard atmosphere the value of n is taken as 1·235.

EXAMPLE 1.

If the atmosphere at sea-level has a pressure of 14·7 lb. per sq. in. and a temperature of 60° F., find the pressure at a height of 10,000 ft. (1) Assuming an isothermal change, (2) assuming an adiabatic change. $\gamma = 1·41$ for air.

$T_1 = 460 + 60 = 520°$ F. abs.

(1) ASSUMING ISOTHERMAL CHANGE.

Using Equation (6),

$$h_2 = -RT\log_e\frac{p_2}{p_1}$$

that is,
$$10{,}000 = -53·3 \times 520 \log_e \frac{p_2}{14·7}$$

or
$$-·361 = 2·303 \log_{10}\frac{p_2}{14·7}$$

$$-·1565 = \log\frac{p_2}{14·7}$$

Taking antilogs, $\cdot 6974 = \dfrac{p_2}{14\cdot 7}$

From which, $p_2 = 10\cdot 24$ lb per sq. in.

(2) Assuming Adiabatic Change.

Using Equation (12).

$$h_2 = \left(\frac{\gamma}{1-\gamma}\right) RT_1 \left[\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$$

That is,

$$10{,}000 = -\left(\frac{1\cdot 41}{\cdot 41}\right) 53\cdot 3 \times 520 \left[\left(\frac{p_2}{14\cdot 7}\right)^{\frac{\cdot 41}{1\cdot 41}} - 1\right]$$

then, $$\cdot 1048 = 1 - \left(\frac{p_2}{14\cdot 7}\right)^{\cdot 2905}$$

or $$\cdot 8952 = \left(\frac{p_2}{14\cdot 7}\right)^{\cdot 2905}$$

Taking logs, $$-\cdot 0481 = \cdot 2905 \log\left(\frac{p_2}{14\cdot 7}\right)$$

or $$-\cdot 1654 = \log\left(\frac{p_2}{14\cdot 7}\right)$$

Taking antilogs, $\cdot 6832 = \dfrac{p_2}{14\cdot 7}$

From which $p_2 = 10\cdot 03$ lb. per sq. in.

Example 2.

Find the pressure and density of the atmosphere at a height of 12,000 ft. when the pressure, temperature, and density at ground-level are 14·7 lb. per sq. in., 15° C., and ·0765 lb. per cu. ft. respectively. Assume that the temperature of a quiescent atmosphere diminishes with the height at a uniform rate of 2° C. per 1000 ft. For air, $PV = 96\ T$. (London Univ.)

Using Equation (13),

$$\log_e\left(\frac{p_1}{p_0}\right) = \frac{1}{Rt}\log_e\left(1 - \frac{th_1}{T_0}\right)$$

In this example, $t = \cdot 002°$ C.

$T_0 = 288°$ C. abs.

$R = 96$

and $p_0 = 14\cdot 7$ lb. per sq. in.

then, $$\log_e\left(\frac{p_1}{14 \cdot 7}\right) = \frac{1}{96 \times \cdot 002}\log_e\left(1 - \frac{\cdot 002 \times 12{,}000}{288}\right)$$

from which $$\frac{p_1}{14 \cdot 7} = 638$$

hence, $$p_1 = 9 \cdot 38 \text{ lb. per sq. in.}$$

From the characteristic equation of a gas

$$\frac{p_1}{T_1} = \frac{96}{V_1}$$

hence, final density $$= \frac{1}{V_1} = \frac{144 \times 9 \cdot 38}{96 \times 264}$$

$$= \cdot 0534 \text{ lb. per cu. ft.}$$

190. **Flow of Gas Through Venturi Meter.** The flow of a gas through a Venturi may be regarded as an isothermal process if the pressure drop is small, in which case the flow can be

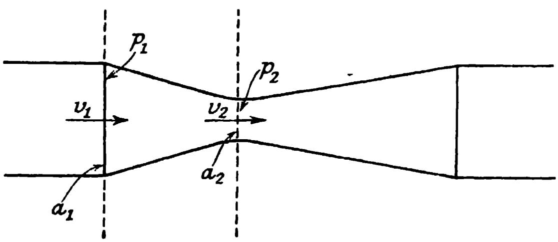


FIG. 226

calculated by the method given in Art. 27. If the pressure drop is appreciable, the flow will be adiabatic, and there is a rapid fall of temperature at the throat. In this case Equation (3) of Art. 185 can be used for calculating the quantity of flow.

Imagine a quantity of gas to be flowing through the Venturi meter of Fig. 226. Let suffix 1 apply to the conditions of the gas at entrance, and suffix 2 to the conditions at the throat. Neglect all losses, and assume flow to be adiabatic.

Let a_1 and a_2 be the areas of cross-section at inlet and throat, in sq. ft., and v_1 and v_2 the corresponding velocities.

Weight of gas flowing per sec. $= W = \dfrac{a_1 v_1}{V_1} = \dfrac{a_2 v_2}{V_2}$

where V_1 and V_2 represent the volume of 1 lb. gas at pressures p_1 and p_2 respectively.

hence, $$v_1 = \frac{a_2}{a_1}\left(\frac{V_1}{V_2}\right) v_2 \quad . \quad . \quad . \quad (1)$$

But, from Equation (5), Art. 180,

$$\frac{V_1}{V_2} = \left(\frac{p_2}{p_1}\right.$$

$$= n^{\gamma}$$

hence, from Equation (1), $$v_1 = \frac{a_2}{a} n^{\frac{1}{\gamma}} v_2 \quad . \quad . \quad . \quad (2)$$

Now, the flow through the meter is assumed to be adiabatic, and is consequently represented by Equation (3), Art. 185. Then, substituting Equation (2) in this equation,

$$(Z_1 - Z_2) + \left(\frac{\gamma}{\gamma - 1}\right) p_1 V_1 \left[1 - n^{\frac{\gamma-1}{\gamma}}\right] = \frac{v_2^2 - \left(\frac{a_2}{a_1}\right)^2 n^{\frac{2}{\gamma}} v_2^2}{2g}$$

If the Venturi meter is horizontal, $Z_1 = Z_2$, then,

$$\left(\frac{\gamma}{\gamma - 1}\right) p_1 V_1 \left[1 - n^{\frac{\gamma-1}{\gamma}}\right] = \frac{v_2^2}{2g}\left[1 - \left(\frac{a_2}{a_1}\right)^2 n^{\frac{2}{\gamma}}\right]$$

From which

$$v_2 = \sqrt{\frac{2g\left(\frac{\gamma}{\gamma - 1}\right) p_1 V_1 \left[1 - n^{\frac{\gamma-1}{\gamma}}\right]}{\left[1 - \left(\frac{a_2}{a_1}\right)^2 n^{\frac{2}{\gamma}}\right]}} .$$

Then, $$W = \frac{a_2 v_2}{V_2}$$

$$= \frac{a_2}{V_2}\sqrt{\frac{2g\left(\frac{\gamma}{\gamma - 1}\right) p_1 V_1 \left[1 - n^{\frac{\gamma-1}{\gamma}}\right]}{\left[1 - \left(\frac{a_2}{a_1}\right)^2 n^{\frac{2}{\gamma}}\right]}} \quad . \quad (3)$$

If the Venturi meter is not horizontal, the $(Z_1 - Z_2)$ term remains in the equation; then Equation (3) becomes

$$W = \frac{a_2}{V_2}\sqrt{\frac{2g\left[(Z_1 - Z_2) + \left(\frac{\gamma}{\gamma - 1}\right) p_1 V_1 \left\{1 - n^{\frac{\gamma-1}{\gamma}}\right\}\right]}{\left[1 - \left(\frac{a_2}{a_1}\right)^2 n^{\frac{2}{\gamma}}\right]}} \qquad (4)$$

Equations (3) and (4) give the weight of gas flowing, in lb. per sec.

EXAMPLE.

A horizontal air Venturi meter is installed in a 6 in. pipe line, its throat diameter being 2 in. The inlet pressure measured 140 lb. per sq. in. abs. and the throat pressure 130 lb per sq. in. abs. The temperature at inlet was 15° C. Calculate the weight of air flowing per second. $R = 96$ ft. lb. units and $\gamma = 1{\cdot}4$.

As $\qquad p_1 V_1 = RT_1$

$$V_1 = \frac{96 \times 288}{144 \times 140}$$

$$= 1{\cdot}37 \text{ cu. ft. per lb.}$$

As expansion is assumed to be adiabatic,

$$V_2 = V_1\left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}}$$

$$= 1{\cdot}37\left(\frac{140}{130}\right)^{\frac{1}{1\cdot4}}$$

$$= 1{\cdot}443 \text{ cu. ft. per lb.}$$

also, $\qquad n = \frac{p_2}{p_1}$

$$= \frac{130}{140} = {\cdot}9285$$

Using Equation (3),

$$W = \frac{a_2}{V_2}\sqrt{\frac{2g\left(\frac{\gamma}{\gamma - 1}\right)p_1 V_1\left[1 - n^{\frac{\gamma-1}{\gamma}}\right]}{\left[1 - \left(\frac{a_2}{a_1}\right)^2 n^{\frac{2}{\gamma}}\right]}}$$

$$= \frac{\frac{\pi}{4} \times \frac{1}{36}}{1{\cdot}443} \sqrt{\frac{64{\cdot}4 \times \frac{1{\cdot}4}{{\cdot}4} \times 144 \times 140 \times 1{\cdot}37 \left[1 - ({\cdot}9285)^{\frac{{\cdot}4}{1{\cdot}4}}\right]}{\left[1 - \frac{1}{81}({\cdot}9285)^{\frac{2}{1{\cdot}4}}\right]}}$$

$$= {\cdot}0151 \sqrt{1{,}323{,}000}$$

$$= 17{\cdot}4 \text{ lb. per sec.}$$

191. **Flow of Gas Through an Orifice or Nozzle.** The flow of gas through an orifice or nozzle may be regarded as an adiabatic process if the pressure drop is large. For a small pressure drop the process may be assumed isothermal; the isothermal flow of a gas through orifices and pipes is dealt with in Arts. 32 and 74 respectively.

The adiabatic flow through an orifice (Fig. 277) is represented by Equation (3), Art. 185. In this case $Z_1 = Z_2$, so that the

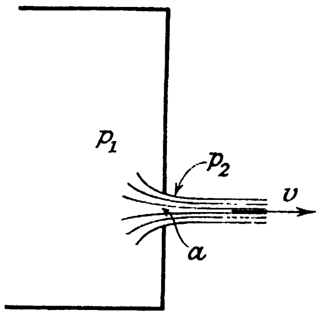


FIG. 227

term $(Z_1 - Z_2)$ is zero. Also, $v_1 = 0$, as the flow commences from rest. Let the final velocity of the gas be represented by v. Then, Equation (3), Art. 185, becomes

$$\left(\frac{\gamma}{\gamma - 1}\right) p_1 V_1 \left[1 - n^{\frac{\gamma - 1}{\gamma}}\right] = \frac{v^2}{2g} \quad . \quad . \quad (1)$$

where p_1 and V_1 apply to the initial state of 1 lb. of the gas and n is the ratio of the pressures $\left(\frac{p_2}{p_1}\right)$ on both sides of the orifice or nozzle. Then, from Equation (1),

$$v = \sqrt{2g\left(\frac{\gamma}{\gamma - 1}\right) p_1 V_1 \left[1 - n^{\frac{\gamma - 1}{\gamma}}\right]}$$

Let a = effective area of orifice or throat area of nozzle in sq. ft.

$= C_d \times$ area of orifice.

Weight of gas flowing per sec. $= W = \dfrac{av}{V_2}$

where V_2 is the volume of 1 lb. of gas at pressure p_2.

Then,

$$W = \frac{a}{V_2}\sqrt{2g\left(\frac{\gamma}{\gamma - 1}\right)p_1 V_1\left[1 - n^{\frac{\gamma-1}{\gamma}}\right]} \quad . \quad . \quad (2)$$

But, it was proved in Art. 190 that

$$\frac{V_1}{V_2} = n^{\frac{1}{\gamma}}$$

hence,
$$V_2 = \frac{V_1}{n^{\frac{1}{\gamma}}}$$

Substituting this value of V_2 in Equation (2),

$$W = \frac{a}{V_1}\sqrt{2g\left(\frac{\gamma}{\gamma - 1}\right)p_1 V_1\, n^{\frac{2}{\gamma}}\left[1 - n^{\frac{\gamma-1}{\gamma}}\right]} \quad . \quad (3)$$

This equation gives the weight of discharge in lb. per sec.

From Equation (3) it will be seen that, for a gas in the given initial condition, the only variable is n. Hence, if this equation is differentiated in terms of n and equated to zero, the value of n for a maximum weight of discharge through the orifice would be obtained.

From Equation (3),

$$W \propto n^{\frac{2}{\gamma}}\left[1 - n^{\frac{\gamma-1}{\gamma}}\right]$$

that is,
$$W \propto n^{\frac{2}{\gamma}} - n^{\frac{\gamma+1}{\gamma}}$$

Differentiating,
$$\frac{dW}{dn} = \frac{2}{\gamma}n^{\frac{2-\gamma}{\gamma}} - \frac{\gamma + 1}{\gamma}n^{\frac{1}{\gamma}} = 0$$

That is,
$$(\gamma + 1)\, n^{\frac{1}{\gamma}} = 2n^{\frac{2-\gamma}{\gamma}}$$

Hence,
$$n^{\frac{\gamma-1}{\gamma}} = \frac{2}{\gamma+1} \quad . \quad . \quad . \quad (4)$$

Equation (4) gives the value of n, the pressure ratio, for a maximum weight of discharge for any orifice and for any gas. Applying this equation to air, $\gamma = 1{\cdot}404$, Equation (4) then becomes

$$n^{\frac{1\cdot404-1}{1\cdot404}} = \frac{2}{1\cdot404+1}$$

From which
$$n = \cdot528$$

hence,
$$n = \frac{p_2}{p_1} = \cdot528$$

or
$$p_2 = \cdot528\, p_1 \quad . \quad . \quad . \quad . \quad (5)$$

Thus, the maximum weight of discharge for air flowing through an orifice is when the final pressure is ·528 of the initial pressure.

EXAMPLE.

A nozzle having a throat diameter of 1 in. is fitted into the side of a tank containing air at a pressure of 120 lb. per sq. in. and at a temperature of 15° C. What should be the back pressure for a maximum discharge through the nozzle? Calculate the weight of this maximum discharge in lb. per second. Assume $C = 96$ and $\gamma = 1{\cdot}404$.

Using Equation (5) for maximum discharge,

$$\begin{aligned} p_2 &= \cdot528\, p_1 \\ &= \cdot528 \times 120 \\ &= 63{\cdot}4 \text{ lb. per sq. in.} \end{aligned}$$

As
$$p_1 V_1 = 96\, T$$

then,
$$\begin{aligned} V_1 &= \frac{96 \times 288}{144 \times 120} \\ &= 1{\cdot}6 \text{ cu. ft. per lb.} \end{aligned}$$

$$\begin{aligned} n &= \frac{p_2}{p_1} \\ &= \cdot528 \end{aligned}$$

Using Equation (3),

$$W = \frac{a}{V_1}\sqrt{2g\left(\frac{\gamma}{\gamma - 1}\right)p_1 V_1\, n^{\frac{2}{\gamma}}\left[1 - n^{\frac{\gamma-1}{\gamma}}\right]}$$

$$= \frac{\pi}{4 \times 144 \times 1{\cdot}6} \times$$

$$\sqrt{64{\cdot}4 \times \frac{1{\cdot}404}{{\cdot}404} \times 144 \times 120 \times 1{\cdot}6 \times {\cdot}528^{\frac{2}{\gamma}}\left[1 - {\cdot}528^{\frac{\gamma-1}{\gamma}}\right]}$$

$$= {\cdot}0034\ \sqrt{420{,}000}$$

$$= 2{\cdot}2 \text{ lb. per sec.}$$

192. **Buoyancy of Balloon.** The *lift* of a balloon is mainly due to the weight of atmospheric air it displaces. Archimedes' principle applies to bodies immersed in a gas in the same way as it applies to bodies immersed in liquids.

A balloon consists of a light fabric container filled with a gas of less density than the atmosphere; its gross lift is equal to the weight of atmosphere it displaces. Its actual lift is the gross lift less the weight of the container and the gas inside the container. This would have its maximum value if the inside of the container was a vacuum, but this condition is impossible as the container would collapse inwards due to the atmospheric pressure on the outside. To prevent this, the container is filled with the lightest gas possible at a pressure approximately equal to that of the atmosphere. The container is thus prevented from collapsing inwards, but its actual lift is reduced by the weight of the gas used. It should be noted that the balloon gets no lift from this gas, which merely acts as an internal structure inserted to resist the external pressure of the atmosphere. The gas used is hydrogen or helium.

Consider the spherical balloon of Fig. 228, and consider a vertical column of gas of height h and cross-sectional area a as shown.

Let p = pressure of atmosphere at base of container

w_a = average weight of 1 cu. ft. of air surrounding container

w_g = average weight of 1 cu. ft. of gas inside container

Assume the pressure of gas at base of container to be the same as that of the atmosphere at the base. Now, the pressure of gas at the top of the column will be less than that at the base by an amount equal to the weight of the column of gas. In the same way, the pressure of the atmosphere at the top of the container is less than that at the base by an amount equal to the weight of a corresponding column of atmosphere. Or,

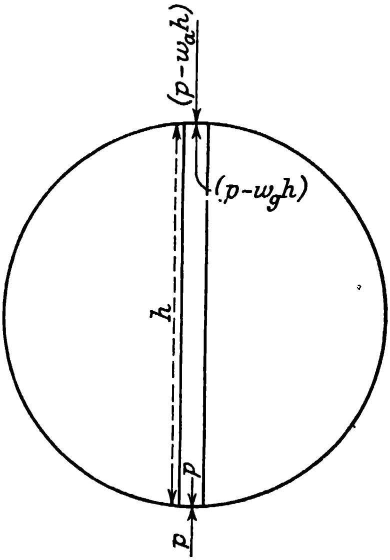


FIG. 228

pressure of gas at top $=$ $p - w_g h$ lb. per sq. ft.

pressure of atmosphere at top $= p - w_a h$ lb. per sq. ft.

As w_a is greater than w_g it follows that the gas pressure at the top is greater than that of the atmosphere at the top; hence, there is an upward force on the container.

Considering the column of gas and ignoring the weight of the container,

upward force on column $= a(p - w_g h)$

downward force on column $= a(p - w_a h)$

Subtracting, net upward force on column

$$= -a\, w_g h + a w_a h$$
$$= (w_a - w_g) ah$$
$$= (w_a - w_g) \times \text{volume of column} \quad . \quad . \quad . \quad (1)$$

If the whole container is imagined to be made up of similar columns, it follows that

lift on container $= (w_a - w_g) \times$ volume of container.

The weight of the container must be subtracted from this amount for the actual lift of the balloon.

It will be seen from Equation (1) that the apparent lift of the gas is equal to $(w_a - w_g)$ lb. per cu. ft.

For hydrogen at N.T.P., w_a $= \cdot 0756$ lb. and $w_g = \cdot 0053$

then, apparent lift of hydrogen $= \cdot 0756 - \cdot 0053$

$= \cdot 0703$ lb. per cu. ft.

As the hydrogen can only be obtained at about 97 per cent pure, this figure is reduced in practice to about ·068 lb. per cu. ft.

For helium at N.T.P., w_g $= \cdot 0116$ lb. per cu. ft.

then, apparent lift of helium $= \cdot 0756 - \cdot 0116$

$= \cdot 064$ lb. per cu. ft.

This is reduced to ·062 lb. if a purity of 97 per cent is assumed.

It will be noticed from Fig. 228 that the buoyancy of the balloon is due to the upward pressure on the lower half being larger than the downward pressure on the upper half; but the lift on the container itself is due to the pressure of the gas on the upper portions. In the case of a rigid airship, the containers press on to a rigid structure, being in direct contact with a network of wires fixed circumferentially around the structure. The manner in which the apparent lift of the gas is distributed around the circumference of the structure will depend on the tautness and on the method of fixing of these wires.

As a balloon rises to a higher altitude, the atmosphere is less dense; consequently, the weight of air displaced by the balloon is less; this reduces the lift. As the balloon rises the inward pressure on the outside becomes less. This produces an excess of internal gas pressure which is liable to burst the container. In order to prevent this, an automatic valve is fitted at the base so that gas may escape as the altitude increases. The containers of an airship are filled to 95 per cent of their volume when on the ground; they will then be just full when the ship has risen to its cruising height; consequently, no gas is wasted.

It should be noted that the lift of a balloon or airship will depend on the barometer reading, the temperature of the atmosphere, and on the heating of its gas by the sun's rays; the latter process is termed *superheating*.

193. **Sensitive Manometers.** (*a*) Chattock Tilting Gauge. A very sensitive instrument for measuring the difference of

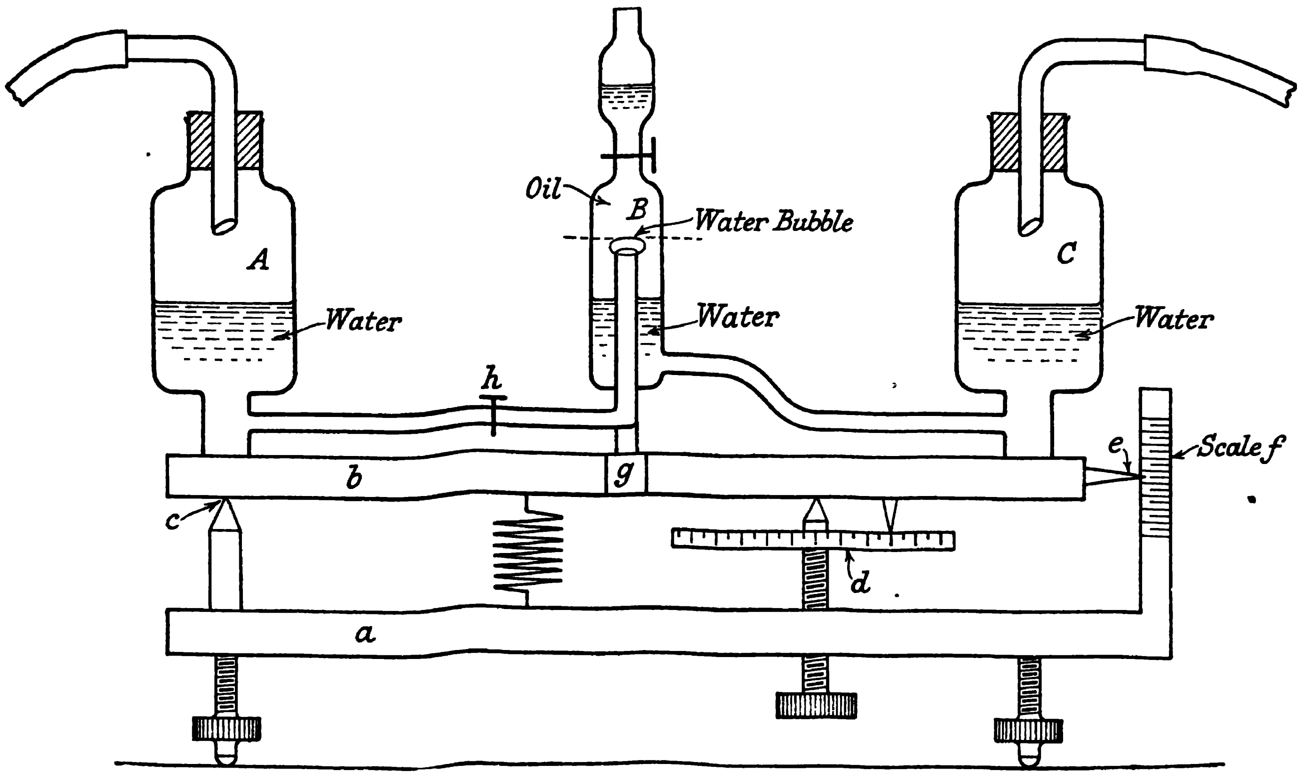


Fig. 229

pressure between the two limbs of a Pitot tube is the Chattock tilting gauge (Fig. 229). The instrument consists of a lever *a*, carrying a scale *f* at one end, which is fixed horizontally by means of three levelling screws. The left-hand end carries a knife edge *c* on which is pivoted another lever *b*. The lever *b* can be tilted by means of the micrometer screw *d* which carries a large disc graduated around its edge. The vertical movement of the right-hand end of lever *b* can be measured by the pointer *e* and by the rotation of the micrometer disc.

Flasks *A* and *C* are fixed to the tilting lever *b*, as shown; a smaller flask *B* is fixed at the centre. The flasks *B* and *C* are connected at their bases by a glass tube, and contain water as shown. The upper portion of *B* is filled with oil. The flask *A* contains water; its base is connected to the oil in *B* by a glass tube which has its outlet in the oil in *B*. The high pressure limb of the Pitot tube is connected to the air in *A*, whilst the low pressure limb is connected to the air in *C*.

The pressure in *A* causes a water bubble to form at the end of the tube in the oil of *B*. The level of the upper surface of this water bubble can be sighted on the cross-wire of a microscope supported at *g*. An increase of pressure in *A* increases the size of this water bubble in B; a decrease of pressure in *C* also causes the size of the bubble to increase.

The instrument is initially set by sighting on to the bubble's upper surface with the cock *h* closed; the readings of the pointer *e* and the micrometer disc being noted. The cock *h* is then opened, the pressure difference between *A* and *C* will now cause an increase in size of the bubble. The lever *b* is now tilted by revolving the micrometer disc until the upper surface of the bubble again coincides with the cross-wire of the microscope. This operation has brought the size of the water bubble back to the original state, showing that the total pressures of the water in *A* and *C* are again equal. The readings of *e* and the micrometer disc are now taken; by subtracting the initial readings from these, the difference of pressure head in the two limbs of the Pitot tube can be calculated.

The instrument is extremely sensitive; differences of pressure can be measured as small as ·0001 in. of water. It is used for Pitot tube measurements of air velocities in wind channels. It may also be modified to be used for measuring pressure differences in other types of fluids.

(*b*) Inclined Tube Manometer. If one arm of a U-tube is

bent so that it has a small inclination to the horizontal, the movement of the liquid in this inclined tube will be considerable for a small change in pressure head. Hence, a U-tube can be made very sensitive by bending one of its arms until it is nearly horizontal; this is the principle of the inclined tube manometer.

A view of the Krall type of inclined tube manometer is shown in Fig. 230. The inclined tube D corresponds to one arm of the U-tube, the other arm being the large reservoir C. The tubes from the Pitot-static tube are connected to the arms C and D at the openings A and B, the higher pressure tube being connected at A. By making the liquid surface in C very many times larger than the cross-sectional area of the tube D, the variation of height of the liquid surface in C will

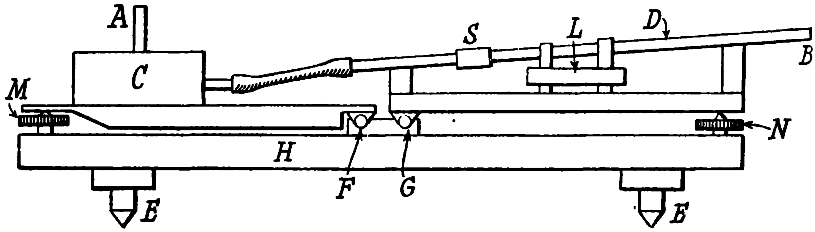


FIG. 230

be extremely small and can, therefore, be regarded as constant. The pressure difference in the two arms will consequently be given by the reading of the liquid surface in the inclined tube D.

The instrument is mounted on a base H, which can be set horizontally by three levelling screws E. The frame supporting the reservoir C is hinged at F and can be raised or lowered by turning the screw M; this adjusts the zero reading of the liquid in the inclined tube D. The frame supporting the inclined tube D is hinged at G; its inclination can be varied by means of the screw N. The liquid level in this tube can be read off a fixed scale by the sliding cross-wire S.

The instrument can be used for measuring two ranges of air speed by fixing the inclined tube at two different slopes. This is done by means of two spirit levels L, one on each side of the tube, which are permanently set at different inclinations. When it is desired to measure low air speeds, the inclined tube D is set at a small inclination by using the spirit level set with

the least permanent slope. If it is desired to measure high air speeds, the inclined tube can be set at a greater slope as larger changes of head will now occur; this is done by setting the tube D to the other spirit level, which has been permanently fixed at a greater slope. It will be seen from this that there are two ranges for the instrument, depending on which spirit level is used for setting the inclination of the tube D.

Alcohol is found to be the best liquid for use in this instrument as it provides a suitable meniscus for the inclined tube. The instrument can be used as a sensitive U-tube for measuring

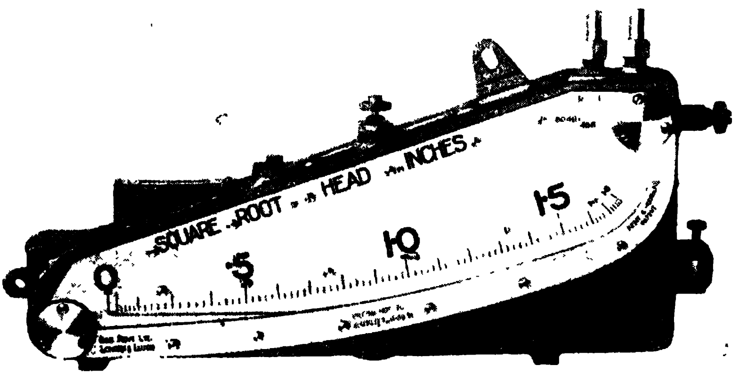


FIG. 231

any fluid pressure differences within its range; its use is not limited to the Pitot-static tube.

(c) THE CURVED TUBE MANOMETER. This is the same in principle as the inclined tube manometer, but the tube is curved in such a form that a uniform scale of velocities is obtained. A view of the Kent-Hodgson Curved Tube Manometer* is shown in Fig. 231. If this instrument is permanently installed in a fixed position for use in connection with a particular apparatus, such as a Pitot-static tube, a velocity scale can be substituted for the pressure-difference scale, from which air velocities can be read off direct. If the installation is of a permanent nature, such as for the measurement of air flow through a duct or pipe of known dimensions, a quantity scale

* By courtesy of Messrs. Geo. Kent, Ltd., Luton.

can be substituted from which the quantity of flow can be obtained directly from the scale reading.

A great advantage of the curved tube manometer is that a wide range of air speeds, or pressure differences, can be quickly read off with the one setting of the instrument. This manometer is for the measurement of low pressures only and will read pressure differences of ·01 in. of water.

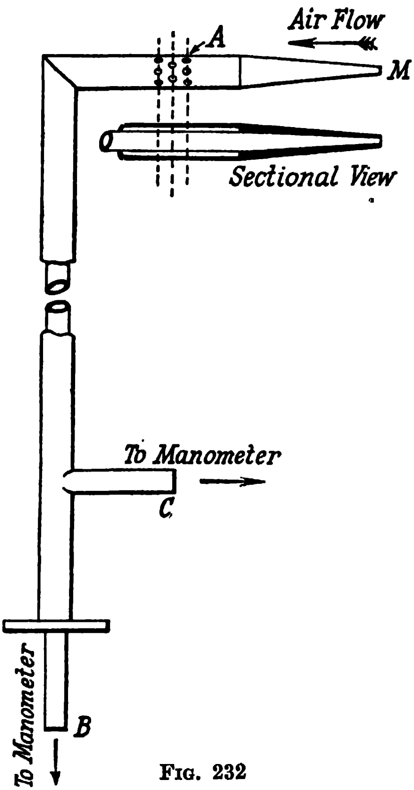


FIG. 232

194. **Measurement of Air Speed.** The following instruments are used for measuring the velocity of an air stream such as wind velocity, the speed of air flow in ducts, or the air flow in an experimental wind tunnel.* They may also be used for measuring the velocity of a body moving in air, such as an aeroplane. In this case, the Pitot-static tube is used and the velocity obtained is the relative velocity of the plane to the air. In order to obtain the absolute velocity of the plane, allowances must be made for the velocity of the wind.

(1) PITOT-STATIC TUBE. A simple Pitot tube cannot be used for the measurement of air velocity if the air is under static pressure, as the latter would affect the reading of the instrument. In order to overcome this, the Pitot-static tube has been designed so that the measured pressure difference gives the required velocity head of the air.

An outside view and a sectional view of a Pitot-static tube are shown in Fig 232. It consists of an internal L-shaped tube which forms the mouth of the instrument at M, the

* For a more detailed account of measurement of air speed, see *The Measurement of Air Flow*, by E. Ower.

other end B being connected to one of the limbs of a suitable manometer. This inner tube is surrounded by an L-shaped outer tube so that an air space is provided between the two tubes; a ring of holes in the outer tube at A admits the air to this air space. The static pressure of the air can thus be transmitted along the air space to its outlet at C, which is connected to the other limb of the manometer. The instrument is placed with its mouth facing the air stream; the head measured is then the velocity head only of the air stream, as its static pressure is transmitted to both limbs of the manometer and is, therefore, eliminated. The instrument must first be calibrated for the range over which it is used in order to obtain the values of its coefficient k. Then,

$$v = k\sqrt{2gh}$$

It is found that good results are obtained with the Pitot-static tube if the centre of the ring of holes at A is about 2 in. away from the mouth. In the N.P.L. Standard Pitot-static Tube the inner tube has internal and external diameter of ·16 in. and ·0204 in. respectively; the thickness of the air space between the tubes is ·032 in.

(2) Vane Anemometer. This instrument, which is the same in principle as the current meter described in Art. 85, is used for measuring wind velocities for meteorological purposes and for measuring air velocities in large ducts such as ventilating shafts. For the latter purpose it usually consists of a rotor containing eight vanes, and is the same in principle as an axial flow turbine. The vanes may be flat plates set at a suitable angle to the direction of the air stream, or they may be curved in the same manner as a turbine blade. The rotor is connected by gearing to a revolution counter, which indicates on the dials incorporated in the instrument the number of revolutions made. As the speed of revolution of the rotor is proportional to the air-stream velocity, the latter can be obtained by noting the number of revolutions made by the rotor over a known interval of time.

To use the instrument, it is placed in the air stream and the initial readings of the dials noted. Then, by using a stop watch, the time is taken for a given number of revolutions of the rotor. From these results the air velocity can be calculated from the calibration curve for the instrument used.

The Negretti and Zambra* vane anemometer, shown in Fig. 233, is a portable instrument for indicating the number of linear feet of air. A number of light vanes are mounted on a spindle running on jewelled bearings; by means of a suitable gearing, the rotation of the spindle is communicated to the pointers.

The instrument is held in the air stream, preferably on a rod, and the number of feet of air passing the instrument is

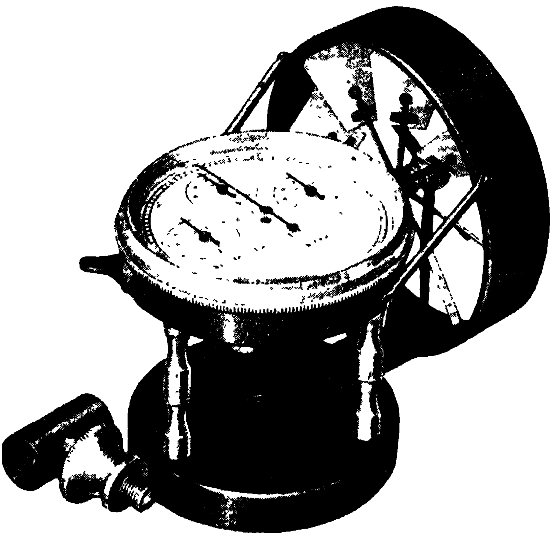


FIG. 233

timed with the aid of a stop watch. A correction is usually required, which is obtained from a calibration factor supplied. A disconnector is provided for throwing the indicating mechanism out of mesh, and a setting device for bringing the hands back to zero.

The vane anemometer gives accurate results over a limited range only; for any given range of air speeds an instrument should be employed which suits that particular range.

(3) HOT-WIRE ANEMOMETERS. Another method of measuring the velocity of an air stream is by measuring the rate of heat loss from an electrically heated body immersed in the air stream; the rate of heat loss is proportional to the velocity

* By courtesy of Messrs. Negretti & Zambra, London.

of the air impinging on the hot body. The hot body usually consists of a short length of platinum or nickel wire which is arranged to form one of the arms of a Wheatstone bridge; a manganin resistance forms the opposite arm.

There are two methods of measuring air velocity with the hot-wire anemometer: one by maintaining a constant temperature in the wire, the other by keeping the electric current constant. In the former method the resistance to the passage of the electric current through the wire remains constant, as the resistance is proportional to the temperature; consequently, the current required to maintain a constant temperature is proportional to the velocity of the air stream. In the

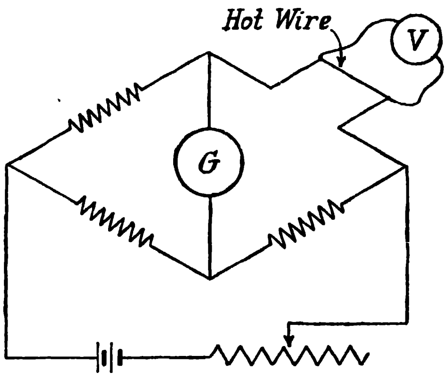


FIG. 234

latter method the electric current is kept constant, consequently the resistance varies with the temperature of the wire; the resistance is thus porportional to the air speed.

The constant current method gives the best results, but the constant resistance method is usually used as it is easier to maintain a constant resistance than a constant current.

(*a*) *Constant Resistance Method.* The arrangement of the Wheatstone bridge is shown in Fig. 234. The hot wire is exposed to the air stream, which tends to cool it; an increase in the electric current is then necessary to restore the temperature of the wire to its former value. By this means the temperature and resistance are maintained constant and the current is proportional to the air speed.

The bridge is kept in balance by varying the current which may be measured by an ammeter in series with the hot wire,

or by a high resistance voltmeter connected across the wire as shown in Fig. 234. Once the instrument is calibrated the air speed can be obtained from the reading of the ammeter or voltmeter.

(*b*) *Constant Current Method.* In this form of the instrument the electric current passing through the hot wire is kept constant. Referring to Fig. 234, the galvanometer G of the constant resistance method is replaced by a milliammeter; this will register any out-of-balance current passing through the bridge. An increase in air speed tends to cool the hot wire, which, in turn, lowers its resistance to the flow of the electric current passing through it. This puts the bridge out-of-balance, the out-of-balance current being registered by the milliammeter. The out-of-balance current is thus proportional to the air speed, the value of which can be obtained from the calibration curve of the instrument. The constant current method is used for low air speeds.

EXAMPLES 16.

(1) Calculate the velocity of a pressure wave transmitted through a liquid having a specific gravity of ·85 and a bulk modulus of 284,000 lb. per sq. in.

Ans.—4985 ft. per sec.

(2) Calculate the velocity of sound in air having a pressure of 9·38 lb. per sq. in. abs. and a temperature of $-$ 9° C.

(*a*) Assuming an isothermal process.

(*b*) Assuming an adiabatic process.

R for air = 96 ft. lb. centigrade units and $\gamma = 1{\cdot}4$.

Ans.—(*a*) 900 ft. per sec.
(*b*) 1065 ft. per sec.

(3) Find the pressure and density of the atmosphere at an altitude of 13,000 ft. assuming an isothermal atmosphere. At sea-level, $p = 14{\cdot}7$ lb. per sq. in., and $t = 15°$ C. R for air = 96 ft. lb. centigrade units.

Ans.—9·2 lb. per sq. in.; ·048 lb. per cu. ft.

(4) If the pressure and temperature of the atmosphere at ground level are 14·7 lb. per sq. in. and 15° C. respectively, calculate the pressure and density at an altitude of 16,000 ft. assuming an adiabatic atmosphere. Find also the mean temperature gradient up to this altitude. $R = 96$ ft. lb. units and $\gamma = 1{\cdot}4$.

Ans.—7·83 lb. per sq. in., ·049 lb. per cu. ft., 3° C. per 1000 ft.

(5) Calculate the weight of air flowing through a horizontal Venturi meter having an inlet diameter of 4 in. and a throat diameter of 2 in. The absolute pressures at inlet and throat were found to be 60 lb. per sq. in. and 50 lb. per sq. in. respectively; the temperature at inlet was 20° C. Assume $R = 96$ ft. lb. centigrade units and $\gamma = 1{\cdot}4$.

Ans.—3·41 lb. per sec.

(6) Air from a large vessel discharges into the atmosphere from a small orifice placed in its side. The pressure and temperature of the air in the vessel are 30 lb. per sq. in. abs. and 15° C. respectively. The diameter of the orifice is 1 in. Assuming R and γ for air to be 96 ft. lb. C. units and 1·4 respectively, calculate the weight of air discharging per second. The atmospheric pressure is 15 lb. per sq. in. and C_d for the orifice = ·64.

Ans.—·351 lb. per sec.

(7) An airship filled with hydrogen has a gas capacity of 7,000,000 cu. ft. If the weight of air at N.T.P. is ·0756 lb. per cu. ft. and the weight of hydrogen at N.T.P. is ·0053 lb. per cu. ft., what is the total lift of the airship at N.T.P.?

Calculate the loss of lift if the airship is filled with helium instead of hydrogen. 1 cu. ft. of helium weighs ·0116 lb. at N.T.P.

Ans.—219·5 tons; 19·5 tons.

(8) Show that a disturbance is propagated with velocity $\sqrt{\frac{k}{\rho}}$ in a fluid of elasticity k and density ρ. Determine this speed for water, k = 300,000 lb. per sq. in. (London Univ.)

Ans.—4730 ft. per sec.

(9) Describe the Pitot tube method of velocity measurement. Comment on its accuracy and working range. Describe a manometer suitable for use with a Pitot tube measuring air speeds of the order of 40 ft. per sec.

A Pitot tube gives a pressure difference of 8 in. of water when placed in an air stream at 750 mm. barometer and 18° C. temperature. What is the speed?

ρ = ·0807 lb. per cu. ft. at N.T.P.

(London Univ.)

Ans.—189·5 ft. per sec.

(10) Given that the barometer pressure is p_0 at ground-level where the temperature is 15° C., prove that the pressure p at a height h ft. is given by the expression

$$\log \frac{p}{p_0} = A \log (1 - Bh)$$

in which A and B are constants. If the temperature of a quiescent atmosphere diminishes with the height at a uniform rate of 2° C. per 1000 ft., and for air $pV = 96T$, find the values of A and B. (London Univ.)

Ans.—A = 5·21; B = ·00000695.

(11) A Venturi meter having an inlet diameter of 3 in. and a throat diameter of 1 in. is used for measuring the rate of flow of air through a pipe. Mercury U-gauges register pressures at the inlet and throat equivalent to 250 mm. and 150 mm. of mercury respectively.

Determine the volume of air flowing through the pipe in cusecs. Assume that flow takes place between the inlet and throat under adiabatic conditions (γ = 1·4) and that the density of the air at inlet is ·10 lb. per cu. ft. The barometric pressure is 760 mm. (London Univ.)

Ans.—1·68 cusecs.

(12) A Venturi meter, whose inlet and throat diameters are 12 in. and 4 in. respectively, is employed for measuring the flow of air.

Calculate the flow in cu. ft. per min. at N.T.P. given the following data: The difference of pressure between the entrance to the meter and the throat is ·6 in. of water; the pressure in the pipe at the entrance to the meter, as

registered by a water manometer, is 5 in., the temperature is 20° C., and the barometric height is 29·83 in. of mercury. The coefficient of discharge for the meter is ·96. Neglect compressibility and take $PV = 96\,T$ for air.

(London Univ.)

***Ans.*—245 cu. ft. per min.**

(13) Find the diameter of a sharp-edged orifice suitable for measuring the discharge from an air compressor which deals with 50 cu. ft. of "free" air at 14·7 lb. per sq. in. and 15° C.

The orifice is to be fitted to the top of a large cylindrical vessel into which air from the compressor passes and is then discharged into the atmosphere through the orifice. The pressure of air inside the vessel is 1 in. of water and the temperature is 20° C. Assume that the density of the air is constant through the orifice, and that $PV = 96T$ for air, and take $C_d = \cdot 6$ for the orifice.

Sketch a manometer suitable for the measurement of the pressure inside the vessel. (London Univ.)

Ans.—2 in.

(14) Sketch an arrangement of a Pitot-static tube combined with a suitable manometer for the measurement of current velocity of an air stream. Give the theory of the Pitot tube and comment on the accuracy to be expected from an instrument of this kind.

If the manometer registers ·2 in. of water when the Pitot-static tube is placed in a current of air at a temperature of 20° C. and pressure of 750 mm. Hg., find the velocity of the air. Take $PV = 96T$. (London Univ.)

Ans.—30·0 ft. per sec.

(15) Prove that $\frac{v^2}{2g} + \int \frac{dp}{\rho} =$ constant for the steady flow of a compressible fluid. Hence, show that Bernoulli's equation for the flow of a gas can be written,

$$\frac{v^2}{2g} + \frac{\gamma}{\gamma - 1} \cdot \frac{p}{\rho} = \text{constant},$$

in which the pressure and density are related by the adiabatic law $\left(\frac{p}{\rho^\gamma}\right) =$ constant. Hence derive an expression for the theoretical flow of a gas in lb. per sec. through a Venturi meter in terms of the pressure and density of the gas, and the sectional area of the meter, at the entrance and at the throat sections. (London Univ.)

(16) State Bernoulli's equation for the frictionless adiabatic flow of a gas, and apply it to calculate the theoretical flow in lb. per hr. of hydrogen gas through a horizontal Venturi meter given the following information: Diameter of meter at inlet, 3 in., and at throat, 1 in.; the pressure is 800 mm. of mercury and the temperature is 15° C. at inlet, and the pressure is 765 mm. of mercury at the throat. For hydrogen $PV^{1\cdot 4} =$ constant for adiabatic expansion, and $R = 1{,}385$ ft. lb. per lb. per °C. (London Univ.)

***Ans.*—386 lb. per hr.**

(17) Prove that the maximum continuous discharge of air through a convergent nozzle, fitted into the side of a large vessel, takes place when the pressure in the throat of the nozzle is ·528 of the constant pressure of the air in the vessel.

Find the diameter of a nozzle suitable for measuring the discharge from an air compressor which deals with 250 cu. ft. per min. of atmospheric air at

14·7 lb. per sq. in. and 15° C. The nozzle is fitted into the side of a large vessel into which air is discharged from the compressor, and the pressure and temperature in the vessel are 33 lb. per sq. in. and 27° C. Assume a coefficient of discharge for the convergent nozzle of ·99, that for 1 lb. of air $PV = 96T$, and that $\gamma = 1 \cdot 4$. (London Univ.)

Ans.—d = ·579 in.

(18) Find the pressure and density of the atmosphere at a height of 3,000 ft. when the pressure and temperature at ground level are 30 in. of mercury and 15° C. Assume that the temperature of the atmosphere diminishes with the height at a uniform rate of 1·5° C. per 1,000 ft. For 1 lb. of air $PV = 96T$. (London Univ.)

Ans.—13·18 lb. per sq. in.; ·0698 lb. per cu. ft.

CHAPTER XVII

NON-DIMENSIONAL FACTORS

195. **Non-dimensional Factors.** In Art. 139 a non-dimensional factor, known as the Reynolds number, was defined. This factor has an important influence on all problems dealing with the viscous resistance of a fluid. In Art. 145 the existence of this non-dimensional factor was also proved by applying Lord Rayleigh's principle of dimensional similarity to the problem of viscous resistance of a fluid. The Reynolds number was proved to have the value $\frac{vd}{\nu}$. By substituting the three fundamental units of mass, length, and time in this equation it is found to be non-dimensional; that is, it is merely a ratio. Consequently, the Reynolds number is known as a non-dimensional factor. It was also shown in Art. 145 that the wave resistance to a body in a fluid also depends on a non-dimensional factor known as the Froude number.

On applying the principle of dimensional similarity to other problems in fluid mechanics, such as orifices, weirs, pressure waves, etc., it is found that they are governed by other non-dimensional factors. The experimental coefficients used in these problems are not true constants for all conditions of flow; their value will depend on the non-dimensional factor of the particular problem considered.

The non-dimensional factors for some of the fluid phenomena already dealt with will now be obtained by the application of the principle of dimensional similarity. In addition to the fundamental units of quantities given in Art. 145, the following values are required—

$$g = LT^{-2}$$

$$\text{Volume per sec.} = Q = L^3T^{-1}$$

$$\text{surface tension} = f = MT^{-2}$$

$$\text{bulk modulus} = K = ML^{-1}T^{-2}$$

196. **Non-dimensional Factor for Viscous Resistance.** This case was dealt with in Art. 145. The non-dimensional factor

obtained is known as the Reynolds number and was found to be

$$\frac{\rho v d}{\mu} \quad \text{or} \quad \frac{vd}{\nu}$$

197. **Non-dimensional Factor for Surface Wave Resistance.** This case was dealt with in Art. 145. The non-dimensional factor obtained is the square of the Froude number (Art. 145) and was found to be

$$\frac{v^2}{Lg}$$

This factor governs the resistance of surface ships to wave formation and has an influence on wave formation in channel flow (Art. 89).

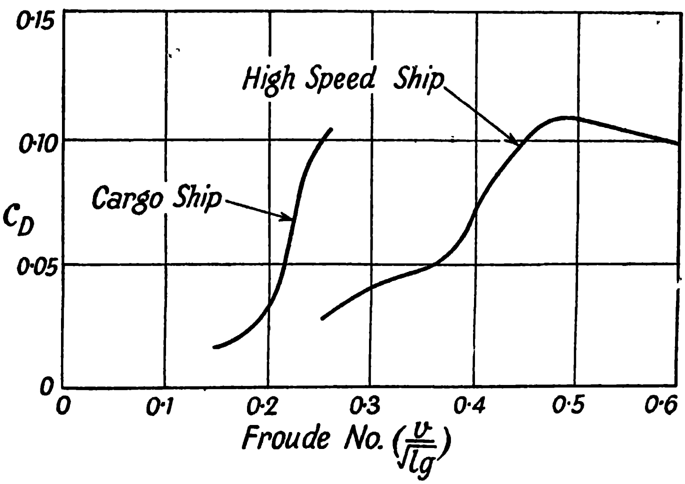


FIG. 234A

When dealing with the wave resistance of surface ships, the resistance can be expressed in the same form as the drag formula used for the resistance of aircraft (Art. 158).

Let D_w = drag or resistance of ship due to wave formation

C_D = drag coefficient due to wave resistance only

A = immersed cross-sectional area amidships.

Then, $D_w = C_D \dfrac{A\rho v^2}{2}$

The drag coefficient C_D is a function of the Froude number.

In Fig. 234A curves have been plotted from experimental results showing the variation of C_R with the Froude number.* The two curves are for a cargo ship and for a high-speed ship respectively; the curves do not coincide, because the two ships are not geometrically similar.

198. **Non-dimensional Factor for Compression Wave Resistance.** Compression waves are formed by the relative movement of a body completely submerged in a fluid; they also occur in the transmission of sound and will depend on the bulk elastic modulus K (Art. 182). Let R be the resistance to the body caused by the formation of the pressure wave. Assume,

$$R = k\rho^a L^b v^c K^d \quad . \quad . \quad . \quad (1)$$

where k is an experimental coefficient, L the linear dimension, v the relative velocity between the body and fluid, and a, b, c, d are unknown indices of which the values are to be found. Substituting the fundamental dimensions in this equation.

$$MLT^{-2} = k(ML^{-3})^a L^b (LT^{-1})^c (ML^{-1}T^{-2})^d$$

then,

$$MLT^{-2} = k(M^a L^{-3a})(L^b)(L^c T^{-c})(ML^{-d}T^{-2d})$$

Equating the indices of M,

$$1 = a + d$$

from which $\quad a = 1 - d$

Equating the indices of T,

$$-2 = -c - 2d$$

from which $\quad c = 2 - 2d$

Equating the indices of L,

$$1 = -3a + b + c - d$$

substituting for the values of a and c,

$$1 = -3 + 3d + b + (2 - 2d) - d$$

hence, $\quad b = 2$

* From "Experimental Fluid Dynamics Applied to Engineering Practice," by G. A. Hankins, *Engineering*, Vol. 157, 25th February and 3rd March, 1944.

Substituting these values of a, b, and c in Equation (1),

$$R = k\rho L^2 v^2 \left(\frac{K}{\rho v^2}\right)^d$$

$$= k\rho L^2 v^2 \phi\left(\frac{K}{\rho v^2}\right) \quad . \quad . \quad . \quad . \quad (2)$$

where ϕ means "a function of."

Hence, the constant k is a function of $\left(\frac{K}{\rho v^2}\right)$, which is the non-dimensional factor governing this type of resistance.

It was shown in Art. 182 that the velocity of sound in a fluid was given by the equation

$$v_s = \sqrt{\frac{K}{\rho}}$$

where v_s is the velocity of sound in the fluid under consideration. Substituting this value in the non-dimensional factor, then,

$$k \text{ is a function of } \left(\frac{v_s^2}{v^2}\right)$$

or it may be written as a function of $\frac{v}{v_s}$

The non-dimensional factor in this form is known as the *Mach number*, and is used as a criterion when dealing with bodies having velocities in the vicinity of the velocity of sound (Art. 220).

It is evident from Equation (2) that the experimental coefficient k is not a true constant for all fluids, but varies with the elastic constant, the density, and the velocity. For any given fluid, k will vary with the velocity.

199. **Non-dimensional Factor for Weirs.** The experimental coefficients used in problems on rectangular and triangular weirs or notches (Arts. 52–56) are not true constants for all heads, but can be shown to vary with such quantities as μ, ρ, H, g and the surface tension f. It will be assumed that the weirs or notches considered are geometrically similar. Then, for a rectangular weir or notch, the breadth B is proportional to H. For a triangular notch, the angle θ is constant (Art. 53).

Assume the equation for the discharge is in the form,

$$Q = k\,H^a g^b \mu^c \rho^d f^e \quad . \quad . \quad . \quad . \quad (1)$$

Substituting the fundamental dimensions in this equation,

$$L^3T^{-1} = kL^a(L^bT^{-2b})(M^cL^{-c}T^{-c})(M^dL^{-3d})(M^eT^{-2e})$$

Equating the indices of M,

$$0 = c + d + e$$

from which $\quad d = -c - e$

Equating the indices of T,

$$-1 = -2b - c - 2e$$

from which $\quad b = \frac{1}{2} - \frac{1}{2}c - e$

Equating the indices of L,

$$3 = a + b - c - 3d$$

Substituting for b and d,

$$3 = a + (\tfrac{1}{2} - \tfrac{1}{2}c - e) - c + (3c + 3e)$$

from which $\quad a = 2\frac{1}{2} - 1\frac{1}{2}c - 2e$

Substituting these values of a, b, and d in Equation (1),

$$Q \propto H^{\frac{5}{2}} g^{\frac{1}{2}} \times \left(\frac{\mu}{H^{\frac{3}{2}} g^{\frac{1}{2}} \rho}\right)^c \left(\frac{f}{H^2 g \rho}\right)^e \quad . \quad . \quad (2)$$

Hence, $\quad k$ is a function of $\left(\frac{\mu}{H^{\frac{3}{2}} g^{\frac{1}{2}} \rho}\right)\left(\frac{f}{H^2 g \rho}\right)$

Thus, the coefficient of discharge of a rectangular or triangular notch, or weir, will vary with the head, the viscosity, the density and the surface tension. It is not, therefore, a constant as was assumed in Arts. 52–56. The actual variation of C_d is small, and it is of sufficient accuracy for practical purposes to assume C_d to be a constant within reasonable limits of head.

As the breadth B has been assumed to be proportional to H in a rectangular weir, Equation (2) when applied to a rectangular weir becomes

$$Q = k\,BH^{\frac{3}{2}} g^{\frac{1}{2}} \left(\frac{\mu}{H^{\frac{3}{2}} g^{\frac{1}{2}} \rho}\right)^c \left(\frac{f}{H^2 g \rho}\right)^e$$

For true dynamical similarity to hold for geometrically similar weirs it follows that

$$\frac{\mu}{H^{\frac{3}{2}} g^{\frac{1}{2}} \rho} \text{ is a constant for both weirs, and}$$

$$\frac{f}{H^2 g \rho} \text{ is a constant for both weirs}$$

It will be noticed that these two conditions cannot occur simultaneously. The first of these is the Reynolds number, as $v \propto \sqrt{2gH}$.

From the results of the two weir experiments shown in Figs. 59 and 61 it will be seen from the deviation of the points from the straight line that C_d is not actually a constant, but is varying slightly with the head. This confirms the above analytical results.

EXAMPLE.

Calculate the values of the two non-dimensional constants of a rectangular weir, given in Equation (2). The water flowing over the weir has a head of 4 ft., a kinematic viscosity of $8{\cdot}42 \times 10^{-6}$ f.p.s. units, and a surface tension of $5{\cdot}04 \times 10^{-3}$ lb. per ft.

From Equation (2), the two non-dimensional constants are

(*a*) $$\frac{\mu}{H^{\frac{3}{2}} g^{\frac{1}{2}} \rho} \quad \text{or} \quad \frac{\nu}{H^{\frac{3}{2}} g^{\frac{1}{2}}}$$

and (*b*) $$\frac{f}{H^2 g \rho}$$

Hence, the value of (*a*) is

$$\frac{H^{\frac{3}{2}} g^{\frac{1}{2}}}{\nu} = \frac{8{\cdot}42 \times 10^{-6}}{(4)^{\frac{3}{2}} \times \sqrt{32{\cdot}2}}$$

$$= 1{\cdot}86 \times 10^{-7}$$

The value of (*b*) is

$$\frac{f}{H^2 g \rho} = \frac{5{\cdot}04 \times 10^{-3}}{16 \times 32{\cdot}2 \times \frac{62{\cdot}4}{32{\cdot}2}}$$

$$= 5{\cdot}05 \times 10^{-6}$$

200. Non-dimensional Factor for Small Orifice. It was

shown in Art. 37 that the discharge through a small orifice under a constant head H (Fig. 235) is given by the equation

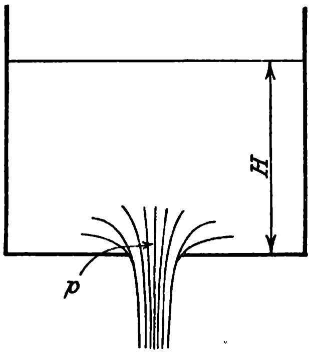


FIG. 235

$$Q = C_d A\sqrt{2gH}$$

where A is the area of the orifice in sq. ft. and C_d is the coefficient of discharge. This equation may be written

$$Q = kD^2\sqrt{gH} \quad . \quad . \quad (1)$$

where k is a constant and D the diameter of the orifice.

If the principle of dimensional similarity be applied to this problem it is found that k is not a true constant, but varies with such factors as the head, viscosity, density, gravity and surface tension.

Let p = mean pressure at orifice

then, $p = wH$

$= \rho gH$

Applying the principle of dimensional similarity to this problem, and ignoring the effect of surface tension, as small,

$$Q = kp^aD^bg^c\rho^d\mu^e$$

Then, inserting the fundamental dimensions,

$$L^3T^{-1} = k(M^aT^{-2a}L^{-a})L^b(L^cT^{-2c})(M^dL^{-3d})(M^eL^{-e}T^{-e})$$

Equating the indices of M,

$$0 = a + d + e$$

hence, $\quad a = -d - e$

Equating the indices of T,

$$-1 = -2a - 2c - e$$
$$= (+2d + 2e) - 2c - e$$

from which $\quad d = -\frac{1}{2} - \frac{1}{2}e + c$

hence, $\quad a = \frac{1}{2} - \frac{1}{2}e - c$

Equating the indices of L,

$$3 = -a + b + c - 3d - e$$
$$= (-\tfrac{1}{2} + \tfrac{1}{2}e + c) + b + c + (\tfrac{3}{2} + \tfrac{3}{2}e - 3c) - e$$

from which $b = 2 + c - e$

Hence, substituting for a, b, and c in the fundamental equation, and separating all terms containing e and c,

$$Q = kp^{\frac{1}{2}}D^2\rho^{-\frac{1}{2}}\left(\frac{\mu}{D\rho^{\frac{1}{2}}p^{\frac{1}{2}}}\right)^e\left(\frac{D\rho g}{p}\right)^c$$

Substituting $p = \rho gH$,

$$Q = k\sqrt{\frac{\rho gH}{\rho}}\,D^2\left(\frac{\mu}{D\rho^{\frac{1}{2}}\rho^{\frac{1}{2}}g^{\frac{1}{2}}H^{\frac{1}{2}}}\right)^e\left(\frac{D\rho}{H\rho}\right)^c$$

$$= k\sqrt{gH}\;D^2\left(\frac{\mu}{D\rho\sqrt{2gH}}\right)^e\left(\frac{D}{H}\right)^c$$

Hence, the coefficient C_d is a function of $\left(\frac{\mu}{D\rho\sqrt{2gH}}\right)\left(\frac{D}{H}\right)$

The second term shows that for true geometrical similarity $D \propto H$; then, if the head and orifice diameter are varied to suit this condition, the value of C_d should be a function of

$$\left(\frac{\mu}{D\rho\sqrt{2gH}}\right)$$

Substituting for $v = \sqrt{2gH}$, this constant may be written

$$\frac{\rho Dv}{\mu}$$

It will be noticed that this form of the non-dimensional constant is the same form as the Reynolds number.

Fig. 236 shows the results of experimental measurements of C_d for a certain orifice. In this figure C_d is plotted against $\log\left(\frac{\rho Dv}{\mu}\right)$. It will be noticed that a smooth curve was not obtained. In these experiments D was not proportional to H; hence true geometrical similarity was not attained. If the experiments had been conducted such that the diameter of the orifice was varied with the head, $\frac{D}{H}$ being a constant, a more uniform result would have been obtained.

The curve of Fig. 236 is probably also affected slightly by other factors, such as surface tension.

This application of the principle of similarity to an orifice

shows that the coefficient of discharge is not a constant, but varies with the head, the density, the viscosity, and consequently with the temperature.

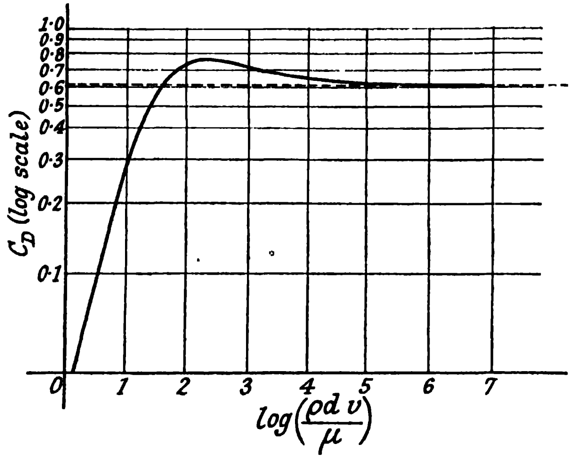


FIG. 236

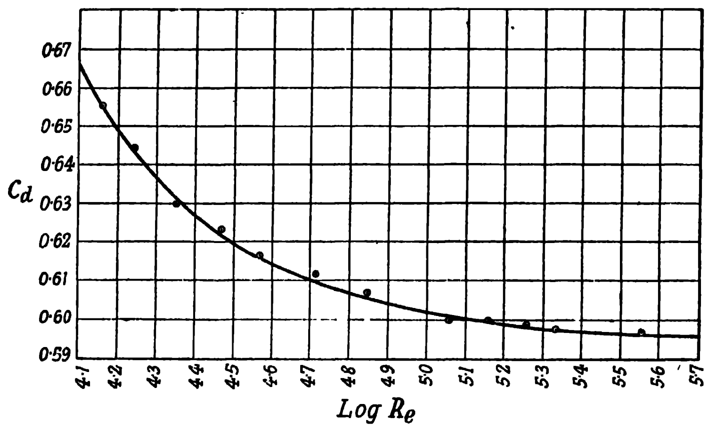


FIG. 237

In Fig 237 the author has plotted, on a base representing R_e, the results of Hamilton-Smith's experiments on round orifices. It will be noticed that the points lie on a smooth curve and that the value of C_d falls as R_e increases, becoming stabilized at a value of ·597. This curve corresponds to the portion of curve of Fig. 236 situated to the right of $R_e = 4$.

EXAMPLE 1.

One of the non-dimensional constants for the discharge through a small orifice was found to be $\frac{\mu}{D\rho\sqrt{2gH}}$. Calculate the value of this constant for water, having a coefficient of viscosity of ·01 C.G.S. units, discharging through a 1 in. diameter orifice under a head of 2·0 ft.

From Art. 138,

$$\mu = \frac{\cdot 01 \times 30\cdot 5}{453\cdot 6 \times 32\cdot 2} \text{ ft. lb. sec. units}$$

and $$\rho = \frac{62\cdot 4}{32\cdot 2}$$

hence, $$\nu = \frac{\mu}{\rho} = \frac{\cdot 01 \times 30\cdot 5}{453\cdot 6 \times 32\cdot 2} \times \frac{32\cdot 2}{62\cdot 4}$$

$$= 10\cdot 79 \times 10^{-6} \text{ ft. units}$$

Then, the value of the non-dimensional constant is

$$\frac{\nu}{D\sqrt{2gH}} = \frac{10\cdot 79 \times 10^{-6}}{\frac{1}{12}\sqrt{64\cdot 4 \times 2}}$$

$$= 11\cdot 42 \times 10^{-6}$$

EXAMPLE 2.

Using the curve of Fig. 236 find the value of C_d for a circular orifice of 1 in. diameter through which water is discharging under a head of 2 ft. The viscosity of the water is $3\cdot 18 \times 10^{-5}$ f.p.s. units.

The non-dimensional constant for an orifice was found to be of the form

$$\frac{\rho D v}{\mu}$$

where $$\rho = \frac{w}{g}$$

Now, $$v = \sqrt{2gH}$$

$$= \sqrt{64\cdot 4 \times 6}$$

$$= 19\cdot 62 \text{ ft. per sec.}$$

Then, $$\frac{\rho D v}{\mu} = \frac{62\cdot 4 \times \frac{1}{12} \times 11\cdot 33}{32\cdot 2 \times 3\cdot 18 \times 10^{-5}}$$

$$= 5\cdot 74 \times 10^{4}$$

hence, $$\log \frac{\rho D v}{\mu} = 4\cdot 759$$

Then, from the curve of Fig. 236,

$$C_d = \cdot 62$$

201. **Non-dimensional Factor for Orifice in Pipe.** Consider a pipe of diameter D containing an orifice of diameter d fitted in a diaphragm as shown in Fig. 238.

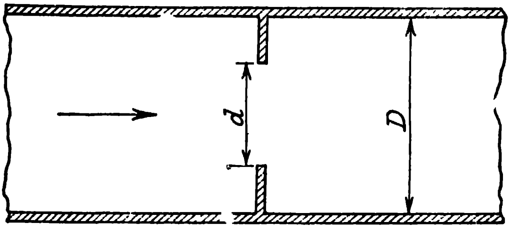


FIG. 238

Let p = difference of pressure between the two sides of the orifice, due to loss of head.

This problem produces non-dimensional factors which can be found by applying the principle of dimensional similarity. Assume

$$Q = kp^a d^b D^c \rho^d \mu^e \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then, substituting the fundamental dimensions in this equation,

$$L^3T^{-1} = k(M^aL^{-a}T^{-2a})\ L^bL^c(M^dL^{-3d})(M^eL^{-e}T^{-e})$$

Equating the indices of M,

$$0 = a + d + e$$

hence, $$d = -a - e \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equating the indices of T,

$$-1 = -2a - e$$

hence, $$a = \tfrac{1}{2} - \tfrac{1}{2}e$$

Substituting this value of a in Equation (2),

$$d = -\tfrac{1}{2} - \tfrac{1}{2}e$$

Equating the indices of L,

$$3 = -a + b + c - 3d - e$$

Substituting for a and d,

$$3 = (-\tfrac{1}{2} + \tfrac{1}{2}e) + b + c + (\tfrac{3}{2} + \tfrac{3}{2}e) - e$$

from which $\quad b = 2 - e - c$

Substituting these values of a, b, and d in Equation (1),

$$Q \propto p^{\frac{1}{2}} d^2 \rho^{-\frac{1}{2}} \left(\frac{\mu}{dp^{\frac{1}{2}}\rho^{\frac{1}{2}}}\right)^e \left(\frac{D}{d}\right)^c \quad . \quad . \quad (3)$$

Hence, the constant k is a function of $\left(\frac{\mu}{dp^{\frac{1}{2}}\rho^{\frac{1}{2}}}\right)$, which corresponds to the Reynolds number, and is also a function of $\left(\frac{D}{d}\right)$.

For true dynamical similarity both of these conditions must hold.

From Equation (3),

$$Q = kd^2 \sqrt{\frac{p}{\rho}}$$

202. **Non-dimensional Factor for Resistance of Oiled Bearings.** The viscous resistance of an oiled bearing depends on the linear dimensions of the bearing, the coefficient of viscosity of the oil, the speed of rotation, and on the pressure on the bearing. This resistance causes the frictional torque or moment on the bearing. A non-dimensional constant for oiled bearings having true dimensional similarity can be obtained by equating the fundamental dimensions.

Let R = frictional resisting torque on bearing

N = speed of shaft in r.p.m.

D = linear dimension of bearing

p = pressure per unit area on bearing

Then, $R \propto \mu N D p$

Let $R = k\mu^a N^b D^c p^d \quad . \quad . \quad . \quad . \quad . \quad (1)$

Inserting the fundamental dimensions, M, L, and T,

$$ML^2T^{-2} = k(M^aL^{-a}T^{-a})(T^{-b})(L^c)(M^dT^{-2d}L^{-d})$$

From M,

$$1 = a + d,$$

hence, $a = 1 - d$

From L,

$$2 = -a + c - d$$
$$= -(1 - d) + c - d$$

hence, $c = 3$

From T,

$$-2 = -a - b - 2d$$
$$= -(1-d) - b - 2d$$

hence, $\quad b = 1 - d$

Substituting these values of a, b, and c in Equation (1),

$$R = k\mu^{1-d}N^{1-d}D^3p^d$$

hence, $$R = k\mu ND^3\left(\frac{p}{\mu N}\right)^d$$

or $$R = \mu ND^3\phi\left(\frac{p}{\mu N}\right)$$

Let suffix 1 and suffix 2 apply to two geometrically similar bearings under comparison. Then

$$\frac{R_1}{R_2} = \frac{\mu_1 N_1 D_1^3\phi\left(\frac{p_1}{\mu_1 N_1}\right)}{\mu_2 N_2 D_2^3\phi\left(\frac{p_2}{\mu_2 N_2}\right)} \quad . \quad . \quad . \quad . \quad (2)$$

For true dimensional similarity the non-dimensional constants $\frac{p_1}{\mu_1 N_1}$ and $\frac{p_2}{\mu_2 N_2}$ must cancel, as explained in Art 145.

Then $$\frac{p_1}{\mu_1 N_1} = \frac{p_2}{\mu_2 N_2}$$

from which $$N_2 = N_1\frac{p_2\mu_1}{p_1\mu_2}$$

This is the corresponding speed for true dimensional similarity; if this value of N_2 is used, the non-dimensional constant in Equation (2) will cancel.

Substituting this corresponding speed in Equation (2),

$$\frac{R_1}{R_2} = \frac{\mu_1 N_1 D_1^3}{\mu_2\left(\frac{N_1 p_2\mu_1}{p_1\mu_2}\right)D_2^3}$$
$$= \frac{p_1 D_1^3}{p_2 D_2^3}$$

Hence, $$R \propto pD^3$$

That is, for similar bearings under the same pressure, the resistance moment is proportional to the cube of the linear dimensions.

It will be seen from this that the non-dimensional factor for geometrical similar oiled bearings is

$$\frac{\mu N}{p}$$

and the frictional coefficient for the oiled bearing will be a function of this parameter.

Fig. 238A shows the variation of the coefficient of friction with the non-dimensional factor, taken from tests*; two curves are shown, one for bearings having an L/D ratio less

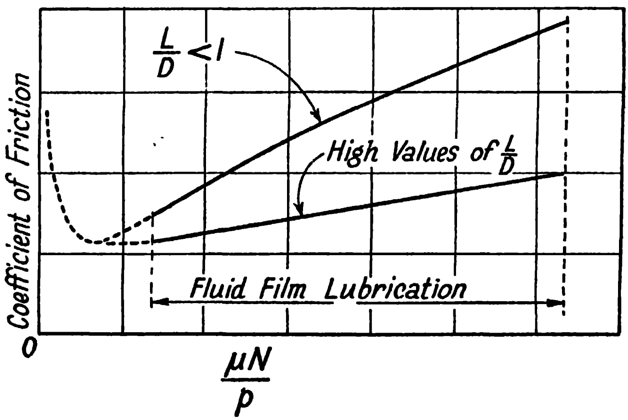


FIG. 238A

than unity, the other for bearings having high values of L/D. It will be noticed that in each case the value of the frictional coefficient is a function of the non-dimensional factor.†

203. **Resistance of Sphere Moving in Fluid.** The motion of a sphere moving through a fluid depends on the density and viscosity of the fluid and on the radius and velocity of the sphere. Hence, the resistance to motion R will be an equation of the form

$$R \propto \rho^a \mu^b r^c v^d \quad . \quad . \quad . \quad . \quad (1)$$

where r is the radius of the sphere and v its linear velocity.

* From "Experiment Fluid Dynamics Applied to Engineering Practice," by G. A. Hankins, *Engineering*, Vol. 157.

† For further information, see "The Film Lubrication of the Journal Bearing," by R. O. Boswall and J. C. Brierley, *Proc. Inst. Mech. Engs.*, Vol. 122 (1932), page 423.

Substituting the fundamental dimensions in the above equation,

$$MLT^{-2} = (ML^{-3})^a (ML^{-1}T^{-1})^b L^c (LT^{-1})^d$$

Equating the indices of M,

$$1 = a + b$$

from which $a = 1 - b$ . . . . . . (2)

Equating the indices of T,

$$-2 = -b - d$$

from which $b = 2 - d$ . . . . . . (3)

Substituting this value of b in Equation (2),

$$a = 1 - 2 + d$$
$$= -1 + d$$ . . . . . . (4)

Equating the indices of L,

$$1 = -3a - b + c + d$$

Substituting the values of b and a from Equations (3) and (4),

$$1 = 3 - 3d - 2 + d + c + d$$

hence, $c = d$ . . . . . . (5)

Substituting in Equation (1) the values of a, b, and c from Equations (3), (4), and (5),

$$R \propto \rho^{-1+d} \mu^{2-d} r^d v^d$$

then, $R \propto \dfrac{\mu^2}{\rho} \phi \left(\dfrac{\rho r v}{\mu}\right)^d$ . . . . . . (6)

or, $R = k_1 \dfrac{\mu^2}{\rho}$

where k_1 is a function of $\dfrac{\rho r v}{\mu}$, which is the Reynolds number.

If the velocity of the sphere is very small, the motion is of a laminar type of flow, in which case $R \propto v$ (Art. 140, Equation (8)). Applying this condition to Equation (1), it follows that the suffix d is unity. Substituting this value of d in Equation (6),

$$R = k \frac{\mu^2}{\rho} \times \frac{\rho r v}{\mu}, \text{ where } k \text{ is a constant}$$

from which $R = k\mu r v$ . . . . . . (7)

An equation of the same form as Equation (7) was deduced by Stokes for the resistance of a sphere moving in a fluid

with a uniform velocity. He calculated the value of k to be 6π. Substituting this value in Equation (7),

$$R = 6\pi\mu r v \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Equation (8) is known as Stokes' law. This law is used in the grading of fine powders, the particles of which are too small to be measured by direct means. The powder is mixed in suspension in a suitable liquid and the time taken for the powder to settle is measured. If the powder is assumed to consist of spherical-shaped particles it should approximately obey Stokes' law in settling to the bottom of the container. The time taken by the powder in settling is thus a function of the diameter of its particles; this provides a method of grading. It will be noticed that the finer powders take a longer period in being completely deposited.

Stokes' law is also used to calculate the amount of silting which may occur in problems dealing with channel flow and weirs. The silting is due to the depositing of fine particles of sand or earth which is in suspension in the flowing water. Whilst the water has a high velocity, these particles remain in suspension, but when the velocity becomes small or stationary the particles are deposited on the stream bed at a rate given by Stokes' law. This is noticeable at the inside edge of river bends, in backwaters, and on the stagnant side of dams and weirs.

204. **Non-dimensional Factor for Propellers.** The principle of dimensional similarity can be applied to screw propellers, which are geometrically similar, by considering the thrust of the propeller T, the diameter of the propeller d, the speed of advance v, and its speed of rotation n. Let ρ and μ apply to the fluid in which the propeller is immersed. Assume T is proportional to ρ, d, v, n, and μ; then

$$T \propto \rho^a d^b v^c n^e \mu^f \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Inserting the fundamental dimensions for these factors,

$$MLT^{-2} = (ML^{-3})^a\, L^b\, (LT^{-1})^c\, (T^{-1})^e\, (ML^{-1}T^{-1})^f$$

then, $MLT^{-2} = M^aL^{-3a}L^bL^cT^{-c}T^{-e}M^fL^{-f}T^{-f}$

Equating the indices of M,

$$1 = a + f$$

hence, $$a = 1 - f \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equating the indices of T,

$$-2 = -c - e - f$$

$$c = 2 - e - f \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equating the indices of L,

$$1 = -3a + b + c - f$$

Substituting the values of a and c from Equations (2) and (3),

$$1 = -3 + 3f + b + 2 - e - f - f$$

from which $$b = 2 - f + e \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting in Equation (1) the values of a, b, and c from Equations (2), (3), and (4),

$$T \propto \rho^{1-f} d^{2-f+e} v^{2-e-f} n^e \mu^f$$

hence, $$T = \rho d^2 v^2 \left(\frac{\mu}{\rho d v}\right)^f \cdot \left(\frac{dn}{v}\right)^e$$

or, $$T = k d^2 v^2$$

where k is a function of $\left(\frac{\mu}{\rho d v}\right)$ and $\left(\frac{dn}{v}\right)$

Hence, for true dynamical similarity, both of these conditions should be satisfied. The first of these conditions is the Reynolds number.

If a geometrically similar model of a propeller is tested in a fluid in order to predict the performance of the propeller, it will be impossible to satisfy both of these conditions in the same test. The method adopted is to neglect the Reynolds number and to use the second non-dimensional factor $\frac{dn}{v}$ as the criterion for dynamical similarity. This means that the model is tested at such a speed that

$$\frac{dn}{v} \text{ for model} = \frac{dn}{v} \text{ for large propeller}$$

The efficiency of the propeller can be obtained by comparing the work done per sec. by the thrust T to the work done per sec. by the torque on shaft. Or,

$$\text{efficiency of propeller} = \frac{T \times v}{\text{torque on shaft} \times \omega}$$

where ω = angular velocity of propeller shaft in radians per sec.

$$= \frac{2\pi \text{ r.p.m.}}{60}$$

205. **Non-dimensional Constants by Group Method.** The method of application of the principle of dimensional similarity demonstrated in Arts. 195–202 can be simplified by dealing with the variables in groups of four. This group method considerably reduces the labour when a large number of variables are being considered. In applying the group method the following rules should be observed—

(1) Choose three variables, preferably those which occur in the fundamental equation, and which contain all three fundamental dimensions M, L, and T.

(2) Form groups containing all the above three plus each of the other variables, in turn.

(3) Give indices a, b, and c to the three variables of (1) only.

As an example, apply the method to the orifice of Art. 200 and Fig. 235. Consider the variables to be Q, p, D, g, ρ, μ, and surface tension f. Now,

$$0 = \phi(QpDg\rho\mu f)$$

Let Q, p and D be the three variables chosen to satisfy Rule (1).

Then, $$0 = \phi(Q^a p^b D^c g\rho\mu f)$$

GROUP 1. $$0 = \phi(Q^a p^b D^c g) \qquad (1)$$

Inserting the fundamental dimensions,

$$0 = (L^{3a}T^{-a})(M^bT^{-2b}L^{-b})L^c(LT^{-2})$$

Equating the indices of M,

$$0 = b$$

Equating the indices of T,

$$0 = -a - 2b - 2$$

hence, $$a = -2$$

Equating the indices of L,

$$0 = 3a - 2b + c + 1$$

hence, $$c = 5$$

Inserting these values in Equation (1) the non-dimensional factor for this group becomes

$$\left(\frac{D^5g}{Q^2}\right) \qquad (2)$$

But $$Q \propto D^2\sqrt{\frac{p}{\rho}}$$

and $$p = \rho g H$$

Substituting these values in Equation (2), the non-dimensional factor becomes

$$\left(\frac{D}{H}\right)$$

GROUP 2. $$0 = \phi(Q^a p^b D^c \rho) \quad . \quad . \quad . \quad (3)$$

Inserting the fundamental dimensions,

$$0 = (L^{3a}T^{-a})(M^bT^{-2b}L^{-b})\ L^c(ML^{-3})$$

Equating the indices of M,

$$0 = b + 1$$

hence, $b = -1$

Equating the indices of T,

$$0 = -a - 2b$$

Substituting for $0 = -a + 2$

hence, $a = 2$

Equating the indices of L,

$$0 = 3a - b + c - 3$$

Substituting for a and b, $0 = 6 + 1 + c - 3$

hence, $c = -4$

Inserting these values of a, b, and c in Equation (3), the non-dimensional factor for this group becomes

$$\left(\frac{Q^2\rho}{pD^4}\right) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

but as $Q \propto D^2\sqrt{\frac{p}{\rho}}$ this factor becomes

$$\frac{D^4 p\rho}{\rho p D^4} = 1$$

This proves that the coefficient k is independent of the density ρ.

GROUP 3. $$0 = \phi(Q^a p^b D^c \mu) \quad . \quad . \quad . \quad (5)$$

Inserting the fundamental dimensions,

$$0 = (L^{3a}T^{-a})(M^bT^{-2b}L^{-b})L^c(ML^{-1}T^{-1})$$

Equating the indices of M,

$$0 = b + 1$$

hence, $b = -1$

Equating the indices of T,

$$0 = -a - 2b - 1$$

Substituting for b, $\quad 0 = -a + 2 - 1$

from which $\quad a = 1$

Equating the indices of L,

$$0 = 3a - b + c - 1$$

Substituting for a and b, $0 = 3 + 1 + c - 1$

hence, $\quad c = -3$

Inserting these values of a, b, and c in Equation (5), the non-dimensional factor for this group becomes

$$\left(\frac{Q\mu}{pD^3}\right)$$

But $Q \propto D^2\sqrt{\frac{p}{\rho}}$ and $p = \rho gH$, hence this factor becomes

$$\frac{D^2\sqrt{p}\mu}{\sqrt{\rho}pD^3}$$

or

$$\frac{\mu}{\sqrt{gH}\rho D}$$

Group 4. $0 = \phi(Q^a p^b D^c f)$ . . . . . (6)

Inserting the fundamental dimensions,

$$0 = (L^{3a}T^{-a})(M^bT^{-2b}L^{-b})L^c(MT^{-2})$$

Equating the indices of M,

$$0 = b + 1$$

hence, $\quad b = -1$

Equating the indices of T,

$$0 = -a - 2b - 2$$

Substituting for b, $\quad 0 = -a + 2 - 2$

hence, $\quad a = 0$

Equating the indices of L,

$$0 = 3a - b + c$$

Substituting for a and b, $0 = 0 + 1 + c$

hence, $\quad c = -1$

Inserting these values of a, b, and c in Equation (6), the non-dimensional factor for this group becomes

$$\left(\frac{f}{pD}\right)$$

Substituting $p = \rho g H$, the factor becomes

$$\left(\frac{f}{\rho g H D}\right)$$

But it was proved by Group 1 that $D \propto H$, hence this factor may be written

$$\left(\frac{f}{\rho g H^2}\right)$$

From the results of Group 2, Equation (4) may be written

$$Q = k\sqrt{\frac{p}{\rho}}D^2 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

As $\quad$ Group 2 $= \phi$(Group 1, Group 3, Group 4)

then, $$\frac{Q}{\sqrt{\frac{p}{\rho}}D^2} = \phi\left[\left(\frac{D}{H}\right)\left(\frac{\mu}{\sqrt{gH}\rho D}\right)\left(\frac{f}{\rho g H^2}\right)\right]$$

Hence, $$Q = \sqrt{\frac{p}{\rho}}D^2\phi\left[\left(\frac{D}{H}\right)\left(\frac{\mu}{\sqrt{gH}\rho D}\right)\left(\frac{f}{\rho g H^2}\right)\right]$$

This proves that the coefficient of discharge of a small orifice depends on the three non-dimensional functions contained inside the square brackets. It may also depend on other factors which have not been included in this solution.

It will be noticed that this group method of solution gives the same results as the method demonstrated in Art. 200 and provides a simpler solution.

Examples 17

(1) Discuss the Froude and Reynolds numbers, giving illustrations of their significance.

Give a rule for the formation of dimensionless groups in problems involving more than four variables. Turbine models are tested under conditions giving the same specific speed as the prototype, yet the efficiency of the model is usually lower. Explain this. (London Univ.)

(2) The quantity of fluid flowing along a pipe is determined from the pressure drop P across a diaphragm having a central circular orifice. Show, by application of the principles of geometrical and dynamical similarity, that the volume flowing per second can be expressed by

$$Q = C.A\sqrt{\frac{P}{\rho}}$$

where ρ is the density of the fluid, A is the area of the orifice, and C is a coefficient which depends upon the pipe and orifice dimensions and the Reynolds number. State the units that must be used for the quantities concerned in both the English and metric systems. (London Univ.)

(3) Show that a rational formula for the resistance to the motion of partially submerged similar bodies through a liquid in which the formation of surface waves is the important factor, viscosity being negligible, is

$$R = \rho l^2 v^2 \,.\, F(gl \,/\, v^2)$$

in which ρ is the density of the liquid, l the length, and v the speed of the body.

In model experiments to determine the resistance of a ship, what is meant by "corresponding speeds"?

Hence show that in similar speed boats in which the resistance is mainly due to wave formation, if S is the ratio of the linear dimensions the resistances at corresponding speeds vary as S^3. (London Univ.)

(4) In two geometrically similar shaft bearings, in which the same lubricant is used, the speeds of rotation are the same and viscous flow occurs. Make use of the method of dimensions and prove that if the loads carried per unit area are the same for each, the moments of the frictional resistances are proportional to the cubes of the linear dimensions. (London Univ.)

(5) In geometrically similar shaft bearings in which the motion of the lubricant is purely viscous, show that a rational expression for the frictional resistances should be of the form

$$R = \mu N D^3 \,\phi\left(\frac{p}{\mu N}\right)$$

in which R is the moment of the frictional resistances, μ the viscosity of the lubricant, N the speed of rotation, D the diameter of the shaft, and p the load carried per unit area of bearing surface. Hence, show that in similar bearings the frictional resistances at corresponding speeds vary as the product pD^3. What is the ratio of the corresponding rotational speeds in two given similar bearings? (London Univ.)

Ans.—$\dfrac{p_2\mu_1}{p_1\mu_2}$

(6) Show that the flow over a 90 degree vee notch for a fluid of kinematic viscosity ν will be

$$Q = H^{\frac{5}{2}} g^{\frac{1}{2}} \phi\left[\frac{H^{\frac{3}{2}} g^{\frac{1}{2}}}{\nu}\right]$$

where H is the head and g the acceleration due to gravity.

A vee notch is employed to measure the flow of a fluid which has a kinematic viscosity 8 times that of water. If the measured head is 10 in., calculate the head for water giving dynamical similarity. From experiments on water the empirical formula $Q = 2{\cdot}48\ H^{2{\cdot}47}$ has been deduced, Q being in cusecs., and H in ft. Calculate the flow of fluid. (London Univ.)

Ans.—2·5 in.; 1·64 cusecs.

(7) In the rotation of similar discs in a fluid in which the motion of the fluid is turbulent, show by the method of dimensions that a rational formula for the frictional torque M of a disc of diameter D rotating at a speed of N in a fluid of viscosity μ and density ρ is

$$M = D^5N^2\rho\ \phi.\left(\frac{\mu}{D^2N\rho}\right)$$

Hence show that in similar discs rotating in the same fluid the frictional torques at the corresponding speeds vary as the diameters of the discs. What is the ratio of the corresponding speeds? (London Univ.)

Ans.—$\dfrac{N_1}{N_2} = \left(\dfrac{D_2}{D_1}\right)^2$

(8) Assuming that the thrust T of a screw propeller is dependent upon the diameter d, speed of advance v, fluid density ρ, revs. per sec. n, and the coefficient of viscosity μ, show, using the principle of dimensional homogeneity, that A can be represented by

$$T = \rho d^2 v^2 \phi\left\{\frac{\mu}{\rho dv},\ \frac{dn}{v}\right\}$$

and hence explain the condition of dynamical similarity usually assumed for propellers.

The characteristics of a propeller of 12 ft. diameter and rotational speed 100 r.p.m. are examined by means of a geometrically similar model of 18 in. diameter. When the model is rotated at 360 r.p.m. by a torque of 16 lb. ft. the thrust developed is 52 lb. and the speed of advance is 4·8 knots. Determine the torque, thrust, speed of advance and efficiency of the full-scale propeller. (1 knot = 6,080 ft. per hr.) (London Univ.)

Ans.—Torque = 40,600 lb. ft.; T = 16,500 lb.; v = 10·68 knots; eff. = 70 per cent.

(9) A model of an air duct is built to $\frac{1}{30}$ scale and tested with water which is 50 times more viscous and 800 times more dense than air. When tested under dynamically similar conditions, the pressure drop is 33 lb. per sq. in. in the model. Find the corresponding full-scale pressure drop, and express it in inches of water. (I. Mech. E.)

Ans.—·326 in. of water.

(10) If the resistance to the motion of a sphere through a fluid is a function of the density ρ and the viscosity μ of the fluid, and the radius r and the velocity v of the sphere, show that the resistance $R = \left(\dfrac{\mu^2}{\rho}\right)\cdot\phi\,(vr\rho/\mu)$. Hence, show that if at very low velocities $R \propto v$, $R = k\mu rv$, when k is a dimensionless constant. (London Univ.)

(11) A model to one-tenth scale of a broad-crested weir gave these values—

Head, ft.	·1	·3	·7
Flow, cu. ft. per sec.	·23	1·22	5·7

Find the corresponding full-scale values. Discuss the likelihood of discrepancies between such a model and the full-scale weir. (I. Mech. E.)

Ans.—72·8; 386; 1,805 cu. ft. per sec.

(12) What are the requirements of dynamical similarity and how may they be fulfilled when testing a model of a hydraulic structure? (I. Mech. E.)

(13) What are meant by "corresponding speeds" in connection with a model and its original? What would be the ratio of the corresponding speeds: (*a*) for a weir and its model, (*b*) for a closed pipe line and its model? (I. Mech. E.)

Ans.—(*a*) $\dfrac{v_1}{v_2} = \dfrac{H_2 v_1}{H_1 v_2}$; (*b*) $\dfrac{v_1}{v_2} = \dfrac{d_2 v_1}{d_1 v_2}$

(14) For the rotational speeds of similar wheels in a fluid, the power dissipated in windage is dependent upon the diameter D and the speed N revs./sec. of the wheel, the density ρ and the viscosity μ of the fluid. Hence, making use of the principle of dimensional homogeneity, show that the power P can be expressed

$$P = D^5 N^3 \rho \cdot \phi\,(\mu / D^2 N \rho)$$

In similar impulse steam turbines it may be assumed that $\phi\,(\mu/D^2 N\rho) = A\,(\mu/D^2 N\rho)$, in which A is a dimensionless constant; hence, obtain a formula for P in terms of the diameter in feet, and the rotational speed in revs./sec. of the wheel, the density and viscosity of the steam in lb. ft. sec. (abs.) units, given that the resisting torque is 8·75 lb. ft. for a wheel 34 in. diameter rotating at 50 revs./sec. in steam of density ·04 and viscosity 8×10^{-6} lb. ft. sec. (abs.) units. (London Univ.)

Ans.—$P = 10{\cdot}96 D^3 N^2 \mu$.

(15) Obtain from first principles an expression for the power P, developed by a hydraulic turbine in the form

$$P = \rho N^3 D^5 \cdot \phi\left(\frac{N^2 D^2}{gH}\right)$$

where ρ is the mass density of the fluid, g the acceleration due to gravity, N the rotational speed, D the diameter of the rotor, and ϕ denotes an arbitrary function.

Hence, or otherwise, deduce the expression for the "specific speed" in terms of N, H, and P.

Explain briefly the application of specific speed to preliminary design of a turbine. (London Univ.)

CHAPTER XVIII

FURTHER PROBLEMS IN FLUID MECHANICS

206. Surface Tension. When two liquids of different density, or when a liquid and a gas, are in contact, the surface of contact forms a curve called the meniscus. The formation of the curved surface is due to the attraction of the molecules and it is found that there is a slight pressure difference between the fluids on either side of the surface. The surface appears to act as an elastic skin which is in tension in both directions; this tension is called the *surface tension.*

This phenomenon may be observed in the curved surface of a liquid in a tube, in a bubble of a light liquid immersed in a

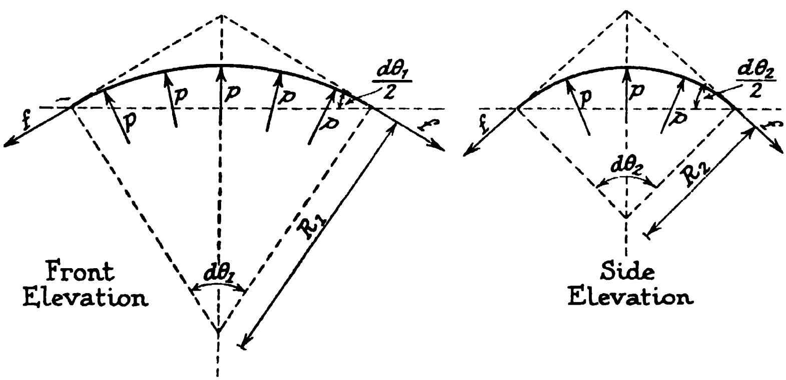


FIG. 239

heavier liquid, or a bubble of gas immersed in a liquid. It is familiar in the bubble of a spirit level and in the soap film of a soap bubble.

Let the curves of Fig. 239 represent the front elevation and end view of a small rectangular portion of a meniscus.

Let $\quad p =$ excess of inside pressure over outside pressure, lb. per sq. in.

R_1 and $R_2 =$ radius, in inches, of surface in front and end view respectively.

$d\theta_1$ and $d\theta_2$ = angle in radians subtended by surface in front and end view respectively

f = surface tension in lb. per inch length in both perpendicular directions.

Then, length of arc in front view $= R_1 d\theta_1$

length of arc in end view $= R_2 d\theta_2$

Now, as the small rectangular surface considered is in equilibrium, the upward pressure due to p must equal the downward pull of the tension f. Hence, resolving vertically

upward force due to p = downward force due to f

$$\text{or} \quad p \times (R_1 d\theta_1 \times R_2 d\theta_2) = 2fR_1 d\theta_1 \frac{d\theta_2}{2} + 2fR_2 d\theta_2 \frac{d\theta_1}{2}$$

$$\text{as} \quad \sin\frac{d\theta}{2} = \frac{d\theta}{2} \text{ for small angles.}$$

$$\text{Then} \quad p = \frac{f(R_2 + R_1)}{R_1 R_2}$$

$$= f\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the surface is spherical of radius R,

$$R_1 = R_2 = R,$$

then,

$$p = \frac{2f}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the surface is cylindrical, R_2 is infinite,

$$\text{Then,} \quad p = \frac{f}{R_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In a large vessel the surface of the liquid is curved near the perimeter only; in this case R_2 of Equation (1) is the radius of the vessel.

It will be seen from these equations that the value of the surface tensiôn depends on the radius of the meniscus, and is found to vary with the nature of the fluids in contact and with the temperature. For a water surface in contact with the atmosphere the surface tension has a value of about ·00042 lb. per inch at a temperature of 60° F.

207. **Capillarity.** If a tube of small bore be inserted in a liquid, the liquid is observed to rise, or fall, by a head h, as is shown in Fig. 240. This, in the case of (a), is caused by the fall in pressure p on the underside of the meniscus, due to the surface tension f. At the point A the pressure is less than atmospheric as the meniscus is sagging; this causes an elevation of the liquid in the tube. In the case of mercury [Fig. 240 (b)]

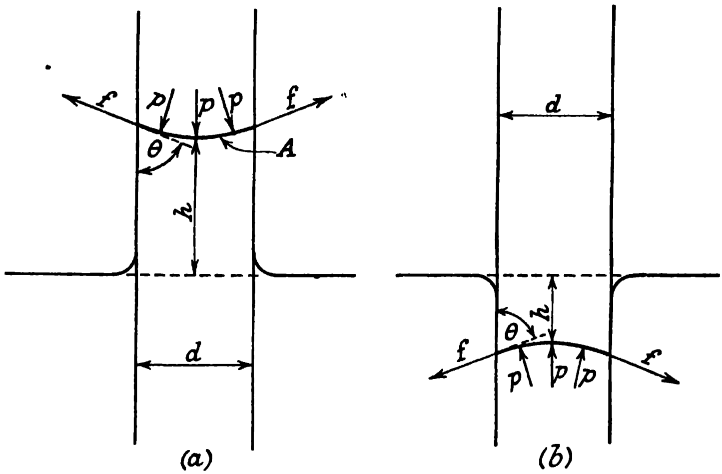


FIG. 240

the meniscus is reversed in curvature, thus causing a capillary depression in the tube.

For a circular-sectioned tube of diameter d inches, the weight of column of liquid elevated [Fig. 240 (a)] is supported by the surface tension f acting around the perimeter of the tube. Then, resolving vertically,

weight of liquid raised = vertical component of f acting on perimeter

or $$w\left(\frac{\pi}{4}d^2h\right) = f\pi d \cos\theta$$

where θ is the angle of the meniscus at the perimeter.

Hence, $$h = \frac{4f\cos\theta}{wd} \quad . \quad . \quad . \quad . \quad (1)$$

For two parallel plates at distance d apart, resolving vertically and considering unit length,

$$whd = 2f \cos \theta$$

hence,
$$h = \frac{2 f \cos \theta}{wd} \quad . \quad . \quad . \quad . \quad (2)$$

This capillary elevation or depression will affect the readings of gauges of small bore, such as piezometer tubes, unless the diameter is of sufficient size to make the value of h negligible. If the diameter is extremely small a correction must be made by the application of Equation (1).

If the liquid wets the walls of the tube, as in the case of water, the value of θ is zero, the Equation (1) becomes

$$h = \frac{4f}{wd} \quad . \quad . \quad . \quad . \quad (3)$$

This equation provides the best method of obtaining the value of the surface tension f experimentally. By measuring the height h of the liquid in a capillary tube of diameter d and having wet walls, the value of f can be calculated.

EXAMPLE.

Find the height through which water is elevated by capillarity in a glass tube of $\frac{1}{4}$ in. bore if the surface tension at the existing temperature is $\cdot 343 \times 10^{-3}$ lb. per inch.

Using Equation (3)

$$h = \frac{4f}{wd}$$

$$= \frac{4 \times \cdot 343 \times 10^{-3} \times 12^3}{62 \cdot 4 \times \cdot 25} \text{ in.}$$

$$= \cdot 152 \text{ in.}$$

208. **Energy Variation Across Streamlines.** Let AB and CD be two adjacent streamlines in a vertical plane (Fig. 241) in a liquid having motion in two perpendicular planes. Consider a short length of the streamlines subtending an angle $d\theta$ at the centre of curvature of AB, and let the radius of this short length of AB be constant and equal to r.

Let p = pressure on streamline AB

$p + dp$ = pressure on streamline CD

dr = distance between AB and CD

v = velocity of AB

$v + dv$ = velocity of CD

Z = height above datum

Let a be the inclination to the vertical of section of streamlines

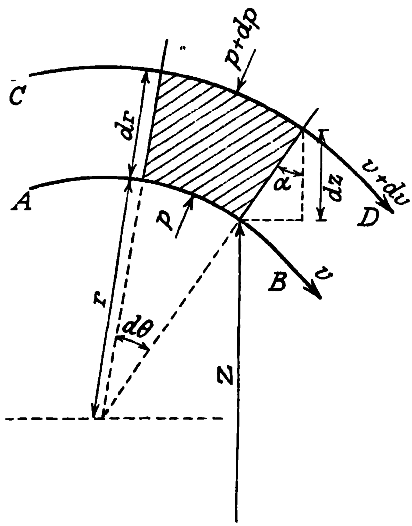


FIG. 241

considered, and dZ be the vertical height of CD above AB. Then, from Fig. 241,

$$\cos a = \frac{dZ}{dr} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

also $$w = \rho g$$

Consider the section of liquid between the streamlines, shown shaded, and consider it to have unit thickness. Then,

weight of section considered $= wr\,d\theta\,dr$

and centrifugal force on section $= \dfrac{(wrd\theta\,dr)\,v^2}{gr}$

In Fig. 242 is shown an enlarged view of this section of liquid, the dimensions and forces acting being inserted in the figure. These forces, plus the centrifugal force and its weight, must balance, as the element is in equilibrium.

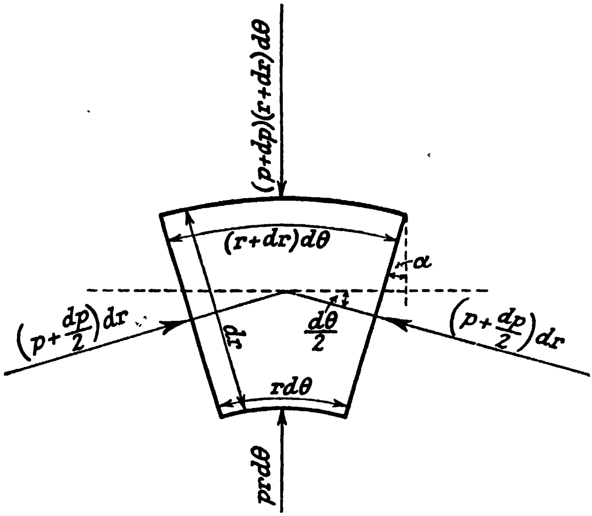


FIG. 242

Hence, resolving radially and writing $\frac{d\theta}{2}$ for $\sin\frac{d\theta}{2}$,

$$prd\theta + 2\left(p + \frac{dp}{2}\right) dr\,\frac{d\theta}{2} - (p + dp)(r + dr)d\theta - (wrd\theta\, dr)\cos a + \frac{(wrd\theta dr)\, v^2}{gr} = 0$$

Substituting for $\cos a$ from Equation (1) and ignoring small quantities of the second order,

$$dp = -wdZ + \frac{wdrv^2}{gr}$$

From which

$$\frac{dp}{w} = -\,dZ + \frac{v^2dr}{gr} \quad . \quad . \quad . \quad (2)$$

Now, applying Bernoulli's equation to any streamline,

$$E = Z + \frac{p}{w} + \frac{v^2}{2g}$$

where E is the total energy of the streamline considered.

Differentiating each term for small changes dZ, dp and dv over the width dr,

$$dE = dZ + \frac{dp}{w} + \frac{vdv}{g}$$

Substituting for $\frac{dp}{w}$ from Equation (2),

$$dE = dZ + \left(-dZ + \frac{v^2dr}{gr}\right) + \frac{vdv}{g}$$

hence,
$$dE = \frac{v^2dr}{gr} + \frac{vdv}{g}$$

or
$$\frac{dE}{dr} = \frac{v}{g}\left(\frac{dv}{dr} + \frac{v}{r}\right) \quad . \quad . \quad . \quad (3)$$

This equation represents the change of energy across any streamline flow in two perpendicular directions. It will be noticed that the Z term does not appear in the final equation; Equation (3) will, therefore, apply to streamlines moving in any plane.

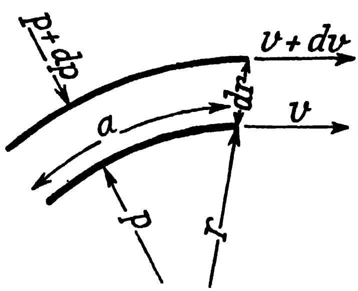


Fig. 243

209. Energy Variation Across Horizontal Streamlines. If the streamlines of a moving fluid are in the horizontal plane only, the effect of the weight of the fluid need not be taken into account, in which case Equation (3) of Art. 208 can be obtained in a simplified manner.

Consider a short length a of the two adjacent streamlines of Fig. 243, having unit depth. Let the fluid at this section be moving in the horizontal plane only with a velocity v at a radius of r. Assume the two streamlines are at a distance of dr apart and over this distance let the velocity v increase by dv and let the pressure p increase by dp; this increase of pressure is due to the centrifugal force on the element considered. Then,

weight of element considered $= wadr$

As difference of radial force on element $\Big\}$ = centrifugal force

then, $$dp \times a = \frac{(wadr)\, v^2}{gr}$$

from which $$\frac{dp}{w} = \frac{v^2 dr}{gr} \quad . \quad . \quad . \quad (1)$$

Total energy of fluid $= E = \dfrac{p}{w} + \dfrac{v^2}{2g}$

Differentiating for small changes of energy, pressure and velocity across the thickness of the element dr,

$$dE = \frac{dp}{w} + \frac{vdv}{g}$$

Substituting for $\dfrac{dp}{w}$ from Equation (1),

$$dE = \frac{v^2 dr}{gr} + \frac{vdv}{g}$$

from which $$\frac{dE}{dr} = \frac{v}{g}\left(\frac{v}{r} + \frac{dv}{dr}\right) \quad . \quad . \quad . \quad . \quad (2)$$

which is the same equation as Equation (3) (Art. 208). It will be noticed that this proof is much simpler than the more general proof given in Art. 208, but is based on a horizontally moving fluid only.

210. **Two-dimensional Flow of a Liquid.** The motion of a liquid in a two-dimensional plane may be in the form of a free cylindrical vortex, a free spiral vortex, a forced vortex, a radial flow, or a straight line motion. The last type is a simple direct flow and need not be dealt with further; the other four types will be treated separately.

(*a*) A FREE CYLINDRICAL VORTEX. In this type of flow the streamlines are moving freely in horizontal concentric circles and there is no variation of the total energy E across the streamlines. Then,

$$dE = 0$$

Hence, applying this to Equation (3), Art. 208,

$$\frac{v}{g}\left(\frac{dv}{dr} + \frac{v}{r}\right) = 0$$

then, $$\frac{dv}{v} + \frac{dr}{r} = 0$$

Integrating, $\log_e v + \log_e r = \text{constant}$

then, $vr = \text{constant}$

or
$$v = \frac{C}{r} \qquad (1)$$

where C is the value of the constant for the liquid considered and is known as the *strength* of the vortex.

Applying Bernoulli's equation to any two concentric horizontal streamlines of radius r_1 and r_2, and letting p_1 and p_2 be the pressures at the two streamlines considered,

$$\frac{p_1}{w} + \frac{v_1^2}{2g} = \frac{p_2}{w} + \frac{v_2^2}{2g}$$

then,
$$\frac{p_1 - p_2}{w} = \frac{v_2^2 - v_1^2}{2g}$$

But, from Equation (1),

$$v_2 = \frac{C}{r_2}$$

and
$$v_1 = \frac{C}{r_1}$$

Hence,
$$\frac{p_1 - p_2}{w} = \frac{C^2}{2g}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right) \qquad (2)$$

If the fluid is a gas, on account of its compressibility, Equation (2) should be written,

$$\frac{p_1}{w_1} - \frac{p_2}{w_2} = \frac{C^2}{2g}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)$$

This equation gives the difference of pressure head between the two streamlines considered. It will be noticed that if a curve representing the pressure variation is plotted on a base representing the radius of the vortex, a parabola is obtained having a maximum value at the outer circumference. If the upper surface of the liquid be free it will assume this parabolic shape.

As the pressure near the outer edge of a free cylindrical vortex is greater than that near the centre, the fluid is caused to flow radially inwards towards its central core, through which it drains away by passing along the core of the cylinder longitudinally, thus causing a suction at the opposite end of

the core. The combination of the circumferential flow and of this inward radial flow results in a spiral motion and converts the free cylindrical vortex into a free spiral vortex [Case (*d*)].

It will be seen from this that a free cylindrical vortex cannot be maintained in nature, as it tends to develop into a free spiral vortex. This latter type of vortex occurs in such phenomena as a tornado, a waterspout, and in the emptying of a vessel of liquid by means of a drain at the base.

(*b*) A FORCED VORTEX. A forced vortex is the name given to a circular stream of liquid the whirl of which is caused by power from an external source. An example of a forced vortex is the stream of water in the casing of a centrifugal pump.

In a forced vortex the liquid has a constant angular velocity; let ω be the angular velocity of the liquid, then

$$\omega = \frac{v}{r} = \frac{dv}{dr} = \text{constant}$$

From Equation (3), Art. 208,

$$\frac{dE}{dr} = \frac{v}{g}\left(\frac{dv}{dr} + \frac{v}{r}\right) \qquad . \quad . \quad . \quad . \qquad (3)$$

Substituting the above value of ω for $\frac{v}{r}$ and $\frac{dv}{dr}$

$$dE = \frac{2v}{g}\omega dr$$

Substituting for $v = \omega r$

$$dE = \frac{2\omega^2 r dr}{g}$$

Using the suffix 1 for the inside radius of the vortex and suffix 2 for the outside, and integrating between these limits,

$$E_2 - E_1 = \frac{\omega^2}{g}\left[r^2\right]_{r_1}^{r_2}$$

$$= \frac{\omega^2}{g}(r_2^2 - r_1^2)$$

$$= \frac{v_2^2 - v_1^2}{g} \qquad . \quad . \quad . \quad . \quad . \qquad (4)$$

But, applying Bernoulli's equation to the limiting streamlines,

$$E_1 = \frac{p_1}{w} + \frac{v_1^2}{2g}$$

and $$E_2 = \frac{p_2}{w} + \frac{v_2^2}{2g}$$

Hence, $$E_2 - E_1 = \frac{p_2 - p_1}{w} + \frac{v_2^2 - v_1^2}{2g} \tag{5}$$

From Equations (4) and (5),

$$\frac{p_2 - p_1}{w} + \frac{v_2^2 - v_1^2}{2g} = \frac{v_2^2 -}{g}$$

from which

$$\frac{p_2 - p_1}{w} = \frac{v_2^2 - v_1^2}{2g} \tag{6}$$

it will be noticed that Equation (6) is the centrifugal head impressed on the liquid and is the same result as that obtained in Art. 30.

If the fluid is a gas, Equation (6) should be written

$$\frac{p_2}{w_2} - \frac{p_1}{w_1} = \frac{v_2^2 - v_1^2}{2g}$$

(c) Radial Flow of a Liquid. In this case it is assumed that the liquid is flowing radially outwards between two horizontal flat discs, placed parallel a fixed distance apart. The liquid is assumed to enter by a hole at the centre and leave at the circumference, as is shown in Fig. 33 (page 59). As the path of the liquid is a straight line, the radius r of the streamline is infinity. Then, applying this to Equation (3), and putting r and dr equal to infinity,

$$\frac{dE}{\infty} = \frac{v}{g}\left(\frac{dv}{\infty} + \frac{v}{\infty}\right) = 0$$

hence, the change of energy across the streamlines is zero. Applying Bernoulli's equation,

$$E = \frac{p}{w} + \frac{v^2}{2g} = \text{constant for all streamlines}$$

Let Q = discharge in cu. ft. per sec.

and t = distance between discs in feet.

Then, $$v = \frac{Q}{2\pi rt} \text{ at any radius } r$$

Using the suffix 1 for the inner radius and suffix 2 for the outer radius,

$$v_2 = \frac{Q}{2\pi r_2 t}$$

and
$$v_1 = \frac{Q}{2\pi r_1 t}$$

As E is a constant for all streamlines,

$$E_2 = E_1$$

that is,
$$\frac{p_2}{w} + \frac{v_2^2}{2g} = \frac{p_1}{w} + \frac{v_1^2}{2g}$$

from which
$$\frac{p_2 - p_1}{w} = \frac{v_1^2 - v_2^2}{2g}$$

$$= \frac{Q^2}{8\pi^2 t^2 g}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) \quad . \quad . \quad . \quad (7)$$

This equation gives the pressure distribution shown in Fig. 28, which is known as Barlow's curve. This type of flow is the same as that dealt with in Art. 29.

(*d*) Free Spiral Vortex. This type of flow is a combination of radial flow [Case (*c*)] and free cylindrical vortex [Case (*a*)]. It will be noticed by comparing Equations (2) and (7) that the pressure variation of these two types of flow are similar.

In a free spiral vortex the liquid is rotating and flowing radially at the same time, thus moving in the form of a horizontal spiral.

Let v_r = radial velocity of liquid

$$= \frac{Q}{2\pi r t} \text{ [Case (c)]}$$

Let v_c = circumferential velocity of liquid

$$= \frac{C}{r} \text{ [Case (a)]}$$

Then, at any radius r,

$$\frac{v_r}{v_c} = \frac{Q}{2\pi r t} \times \frac{r}{C}$$

$$= \frac{Q}{2\pi t C} = \text{a constant for all radii.}$$

Hence, the resultant velocity of the fluid flows at a constant angle to the tangent at all radii. Let this angle be represented by a, then

$$\tan a = \frac{v_r}{v_e}$$

$$= \frac{Q}{2\pi rt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

It will be noticed that the above four types of flow have been applied to liquids only and the density has been assumed constant. The equations obtained may also be applied to gases if the pressure variation is small, in which case the alteration in density is negligible.

EXAMPLE 1.

Calculate the difference of pressure between radii of 6 in. and 3 in. of a forced vortex of water which is rotated at 1450 r.p.m.

Using Equation (6),

$$\frac{p_2 - p_1}{w} = \frac{v_2^2 - v_1^2}{2g}$$

$$= \frac{w^2(r_2^2 - r_1^2)}{2g}$$

$$= \frac{(2\pi\ 1450)^2(\cdot5^2 - \cdot25^2)}{(60)^2 \times 2 \times 32\cdot2}$$

$$= 67\cdot3 \text{ ft. of water}$$

$$p_2 - p_1 = \frac{67\cdot3 \times 62\cdot4}{144}$$

$$= 29\cdot2 \text{ lb. per sq. in.}$$

EXAMPLE 2.

In a free cylindrical vortex of water it is found that at a radius of 3 in. the tangential velocity of the water is 20 ft. per sec. and its pressure 30 lb. per sq. in. Calculate the pressure at a radius of 6 in.

For a free cylindrical vortex,

$$v = \frac{C}{r}$$

hence,
$$C = v_1 \times r_1$$
$$= 20 \times \tfrac{3}{12}$$
$$= 5$$

Using Equation (2),

$$\frac{p_1 - p_2}{w} = \frac{C^2}{2g}\left(\frac{1}{r_2^{\,2}} - \frac{1}{r_1^{\,2}}\right)$$

that is,
$$\frac{144(30 - p_2)}{62 \cdot 4} = \frac{5^2}{64 \cdot 4}\left(\frac{1}{\cdot 5^2} - \frac{1}{\cdot 25^2}\right)$$

then,
$$30 - p_2 = \frac{25 \times (-12) \times 62 \cdot 4}{144 \times 64 \cdot 4}$$

hence,
$$p_2 = 30 + 2 \cdot 02$$
$$= 32 \cdot 02 \text{ lb. per sq. in.}$$

211. **Determination of Coefficient of Viscosity.** The coefficient of viscosity of a liquid can be found experimentally by the three following methods. The first method is also applicable to gases.

(*a*) BY MEASUREMENT OF PRESSURE DROP DURING PIPE FLOW. It was shown in Art. 140 that the viscous resistance to the flow of a fluid along a straight uniform pipe is given by the equation

$$\frac{mig}{v^2} = C\left(\frac{\rho v d}{\mu}\right)^n$$

For a viscous flow it was proved that $C = 8$ and $n = -1\cdot$ hence, the equation becomes

$$\frac{mig}{v^2} = \frac{8\mu}{\rho v d}$$

For a circular sectioned pipe, running full,

$$m = \frac{d}{4}$$

and
$$i = \frac{h_f}{l}$$

hence, by substituting these values in the above equation

$$\frac{d}{4} \times \frac{h_f}{l} \times \frac{g}{v^2} = \frac{8\mu}{\rho v d}$$

from which
$$\mu = \frac{\rho g d^2 h_f}{32 l v}$$

The coefficient of viscosity can be calculated from this equation if the values of h_f and v are measured for a given pipe during viscous flow. The mean velocity v is obtained by measuring the quantity of flow in a known time; the drop in pressure head h_f is measured on a known length of pipe by means of a sensitive pressure gauge. For accurate results a considerable length of pipe should be used, and the fluid must be maintained at a constant temperature. This method can be used for liquids or gases.

(*b*) BY MEASUREMENT OF DISCHARGE THROUGH AN ORIFICE. In Art. 200 it was shown that the rate of flow of a liquid through an orifice partly depends on the non-dimensional constant

$$\frac{\rho D v}{\mu}$$

and, consequently, will vary with the coefficient of viscosity μ. This fact is made use of in a type of viscometer which compares the viscosities of liquids by measuring the time taken for a given quantity of the liquid to discharge through a standard orifice.

This principle is used in the Redwood Viscometer (Fig. 244); it consists of a vertical cylinder containing the liquid which is allowed to discharge through an orifice situated in the centre of its base. The cylinder is surrounded by a water jacket, which can maintain the liquid under test at any required temperature by means of an immersed electric heater. The cylinder is filled to a standard height with the liquid under test and is heated by the water jacket to the required temperature. The orifice is then opened and the time taken for 50 c.c. of the liquid to flow into a measuring flask is noted; the viscosity of the liquid is proportional to this time.

In one type of this viscometer the cylinder is $1\frac{7}{8}$ in. diameter and $3\frac{1}{2}$ in. deep; the orifice is 1·7 mm. diameter and 12 mm. in length.

It should be noticed that this viscometer does not give a direct measurement of μ. It gives a comparison of μ only; the exact value of μ can be obtained by comparison with the times of discharge of liquids of known viscosity.

In this type of viscometer the whole head of the fluid, h, is utilized in overcoming the viscous resistance in the nozzle.

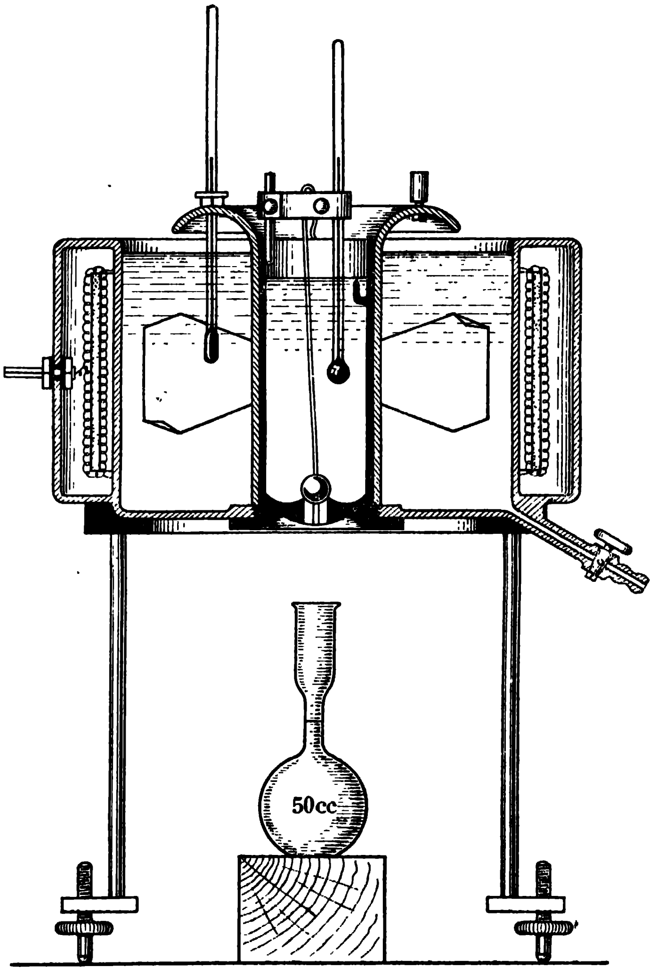


FIG. 244

Hence, in order to obtain good results, the fluid should flow out of the nozzle with no appreciable velocity. In these circumstances, the velocity head of the discharging fluid is extremely small compared with the static head of the fluid in the cylinder.

Using Equation (8) (Art. 140) for the viscous resistance of a cylindrical nozzle,

$$\frac{mig}{v^2} = C\left(\frac{\rho vd}{\mu}\right)^{-1}$$

Let l = length of nozzle.

As the whole head h is assumed to be lost,

$$i = \frac{h}{l}$$

also, $$m = \frac{d}{4}$$

Substituting these values in the above equation,

$$\frac{d}{4} \times \frac{h}{l} \times \frac{g}{v^2} = C\frac{\mu}{\rho vd}$$

But d, h and l are constants for the instrument,

hence, $$\frac{\mu v}{\rho} = \text{constant}$$

Time of discharge $$= T \propto \frac{1}{v}$$

hence, $$T = \text{a constant} \times \frac{\mu}{\rho}$$

If the fluid discharges with an appreciable velocity the above result will not hold; a correction must then be made in order to allow for the velocity head.

(*c*) By Measurement of Time of Fall of Steel Ball in Liquid. Another method of measuring the coefficient of viscosity of a liquid is by allowing a steel sphere to fall freely through a column of the liquid, and by timing its fall through a known height. The falling sphere will at first accelerate; but the resistance to its motion increases with its velocity until it just balances the pull of gravity on the sphere. After this condition is reached the sphere falls with a constant velocity. The time of fall is measured over the constant velocity period only.

Stokes proved by a difficult mathematical analysis that the resistance to a sphere moving through a non-compressible fluid is given by the equation

$$F = 6\pi\mu rv$$

where F = resistance to slow motion (viscous flow)

r = radius of sphere

v = velocity of sphere

By equating this resistance to the pull of gravity on the sphere, after the condition of uniform velocity has been reached, the value of the coefficient of viscosity μ can be calculated.

Let V = final uniform velocity of sphere

ρ_1 = density of sphere used

ρ_2 = density of liquid under test

Then, pull of gravity on sphere

$$= \text{weight of sphere} - \text{weight of liquid displaced}$$

$$= \tfrac{4}{3}\pi r^3(\rho_1 - \rho_2)g$$

Now, pull of gravity = fluid resistance on sphere at velocity V

or
$$\tfrac{4}{3}\pi r^3(\rho_1 - \rho_2)\, g = 6\pi\mu r V$$

from which,
$$\mu = \frac{2r^2 g(\rho_1 - \rho_2)}{9V} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The velocity V is obtained by measuring the time taken by the sphere in falling a known height through the liquid after the condition of uniform velocity has been reached. The coefficient of viscosity μ can then be calculated from Equation (1).

This method of finding the coefficient of viscosity gives accurate results only if the uniform velocity V is low. No eddies should be caused by the falling sphere, and the cylinder containing the liquid should be of sufficient diameter to prevent its surface affecting the motion.

It was shown in Art. 145 that the resistance of a body subject to viscous motion through a fluid is proportional to the density of the fluid, to the square of the linear dimensions, to the square of the velocity, and to the Reynolds number. Or,

$$F = k\rho \times \text{surface area} \times v^2 \times \left(\frac{\mu}{\rho v d}\right)$$

Applying this equation to the falling sphere,

$$F = k\rho\pi d^2 v^2 \left(\frac{\mu}{\rho v d}\right)$$

hence
$$F = k\pi\mu v d$$

This agrees with Stokes's analytical result if the value of k is 3.

As was explained in Art. 203, Stokes's law can be applied to the settlement of fine powders in suspension in a liquid, the fineness of the powder being graded by the time taken by a known amount of powder in settling. It is also used by engineers when dealing with silting problems.

EXAMPLE.

When a sphere of radius r cm. sinks in a liquid at a uniform velocity of v cm. per sec., and μ is the coefficient of viscosity of the liquid in poises, the resistance to the motion of the sphere is $R = 6\pi\mu\, rv$ dynes. Hence find the coefficient of viscosity of a liquid in which a sphere of diameter ·0622 in. sinks 20 cm. in 21·3 sec. The density of the liquid is ·96 and of the sphere 7·8 gm. per cm.³ (London Univ.)

Now, $$V = \frac{20}{21 \cdot 3}$$
$$= \cdot 939 \text{ cm./sec.}$$

and $$r = \frac{\cdot 0622 \times 2 \cdot 54}{2}$$
$$= \cdot 079 \text{ cm.}$$

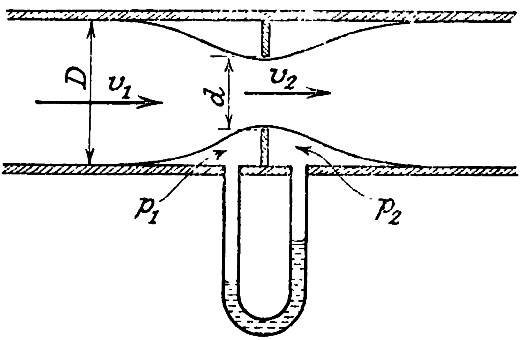


FIG. 245

Using Equation (1),

$$\mu = \frac{2r^2 g\,(\rho_1 - \rho_2)}{9V}$$
$$= \frac{2 \times \cdot 079^2 \times 981\,(7 \cdot 8 - \cdot 96)}{9 \times \cdot 939}$$
$$= 9 \cdot 97 \text{ C.G.S. units}$$

212. **Measurement of Flow by Pipe Orifice.** If a diaphragm containing an orifice be inserted in a pipe through which a fluid is flowing (Fig. 245) there will be a loss of pressure in the

fluid as it passes through the orifice. By measuring this loss of pressure it is possible to calculate the quantity of flow through the pipe. This principle is used in a type of flow meter, the quantity of flow being proportional to the measured pressure drop caused by passing through the orifice.

Let D = dia. of pipe bore
d = dia. of orifice
a_1 = cross-sectional area of pipe
a_2 = area of orifice
v_1 = velocity of fluid approaching orifice
v_2 = velocity of fluid through orifice
p_1 = pressure of fluid immediately before orifice
p_2 = pressure of fluid immediately after orifice
h = measured difference of pressure, ft. of fluid

Then, $Q = a_1 v_1 = a_2 v_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

Applying Bernoulli's equation to both sides of orifice,

$$\frac{p_1}{w} + \frac{v_1^2}{2g} = \frac{p_2}{w} + \frac{v_2^2}{2g}$$

or

$$\frac{p_1}{w} - \frac{p_2}{w} = h = \frac{(v_2^2 - v_1^2)}{2g}$$

Substituting for v_1 from Equation (1)

$$h = \left(v_2^2 - v_2^2 \frac{a_2^2}{a_1^2}\right)\frac{1}{2g}$$

$$= \frac{v_2^2}{2g}\left[1 - \left(\frac{a_2}{a_1}\right)^2\right]$$

$$= \frac{v_2^2}{2g}\left[1 - \left(\frac{d}{D}\right)^4\right]$$

From which

$$v_2 = \sqrt{\frac{2gh}{\left[1 - \left(\frac{d}{D}\right)^4\right]}}$$

Then, $\quad Q = a_2 v_2$

$$= a_2 \sqrt{\frac{2gh}{\left[1 - \left(\frac{d}{D}\right)^4\right]}} \quad . \quad . \quad . \quad (2)$$

Owing to losses due to the passage of the fluid through the orifices, this equation must be multiplied by a coefficient k, the value of k being found from tests for the particular orifice under consideration. The true discharge is thus given by the equation

$$Q = ka_2\sqrt{\frac{2gh}{\left[1-\left(\frac{d}{D}\right)^4\right]}} \quad . \quad . \quad . \quad (3)$$

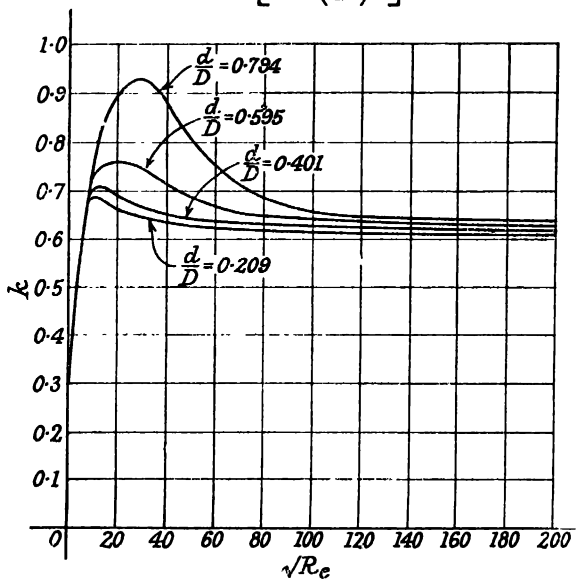


FIG. 246

It was shown in Art. 201 that the coefficient k is a function of the Reynolds number and the ratio $\left(\frac{d}{D}\right)$. In Fig. 246, curves showing the variation of k with the Reynolds number and with the ratio $\left(\frac{d}{D}\right)$ are plotted; these curves were obtained from tests. It will be noticed that k tends to become constant at high values of R_e, when the flow is turbulent, and that its value varies slightly with the ratio $\left(\frac{d}{D}\right)$.

The values of k for low values of R_e are probably affected by surface tension and by the fact that the flow is laminar.

EXAMPLE.

Water having a viscosity of $\mu = \cdot 015$ C.G.S. units flows through a pipe orifice. The diameter of the pipe is 2 in. and that of the orifice 1 in. If the pressure drop when passing through the orifice is found to be ·85 ft. of water, calculate the value of C_d from the curves of Fig. 246 and find the flow through the orifice.

As a first approximation a rough value of C_d is obtained from the curves of Fig. 246 for $\dfrac{d}{D} = \cdot 5$. From these curves an approximate value of C_d is ·63.

Using Equation (3),

$$Q = C_d a_2 \sqrt{\frac{2gh}{\left[1 - \left(\frac{d}{D}\right)^4\right]}}$$

hence,

$$v_2 = \frac{Q}{a_2} = C_d \sqrt{\frac{2gh}{\left[1 - \left(\frac{d}{D}\right)^4\right]}}$$

$$= \cdot 63 \sqrt{\frac{64 \cdot 4 \times \cdot 85}{[1 - (\frac{1}{2})^4]}}$$

$$= \cdot 63 \times 7 \cdot 65 = 4 \cdot 82 \text{ ft. per sec.}$$

Using this value of v_2, the Reynolds number can now be calculated.

From Art. 138,

$$\mu = \frac{\cdot 015 \times 30 \cdot 5}{453 \cdot 6 \times 32 \cdot 2} \text{ f.p.s. units}$$

$$= 3 \cdot 24 \times 10^{-5}$$

and

$$\rho = \frac{w}{g} = \frac{62 \cdot 4}{32 \cdot 2}$$

$$= 1 \cdot 935$$

Then,

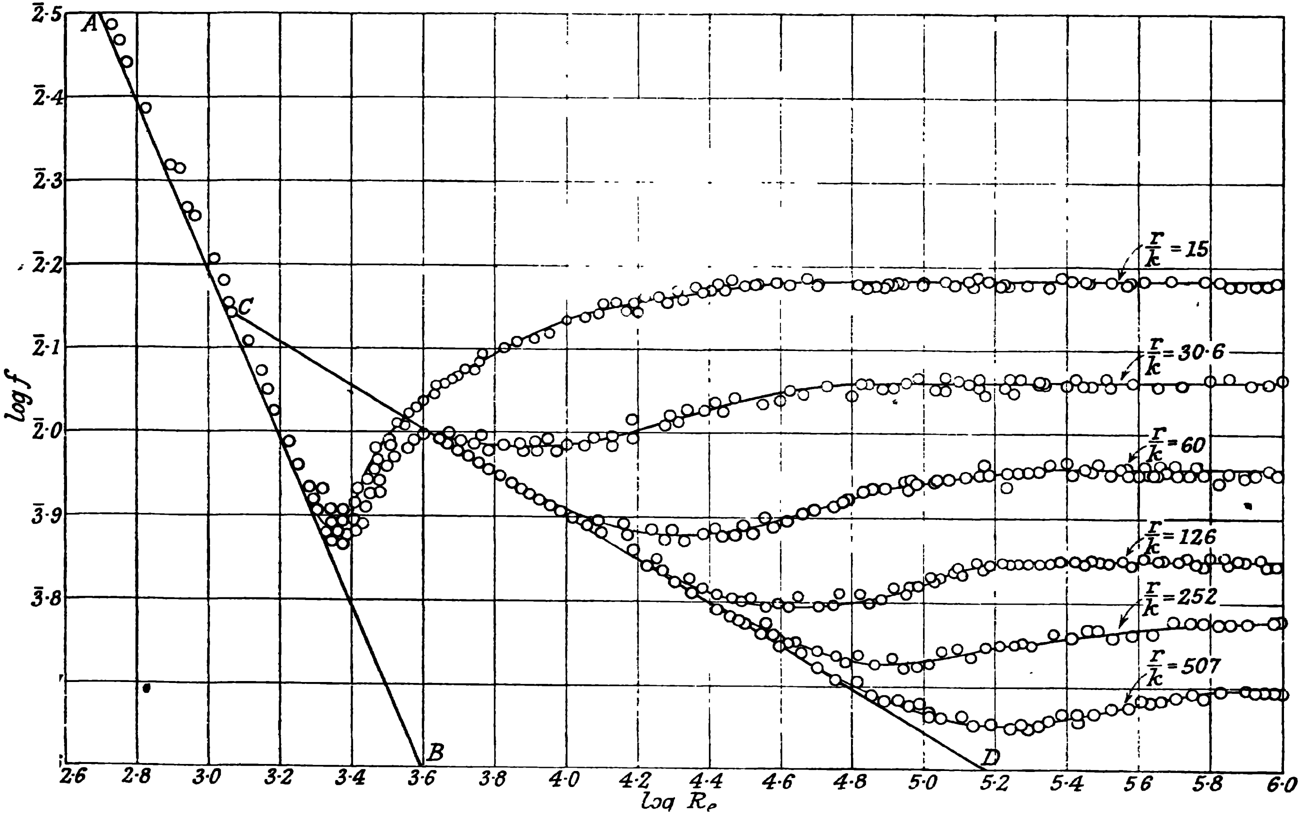
$$R_e = \frac{\rho v d}{\mu}$$

$$= \frac{1 \cdot 935 \times 4 \cdot 82 \times \frac{1}{12}}{3 \cdot 24 \times 10^{-5}}$$

$$= 24{,}000$$

hence,

$$\sqrt{R_e} = 155$$



log f
log R_e
A
B
C
D
$\frac{r}{k}=15$
$\frac{r}{k}=30{\cdot}6$
$\frac{r}{k}=60$
$\frac{r}{k}=126$
$\frac{r}{k}=252$
$\frac{r}{k}=507$
$\bar{2}{\cdot}5$
$\bar{2}{\cdot}4$
$\bar{2}{\cdot}3$
$\bar{2}{\cdot}2$
$\bar{2}{\cdot}1$
$\bar{2}{\cdot}0$
$\bar{3}{\cdot}9$
$\bar{3}{\cdot}8$
2·6
2·8
3·0
3·2
3·4
3·6
3·8
4·0
4·2
4·4
4·6
4·8
5·0
5·2
5·4
5·6
5·8
6·0

Using this value of $\sqrt{R_e}$, the actual value of C_d can now be obtained from the curves of Fig. 246. It will be found that $C_d = \cdot 62$ when $\sqrt{R_e} = 155$ and $\dfrac{d}{D} = \cdot 5$.

Then,
$$Q = C_d\, a_2 \sqrt{\frac{2gh}{\left[1 - \left(\dfrac{d}{D}\right)^4\right]}}$$
$$= \cdot 62 \times \left(\frac{\pi}{4} \times \frac{1}{144}\right) \times 7{\cdot}65 \times 60$$
$$= 1{\cdot}55 \text{ cu. ft. per min.}$$

213. **Nikuradse's Experiments on Rough Pipes.** In order to investigate the effect of the roughness of pipe walls on the resistance to flow Nikuradse* conducted a series of experiments on pipes having diameters of 2·5 cm., 5 cm., and 10 cm. The inner surfaces of these pipes were given different degrees of roughness by coating them with grains of sand of various coarseness.

Let r = radius of pipe
k = average height of roughness projections, or rugosities

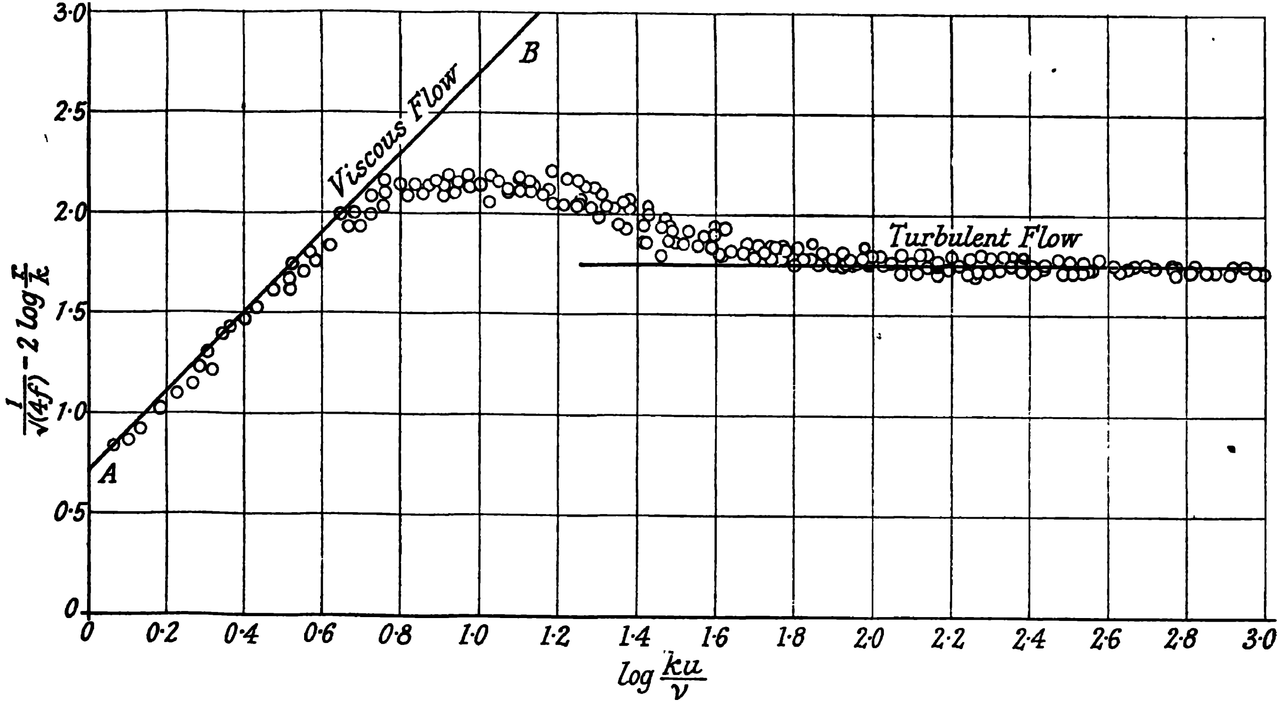
then, $\dfrac{r}{k}$ = roughness factor.

The inner surfaces of the pipes were coated with sand to give six different values of $\dfrac{r}{k}$ varying from $\dfrac{r}{k} = 15$ to $\dfrac{r}{k} = 507$. The resistance of each pipe was measured experimentally for various velocities of flow. The resistance coefficients were thus obtained for various values of the Reynolds number and for various values of $\dfrac{r}{k}$.

Let R_e = Reynolds number for the pipe flow
$= \dfrac{vd}{\nu}$

f = frictional coefficient as used in the Chezy formula;
$$h_f = \frac{4flv^2}{2gd}$$

* See *Forschungsheft V.D.I.* 361 (Berlin). See also "Experiments with Fluid Friction in Roughened Pipes," by C. F. Colebrook and C. M. White. (*Proc. Roy. Soc.*, Vol. 161.)



3·0
2·5
2·0
1·5
1·0
0·5
0
$\frac{1}{\sqrt{(4f)}} - 2 \log \frac{r}{k}$
Viscous Flow
B
A
Turbulent Flow
0
0·2
0·4
0·6
0·8
1·0
1·2
1·4
1·6
1·8
2·0
2·2
2·4
2·6
2·8
3·0
$\log \frac{ku}{\nu}$

Fig. 248

The results of Nikuradse's experiments are shown plotted in Fig. 247; the values of $\log f$ being plotted on a base representing $\log R_e$. It will be noticed from the curves that, for small values of R_e, the flow is viscous or laminar and follows the straight line AB. During this type of flow the resistance was not affected by the roughness factor. As the values of R_e increased the flow passed through a transition stage, and then became turbulent. The turbulent flow for smooth surfaces then appears to follow another straight line CD. The effect of the roughness factor can be seen by the deviation of the experimental points from this straight line. The pipes with the roughest surface cause the points to break away from the line CD at smaller values of R_e, whilst the smoother pipes cause the points to coincide with line CD up to large values of R_e.

It may be concluded from these results that a boundary sub-layer (Art. 173) around the walls of the pipe is in existence even during turbulent flow. If k is small and the rugosities do not project beyond this boundary sub-layer, the pipe wall will correspond to a smooth surface. This explains why the experimental points for small values of k followed the line CD. It was shown in Chapter XV that the thickness of the boundary sub-layer is proportional to $\sqrt{\frac{1}{R_e}}$, hence, as the value of R_e increases the boundary sub-layer becomes thinner; consequently, the experimental results tend to leave the line CD for large values of R_e. Thus, the surface may be regarded as smooth if the rugosities do not penetrate beyond the boundary sub-layer; if this penetration occurs the surface may be regarded as rough.

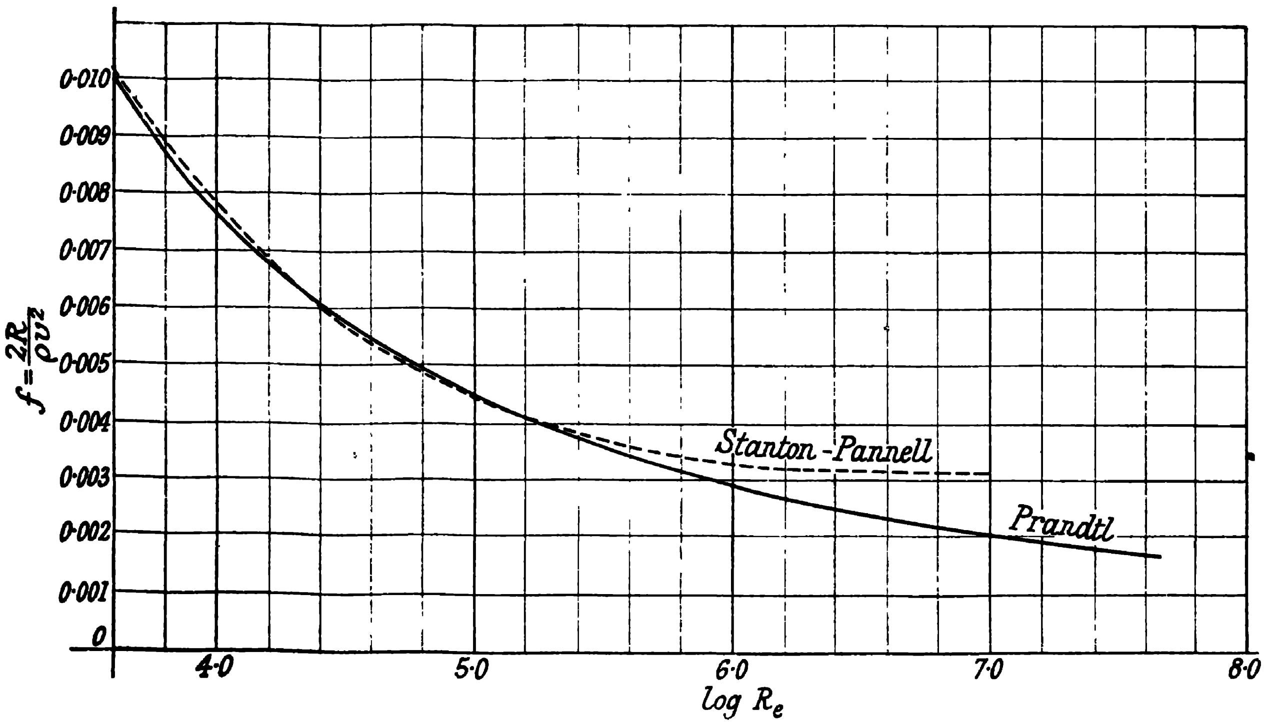
As the experimental points for laminar flow coincided with the line AB it appears that laminar flow is not affected by the roughness of the surface.

214. Prandtl and Von Kàrmàn's Equations for Pipe Flow. Prandtl and Von Kàrmàn have shown* that the results of Nikuradse's experiments on rough pipes (Art. 213) can be represented by an equation of the form

$$\frac{1}{\sqrt{4f}} = 2 \log \frac{r}{k} + \phi \frac{ku}{\nu}\sqrt{4f} \quad . \quad . \quad . \quad . \quad (1)$$

where f, r and k are as defined in Art. 213. They showed that

* See *Forschungsheft V.D.I.* 361 (Berlin).



$f = \frac{2R}{\rho v^2}$
0·010
0·009
0·008
0·007
0·006
0·005
0·004
0·003
0·002
0·001
0
4·0
5·0
6·0
7·0
8·0
Log R_e
Stanton-Pannell
Prandtl

FIG. 249

if u is the local velocity near the pipe wall, the term $\frac{ku}{\nu}$ is a local non-dimensional factor, corresponding to a local Reynolds number, on which the resistance of the surface depends. By plotting the values of $\frac{1}{\sqrt{4f}} - 2\log\frac{r}{k}$ on a base representing $\log\frac{ku}{\nu}$, Nikuradse's results gave the curve shown in Fig. 248. It will be noticed that the experimental points obtained from smooth surfaces tend to follow the straight line AB. As the roughness of the surface increases the points leave the line AB and, after a transition period, appear to approach a horizontal line. When this horizontal line is reached the value of the ordinate $\frac{1}{\sqrt{4f}} - 2\log\frac{r}{k}$ remains constant at 1·74. The surface is then said to be rough.

The equation to the straight line AB, representing a smooth surface laminar flow, is found to be

$$\frac{1}{\sqrt{4f}} - 2\log\frac{r}{k} = \cdot 8 + 2\log\frac{ku}{\nu} \qquad (2)$$

The value of the horizontal line when a completely rough surface is attained is given by the equation

$$\frac{1}{\sqrt{4f}} - 2\log\frac{r}{k} = 1{\cdot}74 \qquad (3)$$

Prandtl has deduced the following equation for turbulent flow through smooth pipes

$$\frac{1}{\sqrt{4f}} = 2\log(R\sqrt{4f}) - \cdot 8 \qquad (4)$$

where R is the resistance per unit area of wetted surface (Art. 140).

This equation is shown plotted in Fig. 249, the ordinate representing f and the base $\log R_e$. The equation is found to agree with experimental results on turbulent flow for all values of R_e.

The dotted line in Fig. 249 represents the turbulent portion of the Stanton-Pannell curve of Fig. 170. It is found that this curve only agrees with experimental results up to a Reynolds number of logarithmic value 5·6. For larger values of R_e,

experimental results agree with the Prandtl equation, which is represented by the full line.

Prandtl's equation is regarded as a great scientific achievement, as experimental tests on fluid flow through pipes of all sizes, and for different types of fluids, are found to lie on this curve, even up to such a large value of Reynolds number as 40×10^6.

EXAMPLE.

Find the diameter for a pipe to transmit 10,000 h.p. 20 miles with 95 per cent efficiency, when the gross head is 150 ft. and the roughness of the pipe wall is equivalent to ·1 in. sand. (London Univ.)

Let $\quad Q =$ quantity of flow in cusecs

then, $\quad \text{gross h.p.} = \dfrac{wQH}{550}$

$$\text{eff.} = \frac{\text{h.p. transmitted}}{\text{gross h.p.}}$$

that is, $\quad \cdot 95 = \dfrac{10{,}000 \times 550}{62{\cdot}4Q \times 150}$

from which $\quad Q = 620$ cusecs.

then, $\quad v = \dfrac{Q}{a}$

$$= \frac{620}{\frac{\pi}{4} d^2}$$

$$= \frac{790}{d^2}$$

$$h_f = \frac{4flv^2}{2gd} = 5\% \text{ of } 150$$

then, $\quad \cdot 05 \times 150 = \dfrac{4fl\left(\dfrac{790}{d^2}\right)^2}{64{\cdot}4d}$

hence, $\quad d^5 = \dfrac{4f \times 20 \times 5280 \times 622{,}000}{\cdot 05 \times 150 \times 64{\cdot}4}$

$$= 5450 \times 10^5 f$$

from which $\quad d = 55{\cdot}9 \times \sqrt[5]{f} \quad . \quad . \quad . \quad . \quad (5)$

Now, $$\frac{r}{k} = \frac{d}{2k}$$

$$= \frac{55\cdot9 \times \sqrt[5]{f}}{2 \times \dfrac{\cdot2}{12}}$$

$$= 3354 \sqrt[5]{f} \quad . \quad . \quad . \quad . \quad (6)$$

Applying Equation (3) for rough pipes,

$$\frac{1}{\sqrt{4f}} - 2 \log \frac{r}{k} = 1\cdot74$$

Substituting for $\frac{r}{k}$ from Equation (6),

$$\frac{1}{\sqrt{4f}} - 2 \log (3354 \sqrt[5]{f}) = 1\cdot74$$

Solving this equation by trial or by plotting,

$$f = \cdot004$$

Substituting this value of f in Equation (5),

$$d = 55\cdot9 \times \sqrt[5]{\cdot004}$$
$$= 55\cdot9 \times \cdot33$$
$$= 18\cdot5 \text{ ft.}$$

If this value of f is substituting in Equation (6),

$$\frac{r}{k} = 1118$$

As a check, using this value of $\frac{r}{k}$, the value of f can be obtained from the curves of Fig. 247 by extrapolation.

From these curves, $\log f = \bar{3}\cdot6$ (approximately)

from which $f = \cdot00398$

which agrees with the value obtained from Equation (3).

215. **Rotating Cylinder in Moving Fluid.** If a cylinder in a transversely moving fluid be rotated about its longitudinal axis, a transverse force is found to act on the cylinder. The cylinder thus develops a lift coefficient and acts in a similar manner to an aerofoil. This phenomenon is known as the *Magnus effect.*

In Fig. 250 a cylinder is shown rotating clockwise in a fluid

which is moving from left to right. The effect of the rotating motion is to deviate the streamlines, as shown in the figure. The linear velocity of the perimeter of the cylinder is assumed to be greater than that of the fluid.

At the edge a the velocity of the stream is increased by the movement of the cylinder; hence, by considering the application of Bernoulli's equation to these streamlines, the pressure at a will be reduced. At the edge b the velocity of the adjacent streamlines are reduced by the movement of the cylinder; hence, from the application of Bernoulli's equation, the pressure

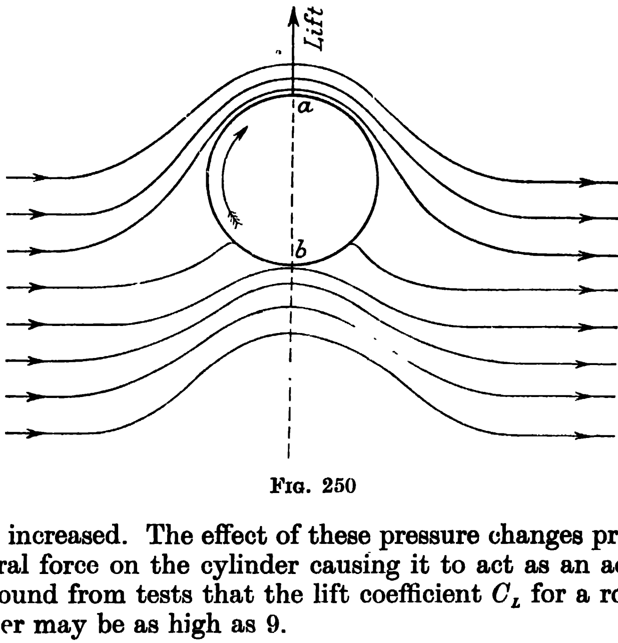


FIG. 250

at b is increased. The effect of these pressure changes produces a lateral force on the cylinder causing it to act as an aerofoil. It is found from tests that the lift coefficient C_L for a rotating cylinder may be as high as 9.

Let V = velocity of fluid stream

u = peripheral velocity of cylinder

The variation of C_L with the ratio V/u is shown plotted in Fig. 251; these results were obtained experimentally by Betz with a rotating cylinder which was fitted with end disks in order to prevent axial end flow.

This aerofoil effect of a rotating cylinder is noticeable in the firing of shells from guns across a transverse wind. The shell leaves the gun barrel with a high linear velocity and also

with a high rotational velocity due to the rifling of the inside of the barrel. It thus becomes a rotating cylinder in a transversely moving fluid. This causes a vertical force on the shell which will affect its range, depending on the strength and direction of the wind.

Another example of a practical use of this phenomenon is in the Flettner rotor ship. A schooner named *Buckau* was

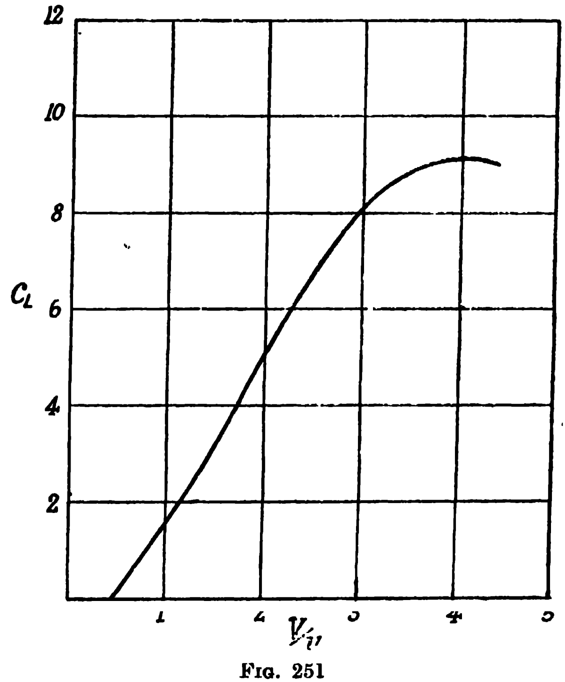


FIG. 251

fitted with two rotating cylindrical towers as a method of propulsion.* These towers were built above the deck and were revolved by an electric motor, the current being produced by a 45 h.p. Diesel engine. They were 9·1 ft. in diameter and 60 ft. high and could be rotated at various speeds up to 125 r.p.m.; the direction of rotation was reversible. The projected area of the towers was only one-tenth of that occupied by the former rigging of the *Buckau* as a sailing schooner. The weight of the towers and driving plant was 7 tons, against

* See *Engineering*, 23rd January, 1925.

a total weight of 35 tons of the former rigging. The ship was fitted in this manner by Flettner, who considered that the propulsion of a ship by rotating cylinders exposed to the wind would be more efficient than the ordinary sails.

The *Buckau*, by making use of the wind on her rotating towers, made a double crossing of the North Sea from Germany to Scotland, but the journey, although of scientific interest, was not regarded as a success.

In 1926, the *Buckau*, renamed *Baden-Baden*, crossed the Atlantic to New York by this method of propulsion. Another ship, the *Barbara*, of 3,000 tons, was fitted with three rotating towers of 13·2 ft. diameter and 56 ft. in height. Each rotor required 35 h.p. for rotation at 150 r.p.m. The weight of the rotors and gear was 40 tons; this was equivalent to 180 tons of sails and rigging. This ship had a speed of 10½ knots and was used as a freight carrier.

Flettner also used this principle in the design of windmills, the sails consisting of four rotating cylinders; each cylinder was rotated about its longitudinal axis by the power of a small engine.

216. **Methods of Controlling Boundary Layer Flow.** It is possible to control the boundary layer flow (Art. 166) of an aerofoil by external means so that an increased lift and a reduced drag are produced. The break-away, or separation, can also be prevented by this method, thus giving greater manœuvring ability to an aeroplane. Many experiments have been carried out showing the various effects of boundary layer control on aerofoils; these effects can be observed by studying the shape of the stream-bands produced when the aerofoil is tested in a wind channel, the air stream being made visible by smoke bands. The shape of the smoke bands are shown in Figs. 252, 254, and 255; these have been reproduced from photographs.

The following five methods have been used to control the boundary layer flow over the surface of an aerofoil.

(1) WING SLOTS. The insertion of a slot through the aerofoil causes either suction or increased pressure on the upper surface, according to the position of the slot. The pressure in the boundary layer can thus be controlled and break-away prevented. The effect of the slot can be observed by comparing the shape of the smoke bands shown in Figs. 252 (*a*) and 252 (*c*).

In Fig. 252 (*a*) is shown a stalled aerofoil; the break-away of

the boundary layer is clearly demonstrated by the shape of the smoke bands. In Fig. 252 (*c*) is shown the same aerofoil under identical conditions, but in this case a slot has been opened near the leading edge. It will be noticed that the opening of the slot has prevented the break-away of the

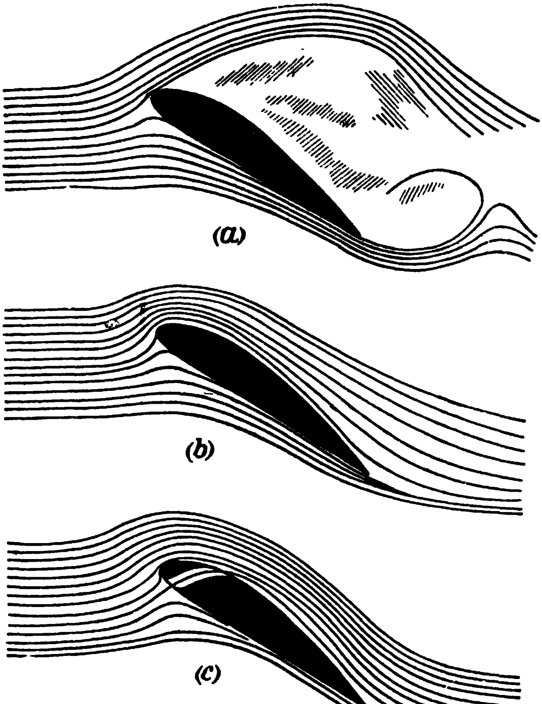


Fig. 252

(*a*) A stalled aerofoil, showing break-away.
(*b*) The same section after boundary layer suction has been applied.
(*c*) The same section fitted with a slot.

boundary layer. The smoke bands now adhere to the upper surface of the aerofoil for the whole of its length; the stalling of the aerofoil has thus been prevented. This shows that by fitting a slot in an aeroplane wing an increased manœuvring power is given to the aircraft, as the condition of break-away of the boundary layer can be prevented and stalling is thus delayed. The principle is used in the Handley-Page slotted wing, and is adopted in many types of aircraft.

The effect of a wing slot on the lift coefficient of a Handley-Page aeroplane wing is shown in the lift coefficient curve of Fig. 253. In this figure the lift coefficient C_L is plotted on a base representing the angle of incidence. The dotted-line graph is the lift coefficient curve for the wing when the slot is closed. It will be noticed that it then stalls at an angle of

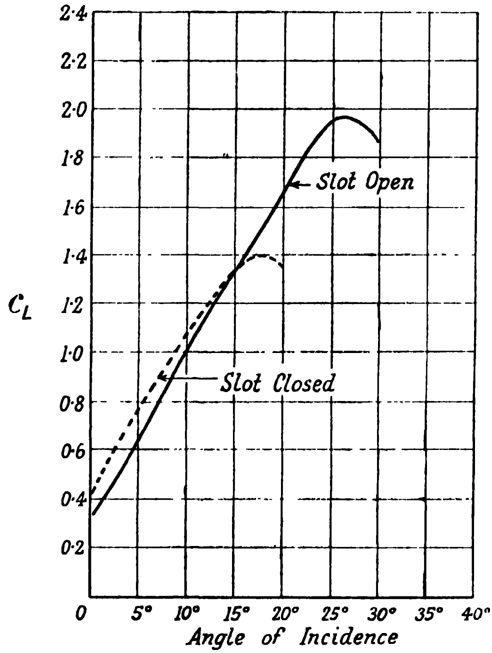


FIG. 253

incidence of $17\frac{1}{2}$ degrees; this corresponds to a break-away of the boundary layer. Also, the greatest lift coefficient is 1·4 under this condition. The full-line graph is the lift coefficient curve when the slot is fully open. The wing now reaches an angle of incidence of 26 degrees before stalling and the greatest lift coefficient is increased to 1·98. From a comparison of these two curves it will be seen that an increased manœuvring power is obtained by the use of the slotted wing. It will be noticed that the slot causes a supply of additional energy to the boundary layer and thus prevents the formation of a reversal of the pressure gradient (Art. 170).

(2) SPOILER FLAPS. The lift coefficient of an aerofoil can be increased by fixing spoiler flaps, or slats, on to its surface. The effect of a flap on the pressure distribution on the surface can be seen from a comparison of the smoke bands of Figs. 254 (*a*) and 254 (*b*). In Fig. 254 (*a*) is shown an aerofoil without the flap raised. The effect of raising the flap is shown in Fig. 254 (*b*), which is the same aerofoil under identical conditions. The negative pressure in the wake of the flap causes an increase of lift. Thus, variation in the pressure within the boundary layer can be caused by the fixing of a flap to the aerofoil surface.

(3) RE-ENERGIZING COMPRESSED AIR STREAM. In this method a stream of compressed air is passed into the boundary

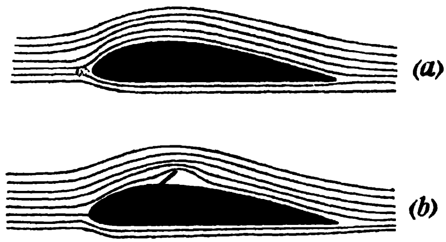


FIG. 254

layer through passages in the wing; the pressure distribution on the surface of the aerofoil, and within the boundary layer, is thus affected. This prevents the formation of the positive pressure gradient which is the cause of separation of the layer (Art. 170). It also causes alteration in the Reynolds number of the flow, consequently, break-away conditions are affected. An air compressor, or blower, is required to produce the energizing air stream.

(4) BOUNDARY LAYER SUCTION. The performance of an aerofoil can be affected by sucking away the boundary layer of the upper surface. This has been performed on the wings of an aeroplane, whilst in flight, by fitting a perforated surface on to the upper surface of the wings. The air of the boundary layer flow is sucked through the wing by the action of a Venturi fitted on to the lower surface, or by a blower driven off a separate engine.

In Fig. 252 (*a*) are shown the smoke bands flowing past a stalled aerofoil, break-away of boundary layer having occurred.

In Fig. 252 (*b*) is shown the same aerofoil under the same conditions, but in this case boundary layer suction is being applied. It will be noticed that the smoke bands now pass along the surface, showing that the break-away condition has been prevented by the suction. The suction draws away the accumulation of fluid due to the reversal of the pressure gradient (Art. 170); separation is thus prevented.

Phillips and Powis* obtained a 26 per cent reduction in profile drag, and an increased lift, by sucking away the boundary layer from the upper surface of an aeroplane's wings by means of a Venturi fitted under each wing. This was done whilst the aircraft was in flight. In another test by the same experimenters, the boundary layer was sucked away from the upper surface of the wings by means of a blower driven by an 8 h.p. engine. It was found that the rate of climb was increased by 29 per cent by this latter method, which was equivalent to another 17 h.p. on the aircraft engines. It was also found that a decrease of 22 per cent in profile drag was obtained when the plane was travelling at top speed.

(5) ROTATING CYLINDER AS LEADING EDGE. The boundary layer flow is affected by incorporating a rotating cylinder as the leading edge of the aerofoil, thus making use of the Magnus effect (Art. 215). This method causes a variation in the pressure distribution on the upper and lower edges of the aerofoil; an increased lift and the prevention of break-away can be accomplished by this method. Power of some type must be provided for the rotation of the cylinder. Lippisch† has shown experimentally that by this means the angle of incidence of an aerofoil can be greatly increased without break-away occurring. Views of his aerofoil, fitted with a rotating cylinder for its leading edge, are shown in Fig. 255. In Fig. 255 (*a*) the cylinder is at rest; the smoke bands show that break-away has occurred at the angle of incidence adopted. In Fig. 255 (*b*) is shown the same aerofoil under the same conditions, but the cylinder is now being rotated. From the new positions taken by the smoke bands it will be seen that the break-away of the boundary layer has been prevented.

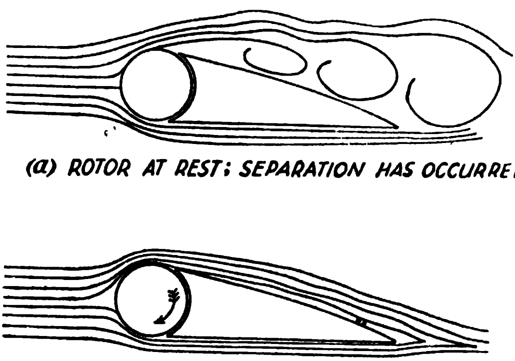
The effect of the rotating cylinder is to cause a stream of fluid of high velocity to enter the boundary layer which is thus re-energized; this prevents the formation of a reversal

* See article by F. G. Miles in *Flight*, January 26th, 1939.

† A. Lippisch in *Journal of Royal Aeronautical Society*, Vol. 43, 1939, p. 653.

of pressure gradient and so prevents the occurrence of separation (Art. 170)*

217. **Application of Boundary Layer Control to Diffuser.** An improvement of the efficiency of diffusers can be obtained by the application of boundary layer suction to the diverging passage. If the throat diameter of a Venturi meter is very small compared with that of the mouth, there is too rapid a rate of expansion of the jet area for the diverging cone to run full; this causes a loss of pressure. In the same way, the diffuser of a centrifugal pump may have a reduced efficiency due to this cause. By applying boundary layer suction at the periphery of



(a) ROTOR AT REST: SEPARATION HAS OCCURRED

(b) ROTOR REVOLVING: SEPARATION PREVENTED

FIG. 255

the diverging passage of a diffuser, the full expansion of the jet can be brought about.

The effect of this suction is shown in the photographs of Fig. 256. These photographs are due to Prandtl; they show a fluid stream flowing through a diffuser which first contracts to a throat and then diverges rapidly. Two slots, which can be opened or closed at will, have been fitted to each side of the diverging portion; by opening the slots boundary layer suction is applied to the jet. In the view shown in Fig. 256A the slots are closed; it will be noticed that the jet has not expanded sufficiently to fill the whole width of the diffuser passage. The same jet is shown in Fig. 256B, the slots are now open and suction is applied. It will be noticed that the boundary layer

* For a detailed account of boundary layer control, see *Modern Developments in Fluid Mechanics*, by Goldstein (Oxford University Press).

FIG. 256A

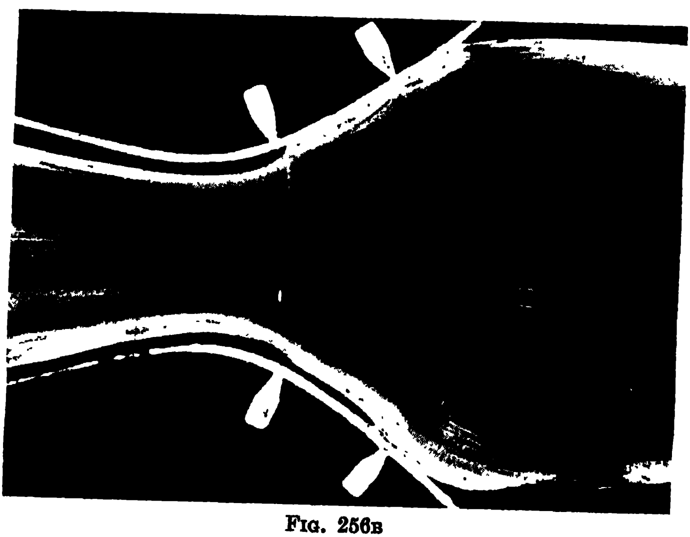


FIG. 256B

is now drawn to the sides of the diffuser and a full expansion of the jet is obtained.

Ackeret* found that a particular diffuser had an efficiency of 50 per cent when no boundary layer suction was applied. This was increased to 81 per cent by applying suction to one annular slot situated between the throat and outlet. A quantity of water equal to 5 per cent of the total flow was withdrawn through the slot by the suction. On the assumption that the efficiency of the suction pump was 75 per cent, the work done in removing this quantity of water was 3·4 per cent of the kinetic energy of the jet at the throat. Ackeret suggests that by the application of this method the efficiency of diffusers

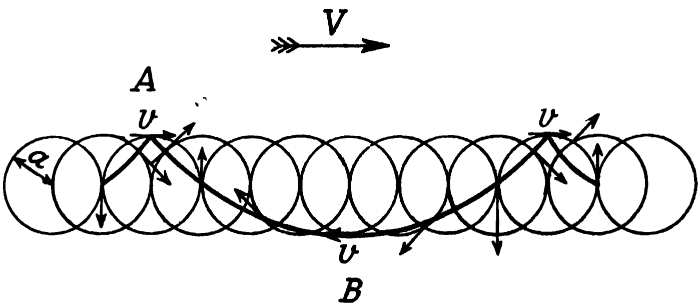


FIG. 257

can be increased, and its use may produce important improvements in hydraulic and ventilating engineering.

218. **Deep Water Surface Waves.** The transmission of a surface wave over deep water is brought about by a local circulation of the surface water, as shown in Fig. 257. The restoring forces acting on the wave are mainly due to gravity, but surface tension also tends to resist wave motion, especially in small waves. In most waves the effect of surface tension is small and may be neglected.

Consider the deep water wave shown in Fig. 257. Each particle of water on the surface describes circles as shown in the figure. When the particle is at the crest A it has a velocity v in the same direction as the wave. By the time the trough of the wave reaches this section, the velocity of the particle is now in the opposite direction to that of the wave. Thus, this

* See Ackeret, *Zeitschr. des Vereines deutscher Ingenieure*, 70 (1926), 1155-1157.

local circulation causes the water to flow upwards to form the crest of the wave and downwards to cause the trough. The particles of water below the surface also flow in circles, but the diameter of these circles decreases with the depth until, at a certain depth, the local motion ceases. As the total energy at the surface remains constant, the decrease of kinetic energy between the trough and the crest must equal the increase of potential energy between these sections.

Let V = velocity of wave

a = radius of circle described by particle

T = period of wave in secs.

λ = wavelength

v = velocity of particle

then, $$v = \frac{2\pi a}{T}$$

and $$T = \frac{\lambda}{V} \qquad (1)$$

Bring the wave to rest by giving the whole mass of water a velocity of V backwards. Then,

$$\text{Velocity of particle at crest } A = -V + v = -V + \frac{2\pi a}{T}$$

$$\text{Velocity of particle at trough } B = -V - v = -V - \frac{2\pi a}{T}$$

Consider unit mass of water at crest A and trough B;

$$\left.\begin{array}{c}\text{decrease of kinetic energy}\\ \text{between } B \text{ and } A\end{array}\right\} = \left\{\begin{array}{c}\text{decrease of potential energy}\\ \text{between } A \text{ and } B\end{array}\right.$$

or, $$\frac{1}{2g}\left[\left(-V - \frac{2\pi a}{T}\right)^2 - \left(-V + \frac{2\pi a}{T}\right)^2\right] = 1 \times 2a$$

from which $$V = \frac{gT}{2\pi}$$

Substituting for T from Equation (1),

$$V = \sqrt{\frac{g\lambda}{2\pi}} \qquad (2)$$

It will be noticed that this equation is independent of the density of the liquid.

219. **Shallow Water Surface Waves.** If the depth of the liquid is considerably less than the wavelength λ, the equation for the wave velocity given in Art. 218 will not apply. Under these circumstances the liquid may be regarded as shallow; the velocity of the wave is affected by the action of the bed in destroying the local circulation immediately above.

Consider the shallow water wave shown in Fig. 258; the wave is transmitted by means of a local circulation near the surface.

Let V = velocity of wave

h = mean depth of water

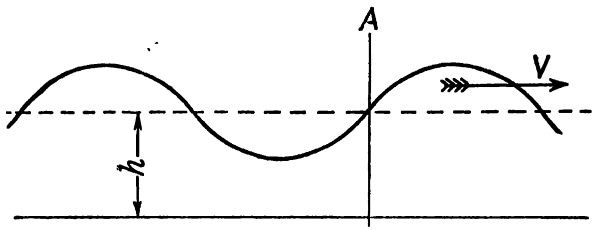


FIG. 258

Bring the wave to rest by giving the whole mass of water a velocity V in the opposite direction. Let Q be the flow through a section of unit width measured perpendicular to the direction of flow. Consider a vertical section A (Fig. 258) at which the depth is the mean depth h.

Then, $$V = \frac{Q}{h}$$

At any vertical section the total energy of the water is constant, because the increase of kinetic energy between the trough of the wave and its crest must equal the loss of potential energy between these sections.

At section A,

$$\text{Total energy of water} = E = h + \frac{V^2}{2g}$$

$$= h + \frac{Q^2}{2gh^2} \quad . \quad . \quad . \quad (1)$$

As this is the same at all vertical sections, it follows that $\frac{dE}{dh} = 0$; hence, differentiating Equation (1),

$$\frac{dE}{dh} = 1 - \frac{2Q^2}{2gh^3} = 0$$

that is,
$$1 - \frac{V^2}{gh} = 0, \text{ as } \frac{Q}{h} = V$$

hence,
$$V = \sqrt{gh} \qquad . \qquad . \qquad (2)$$

It will be noticed that this equation is proportional to the Froude number (Art. 89) and is the same as the critical velocity in a channel; hence, it follows, that waves are formed when water flows in a channel with this velocity and depth.

220. **Supersonic Velocities in a Fluid.** If the velocity of a body moving in a fluid is less than the velocity of a pressure wave in the fluid, the velocity is said to be *subsonic*, as the velocity of a pressure wave is the velocity of sound in the fluid. If the body's velocity is greater than that of a pressure wave, it is termed *supersonic*. When the velocity of the body increases from a subsonic to a supersonic velocity there is found to be a sudden increase in resistance, or drag.

Let v = velocity of body in fluid

v_s = velocity of sound in fluid

Referring to Fig 259, let the point A represent the position of the nose of a body moving through a fluid, from left to right, with a subsonic velocity. The nose of the body impinging on the fluid causes a disturbance in the form of a pressure wave which travels radially outwards with the velocity of sound. After a short interval of time t the nose of the body reaches B and has caused other pressure waves to radiate outwards as it travelled along the path AB. The pressure wave from A reached a radius of $v_s \times t$ when the body is at B. When the nose of the body had traversed one-quarter of the distance AB a pressure wave was radiated which reaches a radius of $\frac{3}{4}\, v_s t$ when the nose is at B.

In the same way, other pressure waves have reached radii of $\frac{1}{2}\, v_s t$ and $\frac{1}{4}\, v_s t$ respectively when the nose of the body has traversed one-half and three-quarters of the distance AB. The pressure waves are thus in the positions shown in the figure when the nose of the body reaches B. It will be seen from the figure that the nose of the body is always behind the

wave front of the pressure waves it has caused. The body is thus always penetrating an area of disturbed fluid; this has an effect on the fluid resistance to the body's motion.

Now consider the body when its velocity is supersonic; the wave disturbance it now causes in the fluid is shown in Fig. 260. Consider the instant when the nose of the body is at A;

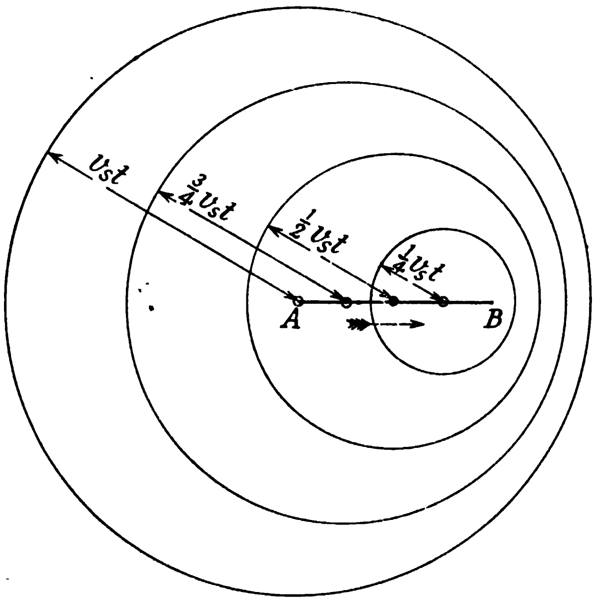


Fig. 259

its velocity v is now greater than the wave velocity v_s. Let the nose reach the point B in the time t. Then,

$$AB = vt$$

When the nose of the body was at A a pressure wave was caused which moved radially outwards from A with a velocity v_s; this wave has travelled a distance of $v_s t$ by the time the nose reached B. When the nose reached distances of $\frac{1}{4}$ AB, $\frac{1}{2}$ AB and $\frac{3}{4}$ AB from A, other pressure waves were caused which, at the time the nose reached B, had radiated to distances of $\frac{3}{4}$ $v_s t$, $\frac{1}{2}$ $v_s t$ and $\frac{1}{4}$ $v_s t$ respectively. Hence, these spherical wave fronts are in the position shown in the figure when the nose is at B. It will be noticed in this case that the nose of the body is always in advance of the pressure waves it has previously

caused; hence, it is always penetrating a region of undisturbed fluid. The resistance of the undisturbed fluid is found to be greater than that due to the body moving with a subsonic velocity.

Draw the straight line CB as a common tangent to all the pressure waves shown; let α be the inclination of this line to

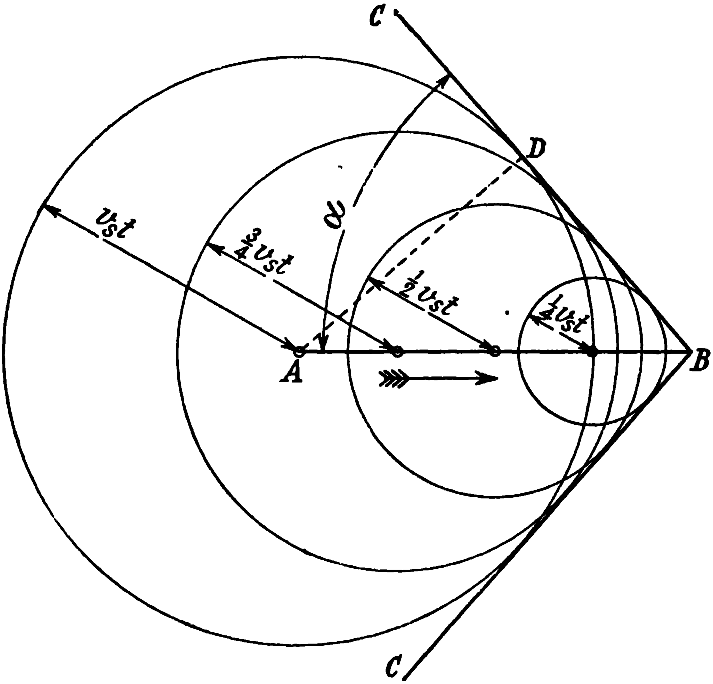


FIG. 260

the line AB. The line CB forms the common wave front advancing with the nose of the body into the undisturbed fluid and is known as a shock wave. Let D be the tangent point on CB of the pressure wave radiating from A; then,

$$\sin \alpha = \frac{AD}{AB}$$

$$= \frac{v_s t}{vt}$$

$$= \frac{v_s}{v}$$

The inverse of this ratio, $\frac{v}{v_s}$, is called the Mach number (Art. 198); it is an important criterion which is used when dealing with the resistance of bodies in a compressible fluid. The angle α is known as the Mach angle.

A photograph showing a shock wave caused by a bullet travelling with a supersonic velocity in air is shown in Fig.

FIG. 261

261. The shock wave can be clearly seen in front of the nose of the bullet. A line of eddies is visible in the wake of the bullet; these are due to disturbances caused by the air rushing into the space vacated by the bullet.

The photograph of the waves in the atmosphere is obtainable because the high pressure air of the wave front has an increased density; this, in turn, affects the refraction of light. It can thus be registered on a photographic plate by means of suitable illumination.

In Fig. 262 is shown the photograph* of the waves from a blade of a two-bladed aeroplane propeller when moving at such a speed that the blade tip velocity was supersonic, having a Mach number of 1·21. The direction of rotation is anti-clockwise. S is the supersonic shock wave from the leading edge and T is the trailing vortex wave. V_1 is the vortex from the blade tip, V_2 is the blade tip vortex from the other blade

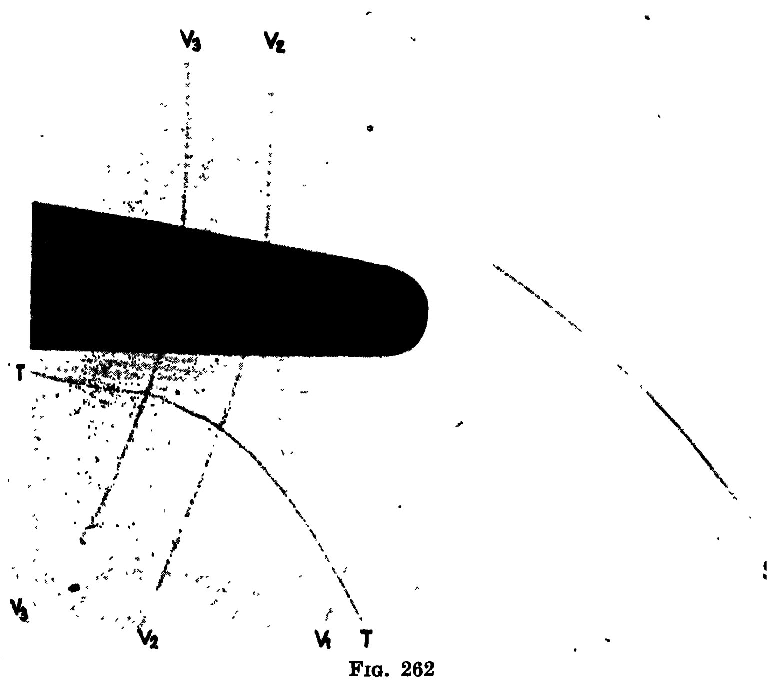


FIG. 262

occurring half of a revolution previously, and V_3 is the blade tip vortex of the blade shown, due to the previous revolution. The eddies in the wake of the blade can be clearly seen.

Oblique shock waves, inclined at the Mach angle, also occur in nozzles through which a jet of fluid is flowing with a supersonic velocity. These are shown in Fig. 263, which has been copied from an actual photograph.

The effect of the variation of resistance when a body changes its velocity from subsonic to supersonic is shown in the drag

* See paper by Dr. W. F. Hilton, *Proc. Royal Society*, Vol. 169, p. 174.

coefficient curve of Fig. 264. These results were obtained experimentally from an artillery shell of the shape shown in the figure; the values of the drag coefficient C_D are plotted on a base representing the Mach number. It will be noticed that

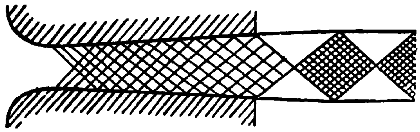


FIG. 263

when the Mach number is unity, that is, when the velocity reaches the velocity of sound, there is a sudden increase in the value of C_D. From this fact it is predicted that there would be a sudden falling off in the performance of an aeroplane if its velocity approaches the velocity of sound. From the results of aerofoil model tests in a wind tunnel, the lift coefficient

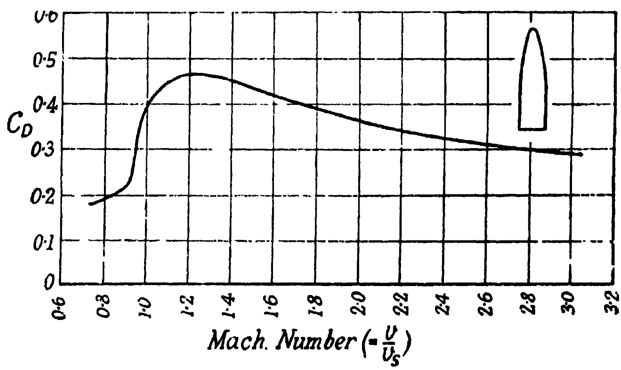


FIG. 264

commences to fall rapidly when the Mach number is 0·8, this is also accompanied by a rapid increase in the drag coefficient.

The rapid drop in the lift coefficient of an aerofoil as it approaches the region of the velocity of sound is shown in Fig. 264A. These results, which are due to Durand, show that there is a considerable drop in the lift of an aeroplane when its Mach number approaches 0·6 to 0·8. From the behaviour of the curves, the value of C_L appears to be approaching zero at the velocity of sound.

221. **Energy Stored in Water by Compression.** It was

explained in Art. 181 that the bulk modulus of a fluid is the ratio of the volumetric stress to the volumetric strain,

or,
$$K = \frac{P}{\frac{dV}{V}} \quad \ldots \quad (1)$$

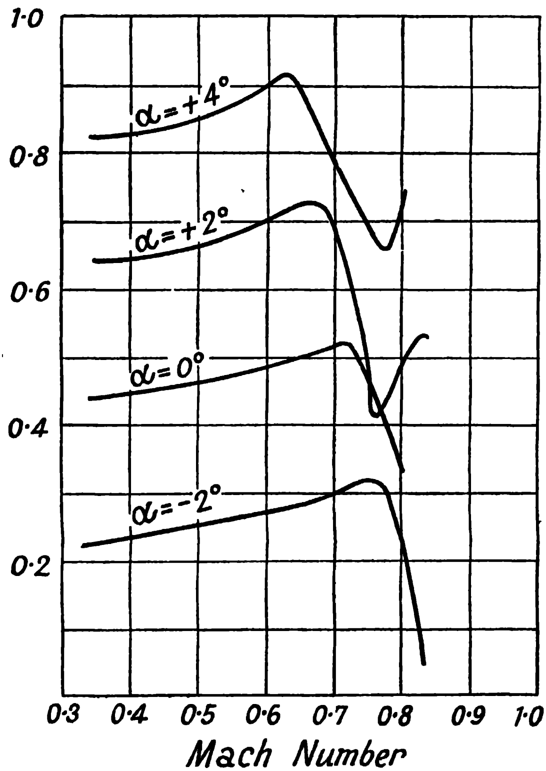


FIG. 264A

where P = pressure of fluid in lbs. per sq. ft.
V = original volume of fluid in cu. ft.
and dV = change in volume due to pressure P

When this equation is applied to a liquid the term $\frac{dV}{V}$ is known as the volumetric strain.

The applied pressure P causes the volume to decrease by dV during its application, thus doing work on the liquid. The liquid receives an amount of energy of the same magnitude

as the work done; this energy is stored in the liquid in the form of elastic energy, or strain energy. The pressure P is assumed to be applied gradually, so that whilst the water is being compressed the pressure commences at zero and gradually increases to P, the increase following a straight line law.

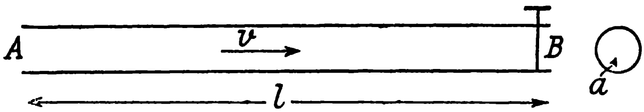


FIG. 265

Hence, the average pressure acting during the decrease of volume dV is $\frac{1}{2}P$.

$$\left.\begin{array}{c}\text{Strain energy stored}\\ \text{in liquid}\end{array}\right\} = \text{work done by } P$$

$$= \text{average pressure} \times \text{change in volume}$$

$$= \tfrac{1}{2} P \times dV$$

But, from Equation (1),

$$dV = \frac{PV}{K}$$

hence,

$$\text{Strain energy stored} = \frac{1}{2}\frac{P^2}{K} \times \text{Volume} \quad . \quad . \quad . \quad (2)$$

where K is in foot units.

222. **Hammerblow in Pipes.** As explained in Art. 77, hammerblow in pipes is due to the pressure wave caused by the sudden closing of a valve.

Consider the pipe AB (Fig. 265) of length l and cross-sectional area "a" having water flowing in the direction AB with a velocity v. If the valve at B is suddenly closed there will be a sudden rise of pressure at B due to the inertia of the water column in the pipe. Let P be the maximum pressure at B due to this cause. The sudden rise of pressure at B causes a pressure wave to travel backwards along the pipe from B to A with a velocity v_s, which is the velocity of sound in water (Art. 182). The time taken for the pressure wave to reach A is

$$\frac{l}{v_s} \text{ secs.}$$

When the pressure wave reaches A the water in the pipe commences to surge backwards from B to A, thus causing the pressure to fall. The wave front of this falling pressure travels

from A to B with a velocity v_s, the pressure falling to the normal pressure of the water when B is reached. Owing to the inertia of this backwards flow, the pressure at B now falls to $-P$; this causes a negative pressure wave to travel from B to A with the velocity v_s. When A is reached the water commences

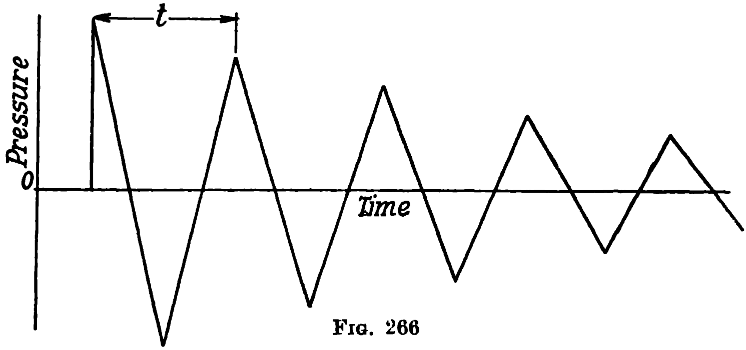


FIG. 266

to flow towards B, thus causing the pressure to rise. The wave front of this rising pressure then travels from A to B at a velocity v_s, the pressure in the pipe again reaching normal when the wave reaches B. Thus, a cycle is completed during which a pressure wave has travelled the length of the pipe four times.

Let t = time taken for complete operation of cycle

$$= \frac{4l}{v_s} \text{ secs.}$$

The cycle is then repeated, but owing to frictional resistances, the maximum positive and negative pressures are continually being reduced during each repetition of the cycle. The pressure waves are thus describing a damped vibration, as shown in Fig. 266, which continues until the whole of the energy of the wave has been absorbed.

The pressure wave causes shock and noise as it vibrates backwards and forwards along the pipe. In a long pipe excessive pressure can be produced which may damage the joints. In such cases, practical methods are adopted to absorb the energy of the wave, such as stand pipes and relief valves (Art. 121).

It will be shown in Art. 223 that if the valve is closed within the time taken for the pressure wave to traverse the length of the pipe, the effect is the same as if the valve were closed instantaneously.

The problem of water hammer is further complicated by the elasticity of the walls of the pipe; the increase of water pressure causes the pipe to expand radially, thus absorbing part of the kinetic energy of the water column. This effect is dealt with in Art. 224.

223. **Pressure due to Sudden Stoppage, Neglecting Pipe Expansion.** Consider the pipe of Fig. 265. Water is flowing with a velocity v and is suddenly brought to rest by the sudden closing of the valve at B. Assume the pipe to be perfectly rigid and non-elastic so that there is no radial expansion of its walls. Before the valve is closed the water has a kinetic energy due to its velocity; after the valve is closed this energy is converted to strain energy due to the compression of the water (Art. 221).

Consider a short length dl of the water column; then,

loss of kinetic energy = gain of strain energy

Substituting Equation (2) (Art. 221) for strain energy,

$$\frac{(wadl)\,v^2}{2g} = \frac{1}{2}\frac{P^2}{K}\,adl$$

from which

$$P^2 = \frac{wv^2K}{g}$$

or

$$P = v\sqrt{\frac{Kw}{g}} \quad . \quad . \quad (1)$$

Let ρ = density of water in absolute units

$$\frac{w}{g}$$

then,

$$P = v\sqrt{K\rho} \quad . \quad . \quad . \quad . \quad (2)$$

Equation (1) and (2) give the rise of pressure of the water in lb. per sq. ft. due to instantaneous closing of the valve.

It can be shown that, if the time taken to close the valve is less than that required for the pressure wave to travel the length of the pipe, the pressure produced is the same as if the valve were closed instantaneously. Assume the valve be closed in the period of time at such a rate that the water is brought to rest with a constant retardation f.

Consider the column of water in the pipe of Fig. 265.

Retarding force on end of pipe $\Big\}$ = Mass of water $\times f$

that is,

$$Pa = \frac{wal}{g} \times f$$

from which
$$P = \frac{w}{g} lf$$
$$= \rho lf \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let t = time taken by pressure wave to travel length of pipe
$$= \frac{l}{v_s}$$

hence, $l = tv_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$

but, from Equation (4), (Art. 182),
$$v_s = \sqrt{\frac{K}{\rho}}$$

also,
$$t = \frac{v}{f}$$

Substituting these values in Equation (4),
$$l = \frac{v}{f}\sqrt{\frac{K}{\rho}}$$

Substituting this value of l in Equation (3),
$$P = \rho \times \frac{v}{f}\sqrt{\frac{K}{\rho}} \times f$$
$$= v\sqrt{K\rho}$$

which is the same result as Equation (2). This proves that if the valve is closed within t secs. the effect on the pressure rise is the same as the rise obtained by an instantaneous closing.

224. **Effect of Pipe Elasticity on Hammerblow.** The rise in pressure of the water due to a sudden stoppage of flow in a pipe causes a radial expansion of the latter; the strain energy thus stored in the pipe absorbs some of the kinetic energy lost by the water.

The increase of radial pressure of the water on the pipe causes circumferential and longitudinal stresses in the pipe walls.* If the walls of the pipe are thin compared with its diameter, it is possible to obtain a simple solution to this problem.

Consider a short length dl of the pipe, shown in Fig. 267.

Let P = pressure of water due to hammerblow, lb. per sq. ft.
r = radius of pipe in ft.
t = thickness of pipe in ft.

* See textbooks on Strength of Materials for Stresses in Thin Cylinders.

a = cross-sectional area of pipe's bore in sq. ft.
$= \pi r^2$

f_c = circumferential stress in pipe, lb. per sq. ft.
$= \dfrac{Pr}{t}$

f_L = longitudinal stress in pipe, lb. per sq. ft.
$= \dfrac{Pr}{2t}$

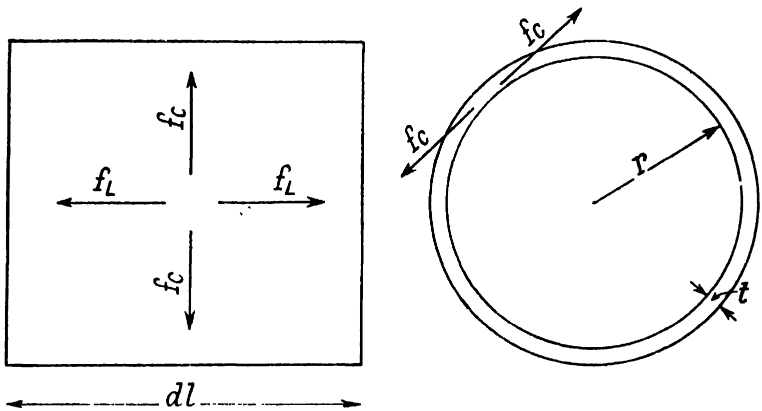


FIG. 267

E = linear elastic modulus of pipe walls, lb. per sq ft.
K = bulk modulus of water, lb. per sq. ft.

Then, volume of pipe walls $= 2\pi r t dl$ cu. ft

Working in foot units,

$$\left.\begin{matrix}\text{strain energy in}\\ \text{pipe walls}\end{matrix}\right\} = \left(\tfrac{1}{2}\frac{f_c^2}{E} \times \text{volume}\right) + \left(\tfrac{1}{2}\frac{f_L^2}{E} \times \text{volume}\right)$$

$$= \frac{1}{2E} \times 2\pi r t dl\,(f_c^2 + f_L^2)$$

$$= \frac{\pi r t dl}{E}\left(\frac{P^2r^2}{t^2} + \frac{P^2r^2}{4t^2}\right)$$

$$= \tfrac{5}{4}\,\frac{P^2\pi r^3 dl}{Et}$$

$$= \tfrac{5}{4}\,\frac{(adl)rP^2}{Et}\ \text{ft. units} \qquad (1)$$

From Equation (2) (Art. 221) the strain energy stored in water of the short length of pipe considered is—

$$\tfrac{1}{2}\frac{P^2}{K} \times (adl) \text{ ft. units} \qquad . \qquad . \qquad (2)$$

Then,

$$\left\{\begin{matrix}\text{loss of kinetic energy}\\ \text{of water}\end{matrix}\right\} = \left\{\begin{matrix}\text{strain energy}\\ \text{stored in water}\end{matrix}\right\} + \left\{\begin{matrix}\text{strain energy}\\ \text{stored in pipe}\end{matrix}\right\}$$

Substituting the values of Equations (1) and (2),

$$\frac{w(adl)v^2}{2g} = \tfrac{1}{2}\frac{P^2(adl)}{K} + \tfrac{5}{4}\frac{(adl)rP^2}{Et}$$

that is, $\dfrac{wv^2}{g} = P^2\left(\dfrac{1}{K} + \dfrac{5}{2}\dfrac{r}{tE}\right)$

from which

$$P = v\sqrt{\frac{w}{g\left(\dfrac{1}{K} + \dfrac{5}{2}\dfrac{r}{tE}\right)}} \text{ lb. per sq. ft.} \qquad . \qquad . \qquad (3)$$

Having obtained the value of P from Equation (3), the stresses f_c and f_L can then be calculated.

It will be noticed from Equation (3) that if the pipe is inelastic, E is infinity, then the equation becomes the same as Equation (1) of Art. 223.

EXAMPLE.

Obtain a formula for the rise in pressure in a thin elastic pipe of circular section in which a flow of water is stopped by the sudden closing of a valve Water flows in a long pipe which is 6 in. diameter and $\frac{1}{4}$ in. thick with a velocity of 4 ft. per sec., when it is suddenly brought to rest by the closing of a valve. Calculate the theoretical stress produced in the pipe near the valve taking K for water as 300,000 lb. per sq. in. and E for the pipe material as 30×10^6 lb. per sq. in. (London Univ.)

In this question,

$$r = \tfrac{1}{4} \text{ ft.}, \; t = \tfrac{1}{48} \text{ ft.}, \; v = 4 \text{ ft. per sec.}$$

$$K = 300{,}000 \times 144, \; E = 30{,}000{,}000 \times 144$$

First calculate P from Equation (3),

$$P = v\sqrt{\frac{w}{g\left(\dfrac{1}{K} + \dfrac{5}{2}\dfrac{r}{tE}\right)}} \text{ lb. per sq. ft.}$$

Substituting the above values in this equation,

$$P = 4\sqrt{\frac{62.4}{32.2} \Big/ \left(\frac{1}{300{,}000 \times 144} + \frac{5 \times \frac{1}{4} \times 48}{2 \times 30{,}000{,}000 \times 144}\right)}$$

$= 36{,}050$ lb. per sq. ft.

Then,

$$f_c = \frac{Pr}{t} \times \frac{1}{144} \text{ lb. per sq. in.}$$

$$= \frac{36{,}050 \times \frac{1}{4}}{\frac{1}{48} \times 144}$$

$= 3006$ lb. per sq. in.

225. **Electric Heater Airflow Meter.** A method of measuring the flow of air through a passage is by supplying the air

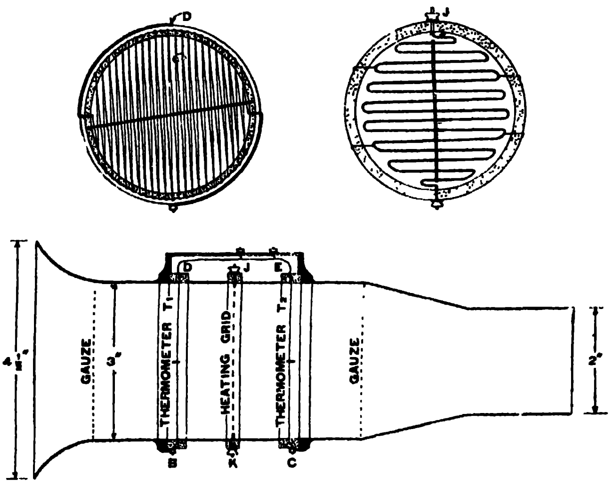


Fig. 268

with a known quantity of heat and measuring the rise in temperature.

A view of an electric heat airflow meter is shown in Fig. 268; this instrument was designed to measure the air supply to a petrol engine, and had a diameter of 3 in. at the heater.* The air to be measured is passed through an electric grid heater JK which is consuming a known amount of electric current; a view of the heater is shown in the top right-hand portion of

* For details, see "Air Consumption in I.-C. Engines," by Dr. H. Moss, *Proceedings Inst. Mech. Engineers*, Vol. 1, 1924, p. 345.

the figure. Equally spaced on each side of the heater are placed nickel wire resistance thermometers DB and EC which are made in the form of a grid. A view of a resistance thermometer is shown in the top left-hand portion of Fig. 268. The temperature of the air flowing through a resistance thermometer is measured by passing a known electric current through the nickel wire grid and measuring its resistance by the Wheatstone bridge method. As the resistance is a function of the temperature of the wire, the instrument can be calibrated to give the temperature of the air flowing through it.

The air is passed through the meter from right to left. It first flows through the resistance thermometer EC and its temperature is measured. It next passes through the electric heater JK and has its temperature increased. This increased temperature is measured as it flows through the resistance thermometer DB.

Let W = weight of air flowing per hour.

K_p = specific heat of air at constant pressure.

t = rise in temperature due to heater, degrees F.

C = no. of kilowatts consumed by heater per hour.

1 kilowatt = 3412 B.Th.U.

Then, heat gained by air = heat supplied to heater

that is, $W \times K_p \times t = 3412 \times C$

from which $$W = \frac{3412 \times C}{K_p \times t} \text{ lb. per hour}$$

The heater method was first used by Callendar for measuring the specific heat of gases. It was later developed by Professor Thomas as an airflow meter for measuring the air supply to the furnaces of steam boilers; this meter had a diameter of 5 ft.

The electric heater airflow meter does not cause any backwards and forwards surging of the air, which often occurs in the orifice method of measurement when applied to I.-C. engines.

EXAMPLES 18.

(1) A capillary tube of $\frac{1}{8}$ in. bore is placed vertically in water. Calculate the elevation of the water in the tube due to capillarity. The surface tension is $\cdot 42 \times 10^{-3}$ lb. per inch.

Ans.—·372 in.

(2) Water is rotating in the form of a free cylindrical vortex. At a radius of 1 in. its tangential velocity is 5 ft. per sec., and its pressure 20 lb. per sq. in. Calculate the pressure at a point in the vortex at a radius of 2·5 in. and in the same horizontal plane. If this vortex has a free surface, find the difference in height of the water surface above these two points.

Ans.—20·141 lb per sq. in.; 3·92 in.

(3) The flow through a 2 in. diameter pipe is measured by means of a pipe orifice of $\frac{1}{2}$ in. diameter. The drop in pressure of the water in passing through the orifice was found to be 12·5 in. of water, and the coefficient for the orifice was ·63. Calculate the flow in cu. ft. per min.

Ans.—·422 cu. ft. per min.

(4) Water having a kinematic viscosity of $5{\cdot}05 \times 10^{-6}$ ft. sec. units, flows through a pipe orifice of 2 in. diameter inserted in a 4 in. diameter pipe. The difference of pressure at the orifice was found to be 6·5 in. of water.

Using the curves of Fig. 246 obtain the value of C_d for the orifice under this condition of flow and, using this value of C_d, calculate the quantity of flow.

Ans.—·63; 5·02 cu. ft. per min.

(5) A cylindrical vessel contains water and rotates about its vertical axis. Show that the surface of the forced vortex thus formed is a paraboloid.

If the cylinder is 3 in. in diameter and 4 in. deep and is exactly half full of water, find the speed of rotation at which water will just begin to spill over the top edge. (London Univ.)

Ans.—354 r.p.m.

(6) Describe the pressure and velocity distributions near a cylinder placed in a stream of moving fluid.

How is the result modified if the cylinder is rotated about its axis? (London Univ.)

(7) Show that the resistance R experienced by a sphere of diameter d moving with a low velocity v through a fluid of density ρ and viscosity z can be expressed by

$$R = kzdv$$

It is found that a steel ball having a diameter of 0·2 cm. attains a "terminal velocity" of 75 cm. per sec. when falling vertically through a fluid whose viscosity is ·8 poise and density is ·9 gramme per cu. cm. The density of the steel ball is 7·8 gramme per cu. cm. Determine the value for the coefficient k. (London Univ.)

Ans.—$k = 2{\cdot}36$.

(8) Derive an expression for the flow of a liquid through a tube in which the resistance is purely viscous, and apply this expression to prove that in the Redwood viscometer the time in discharging a given quantity of a liquid varies directly as the viscosity and inversely as the density of the liquid. (London Univ.)

(9) Show, by the use of the method of dimensions, that the resistance to the motion of a sphere of radius r falling through a viscous fluid at a slow velocity v is given by $R = k\mu rv$, in which μ is the viscosity of the fluid and k is a dimensionless constant.

A steel ball, of diameter ·0625 in., submerged in a liquid of specific gravity ·91 and viscosity 1·62 poises, is allowed to fall from rest. Find the steady speed attained by the ball given that $k = 6\pi$ and that the specific gravity of steel is 7·85. (London Univ.)

Ans.—·1912 ft. per sec.

(10) Prove that the velocity of a wave of low elevation travelling over the surface of shallow water is given by $C = \sqrt{gh}$, in which h is the depth. Hence, explain the significance of the term "critical depth" as it is used in connection with the flow of water in a channel. (London Univ.)

(11) A fine granular material of specific gravity 2·5 is in uniform suspension in still water of depth 10 ft. Regarding the particles as spheres of diameter ·002 cm., find how long it will take for the water to clear. The resistance of a sphere in a liquid is given by the equation: $R = k\mu rv$, when the velocity v is small. Take $k = 6\pi$ and $\mu = \cdot 013$ poise. (London Univ.)

Ans.—3·38 hours.

(12) Describe the orifice method for metering flow in a pipe. Describe how the coefficient varies with the Reynolds number, and explain why air and water may require different coefficients even at the same Reynolds number. (I. Mech. E.)

(13) Define "coefficient of viscosity" and "kinematic viscosity" of a fluid. Describe any experimental means of determining the viscosity of a fluid. (I. Mech. E.)

(14) Deduce an expression for the velocity at a given radius in a free vortex A point in the free surface at 6 in. radius is 3 in. below the free surface at the boundary of the vessel whose radius is large. What will be the surface level at a radius of 12 in.? (I. Mech. E.)

Ans.—·75 in.

(15) Distinguish between a free and forced vortex. Derive formulæ for the relation between radius, velocity and pressure for both types, assuming a perfect fluid.

A U-tube has a horizontal part 2 ft. long with vertical end limbs. If the whole tube is rotated about a vertical axis 18 in. from one end and 6 in. from the other, calculate the speed when the difference of level in the tubes is 10 in. (London Univ.)

Ans.—49·7 r.p.m.

(16) Deduce the atmosphere pressure at 10,000 ft. when it is 14·7 lb. per sq. in. at sea level, weighs ·075 lb. per cu. ft. at sea level, and the temperature is constant. Find the pressure at 2 ft. radius in a free vortex, at an altitude of 10,000 ft., when the motion is adiabatic and the speed 30 ft. per sec. at 40 ft. radius. (London Univ.)

Ans.—10·32 lb. per sq. in; 2·24 lb. per sq. in.

(17) If the water supply to a Pelton wheel is shut off by means of the spear valve in such a way that the retardation of the water in the pipe is at a uniform rate, show that, if the time of closure of the valve is the same as the time taken for sound to travel the length of the pipe in water, the rise of pressure at the valve is $p = v\sqrt{k\rho}$, in which v is the initial velocity, k the modulus of compressibility, and ρ the density of the water.

If the valve is closed instantaneously, prove that the rise of pressure in the pipe is given by the same formula.

Assume that the velocity of sound in water is $c = \sqrt{k/\rho}$, and the pipe is perfectly rigid. (London Univ.)

(18) Deduce an expression for the rise in pressure in a pipe due to the *sudden* closing of a valve when water is flowing through the pipe, allowing both for the compressibility of the water and the expansion of the pipe.

A steel pipe is 3 in. diameter, and $\frac{1}{4}$ in. thick. What is the highest velocity of the water which can be suddenly stopped without stressing the pipe to more than 10,000 lb. per sq. in.? E for steel $= 30 \times 10^6$ lb. per sq. in. Bulk modulus for water $= 300{,}000$ lb. per sq. in. (London Univ.)

Ans.—13·07 ft. per sec.

APPENDIX

Values of coefficients for pipe flow, open channels, orifices, and weirs compiled from various published results of tests. The values apply to water only.

I. VALUES OF COEFFICIENT f FOR PIPE FLOW FORMULA

$$h_f = \frac{4flv^2}{2gd}$$

(Intermediate values of diameters and velocities may be obtained by interpolation.)

Smooth Pipes

Temperature 60° F. (Colebrook and White).* Glass, drawn lead and brass, bitumen-lined steel or concrete pipes over 4 ft. diameter.

	Velocity in Feet per Second								
Diameter in Inches	1·0	2·0	3·0	4·0	5·0	6·0	8·0	10·0	20·0
1	·0083	·0072	·0067	·0061	·0058	·0055	·0052	·0049	·0043
3	·0066	·0055	·0050	·0047	·0045	·0043	·0042	·0038	·0034
6	·0055	·0047	·0043	·0042	·0038	·0038	·0035	·0034	·0030
12	·0047	·0042	·0038	·0035	·0034	·0033	·0032	·0030	·0027
18	·0044	·0037	·0035	·0033	·0032	·0030	·0029	·0028	·0025
24	·0042	·0035	·0033	·0032	·0030	·0029	·0028	·0027	·0024
36	·0037	·0033	·0031	·0029	·0028	·0027	·0026	·0025	·0023
48	·0035	·0032	·0029	·0028	·0027	·0026	·0025	·0024	—
60	·0034	·0030	·0028	·0027	·0026	·0025	·0024	·0023	—

Clean Asphalted Pipes

	Velocity in Feet per Second					
Diameter in Inches	1·0	2·0	3·0	4·0	6.0	8·0
6	·0102	·0094	·0090	·0086	·0081	·0079
9	·0080	·0074	·0070	·0067	·0064	·0060
12	·0067	·0062	·0059	·0056	·0053	·0050
18	·0056	·0051	·0048	·0046	·0043	·0042
24	·0050	·0046	·0044	·0042	·0039	·0038
30	·0047	·0043	·0041	·0039	·0037	·0036
36	·0044	·0040	·0038	·0036	·0035	·0034
48	·0039	·0037	·0035	·0033	·0031	·0029

* "The Reduction of Carrying Capacity of Pipes with Age," by C. F. Colebrook and C. M. White (*Journal of Institution of Civil Engineers*, 1937–1938, page 99); and "Experiments with Fluid Friction in Roughened Pipes," by Colebrook and White. (*Proc. Roy. Soc.*, Vol. 161.)

New Cast Iron Pipes and Similar Surfaces

Diameter in Inches	Velocity in Feet per Second						
	1·0	2·0	3·0	4·0	6·0	10·0	15·0
1	—	—	·0075	·0074	·0072	·0070	·0068
2	·0071	·0069	·0068	·0067	·0065	·0064	·0062
3	·0067	·0065	·0064	·0063	·0061	·0060	·0059
6	·0061	·0059	·0058	·0057	·0055	·0054	·0053
9	·0057	·0055	·0054	·0053	·0052	·0050	·0049
12	·0054	·0052	·0051	·0050	·0049	·0048	·0047
18	·0052	·0051	·0050	·0048	·0045	·0041	·0040
24	·0050	·0048	·0045	·0043	·0040	·0037	·0036
36	·0042	·0040	·0037	·0036	·0035	·0033	·0032
48	·0036	·0034	·0032	·0032	·0031	·0029	·0028
60	·0032	·0031	·0030	·0030	·0029	·0028	·0027

Old Cast Iron Pipes

Diameter in Inches	Velocity in Feet per Second						
	1·0	2·0	3·0	4·0	6·0	10·0	15·0
3	·0152	·0145	·0139	·0135	·0128	·0122	·0120
6	·0135	·0126	·0117	·0114	·0108	·0103	·0100
9	·0122	·0119	·0115	·0110	·0100	·0092	·0090
12	·0108	·0102	·0096	·0094	·0089	·0084	·0080
18	·0087	·0083	·0078	·0076	·0073	·0069	·0067
24	·0076	·0072	·0067	·0065	·0063	·0060	·0059
36	·0061	·0059	·0056	·0054	·0052	·0050	·0049
48	·0057	·0054	·0051	·0050	·0049	·0046	·0045

II. VALUES OF CONSTANT FOR FLOW IN OPEN CHANNELS

For flow in open channels

$$v = C\sqrt{mi}$$

where . $$C = \frac{157 \cdot 5}{+ \frac{k}{\sqrt{m}}}$$

Values of k for channels.

Type of Lining	k
Concrete-lined Channels—	
Smooth finished surface	0 to ·1
Surface due to planed shuttering	·1 to ·45
Rough concrete	1·7 to 2·7
Masonry and Stoneware Channels—	
Vitrified sewer pipe	0 to ·1
Glazed brickwork	·1 to ·4
Brickwork in cement mortar	·1 to ·55
Rubble masonry	·55 to 1·9
Dry rubble	1·9 to 2·9
Timber-lined Channels—	
Smooth planed boards	0 to ·4
Rough boards	·1 to ·45
Old timber	·45 to ·6
Steel-plate Channels—	
With countersunk rivets and flush joints	0 to ·1
Projecting joints and rivets	·4 to ·9
Artificial Channels—	
Smooth rock surface	1·9 to 2·9
Rough rock surface	3·0 to 3·7
Gravel	1·8 to 2·0
Sand and gravel	1·7 to 1·9
Smooth earth surface	·9 to 1·8

III. COEFFICIENT OF DISCHARGE FOR SHARP-EDGED CIRCULAR ORIFICES

Head in Feet	Diameter of Orifice (inches)										
	·24	·48	·75	1·0	1·5	2·0	2·5	4·8	7·2	9·6	12·0
·2	—	—	·684	·645	·617	·611	·609	—	—	—	—
·4	—	—	·675	·640	·615	·609	·607	—	—	—	—
·8	·649	·626	·666	·636	·613	·607	·606	—	—	—	—
1·0	·644	·623	·662	·635	·612	·607	·606	·597	·595	·593	·591
1·2	·642	·621	·659	·634	·612	·607	·606	·598	·596	·594	·593
1·6	·638	·618	·654	·632	·612	·607	·606	·598	·596	·595	·594
2·0	·635	·616	·651	·630	·611	·607	·606	·599	·597	·596	·595
4·0	·624	·611	·641	·627	·611	·607	·606	·598	·597	·597	·596
6·0	·618	·607	·636	·626	·611	·607	·606	·598	·597	·596	·596
8·0	·613	·605	·635	·626	·611	·607	·606	·597	·596	·596	·595
10·0	·609	·604	·635	·625	·611	·607	·606	·597	·596	·596	·595
20·0	·601	·599	·635	·625	·611	607	·606	·596	·596	·595	·594
100·0	·593	·592	·634	·624	·611	·607	·606	·592	·592	·592	·592

IV. COEFFICIENT OF DISCHARGE FOR SQUARE ORIFICES

Head in Feet	Size of Square Orifice in Inches				
	·24 × ·24	·48 × ·48	·84 × ·84	1·44 × 1·44	2·4 × 2·4
·4	—	·643	·628	·616	—
·6	·660	·636	·623	·613	·605
1·0	·648	·628	·618	·610	·605
6·0	·623	·612	·607	·605	·604
20·0	·606	·604	·602	·602	·602
100·0	·599	·598	·598	·598	·598

V. COEFFICIENT OF DISCHARGE FOR RECTANGULAR WEIRS

Values of constant $m\sqrt{2g}$ of Bazin's formula

$$Q = m\sqrt{2g}LH^{\frac{3}{2}}$$

H in Feet	Height of Crest of Weir above Bed of Channel in Feet						
	1·5	2·0	3·0	4·0	6·0	8·0	10·0
·4	3·53	3·50	3·47	3·46	3·45	3·45	3·45
·6	3·53	3·48	3·44	3·41	3·40	3·39	3·39
·8	3·57	3·50	3·30	3·40	3·37	3·36	3·36
1·0	3·61	3·53	3·44	3·40	3·37	3·35	3·34
1·2	3·67	3·57	3·47	3·41	3·37	3·35	3·34
1·4	—	3·62	3·49	3·43	3·37	3·35	3·34
1·6	—	3·66	3·52	3·45	3·38	3·35	3·34
1·8	—	—	3·55	3·47	3·39	3·36	3·34
2·0	—	—	3·58	3·49	3·40	3·36	3·34

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$$T = -\frac{2A}{k^2}\left[Q \log_e \left(\frac{Q - k\sqrt{H_2}}{Q - k\sqrt{H_1}}\right) + k(\sqrt{H_2} - \sqrt{H_1})\right] \quad \ldots \quad 89$$

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Forced vortex, $E_2 - E_1 = \frac{\omega^2}{g}(r_2^2 - r_1^2)$ 523

$\frac{p_2 - p_2}{w} = \frac{v_2^2 - v_1^2}{2g}$ 524

Radial flow, $\frac{p_2 - p_1}{w} = \frac{Q^2}{8\pi^2 t^2 g}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$ 525

Measurement of viscosity,

$\mu = \frac{\rho g d^2 h_f}{32 l v}$ (pipe flow) 528

$\mu = \frac{2r^2 g(\rho_1 - \rho_2)}{9V}$ (falling sphere) 531

Pipe orifice, $Q = ka_2\sqrt{\frac{2gh}{\left[1 - \left(\frac{d}{D}\right)^4\right]}}$ 534

Prandtl and Von Kàrmàn equations for pipe flow,

For smooth surface, laminar flow,

$\frac{1}{\sqrt{4f}} - 2\log\frac{r}{k} = \cdot 8 + 2\log\frac{ku}{\nu}$ 541

For wholly rough surface, turbulent flow,

$\frac{1}{\sqrt{4f}} - 2\log\frac{r}{k} = 1{\cdot}74$ 541

Prandtl's equation for turbulent flow through smooth pipes,

$\frac{1}{\sqrt{4f}} = 2\log(R\sqrt{4f}) - \cdot 8$ 541

Deep water surface waves, $V = \sqrt{\frac{g\lambda}{2\pi}}$ 554

Shallow water surface waves, $V = \sqrt{gh}$ 556

Supersonic velocities, Mach number $= \frac{v}{v_s}$ 559

Strain energy of liquid $= \frac{1}{2}\frac{P^2}{K} \times$ volume 562

Pressure due to sudden stoppage, $P = v\sqrt{\frac{Kw}{g}}$ 564

when elasticity of pipe is considered, $P = v\sqrt{\frac{w}{g\left(\frac{1}{K} + \frac{5}{4}\frac{r}{tE}\right)}}$. 567

Other Useful Formulae—

1st moment $= A\bar{x} = \int ax$

2nd moment $= I = Ak^2 = \int ax^2$

For moment of inertia about any axis oo at a distance h from centre of area—

$$I_O = I_G + Ah^2$$

Moment of inertia of rectangle about base	$= \dfrac{bd^3}{3}$
Moment of inertia of rectangle about centre line	$= \dfrac{bd^3}{12}$
Moment of inertia of circle about diameter	$= \dfrac{\pi d^4}{64}$
Moment of inertia of triangle about base	$= \dfrac{BH^3}{12}$
Moment of inertia of triangle about an axis through centre of area	$= \dfrac{BH^3}{36}$
Distance of centre of area of semi-circle from diameter	$= \dfrac{4r}{3\pi}$
Distance of centre of area of semi-circular arc from diameter	$= \dfrac{2r}{\pi}$
Distance of centre of area of surface of a hemisphere from diameter	$= \dfrac{r}{2}$
Volume of sphere	$= \dfrac{4}{3}\pi r^3$

Volume of paraboloid $= \frac{1}{2} \times$ volume of circumscribing cylinder

Surface area of sphere $= 4\pi r^2$

Useful Constants—

Weight of 1 cu. ft. of fresh water	= 62·4 lb.
Weight of 1 gallon of fresh water	= 10 lb.
Weight of 1 cu. ft. of sea water	= 64·0 lb.
Bulk elastic modulus of water	= 300,000 lb. per sq. in.
Pressure of atmosphere	= 34 ft. of water
	= 14·7 lb. per sq. in.
	= 29·9 ins. of mercury
1 lb.	= 453·6 grammes
1 inch	= 2·54 cms.
g	= 32·2 ft. per sec^2
1 radian	= 57·3°

USEFUL TABLES AND DATA

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934	0969					3	7	11	14	18	21	25	28	32
							1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553							3	6	9	12	15	18	21	24	28
					1584	1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875	1903					3	6	9	11	14	17	20	23	26
							1931	1959	1987	2014	3	5	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405	2430					3	5	8	10	13	15	18	20	23
							2455	2480	2504	2529	2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
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·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
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·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
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·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	11
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

TRIGONOMETRICAL FUNCTIONS

Angle. Degrees.	Sine.	Tangent.	Co-tangent.	Cosine.	
0°	0	0	∞	1	90°
1	·0175	·0175	57·2900	·9998	89
2	·0349	·0349	28·6363	·9994	88
3	·0523	·0524	19·0811	·9986	87
4	·0698	·0699	14·3007	·9976	86
5	·0872	·0875	11·4301	·9962	85
6	·1045	·1051	9·5144	·9945	84
7	·1219	·1228	8·1443	·9925	83
8	·1392	.1405	7·1154	·9903	82
9	·1564	·1584	6·3138	·9877	81
10	·1736	·1763	5·6713	·9848	80
11	·1908	·1944	5·1446	·9816	79
12	·2079	·2126	4·7046	·9781	78
13	·2250	·2309	4·3315	·9744	77
14	·2419	·2493	4·0108	·9703	76
15	·2588	·2679	3·7321	·9659	75
16	·2756	·2867	3·4874	·9613	74
17	·2924	·3057	3·2709	·9563	73
18	·3090	·3249	3·0777	·9511	72
19	·3256	·3443	2·9042	·9455	71
20	·3420	·3640	2·7475	·9397	70
21	·3584	·3839	2·6051	·9336	69
22	·3746	·4040	2·4751	·9272	68
23	·3907	·4245	2·3559	·9205	67
24	·4067	·4452	2·2460	·9135	66
25	·4226	·4663	2·1445	·9063	65
26	·4384	·4877	2·0503	·8988	64
27	·4540	·5095	1·9626	·8910	63
28	·4695	·5317	1·8807	·8829	62
29	·4848	·5543	1·8040	·8746	61
30	·5000	·5774	1·7321	·8660	60
31	·5150	·6009	1·6643	·8572	59
32	·5299	·6249	1·6003	·8480	58
33	·5446	·6494	1·5399	·8387	57
34	·5592	·6745	1·4826	·8290	56
35	·5736	·7002	1·4281	·8192	55
36	·5878	·7265	1·3764	·8090	54
37	·6018	·7536	1·3270	·7986	53
38	·6157	·7813	1·2799	·7880	52
39	·6293	·8098	1·2349	·7771	51
40	·6428	·8391	1·1918	·7660	50
41	·6561	·8693	1·1504	·7547	49
42	·6691	·9004	1·1106	·7431	48
43	·6820	·9325	1·0724	·7314	47
44	·6947	·9657	1·0355	·7193	46
45°	·7071	1·0000	1·0000	·7071	45°
	Cosine.	Co-tangent.	Tangent.	Sine.	Degrees. Angle.

METRIC SYSTEM

1 metre	=	39·370113 in.
1 metre	=	3·28083 ft.
1 sq. metre	=	10·7639 sq. ft.
1 sq. cm.	=	0·15500 sq. in.
1 cub. metre	=	35·3145 cub. ft.
1 cub. cm.	=	0·061023 cub. in.
1 inch	=	2·54001 cm.
1 foot	=	0·3048 m.
1 sq. ft.	=	0·092903 sq. metre
1 sq. in.	=	6·4516 sq. cm.
1 cub. ft.	=	0·028317 cub. metre
1 cub. in.	=	16·387 cub. cm.
1 kg.	=	2·204622 lb.
1 lb.	=	0·453592 kg.
g. (standard)	=	32·1740 ft. per sec.^{2}
g. (standard)	=	980·665 cm. per sec.^{2}
1 kg. per cm.^{2}	=	14·223 lb. per in.^{2}
1 kg. per cm.^{2}	=	28·958 in. Hg.
1 kg. per cm.^{2}	=	735·54 mm. Hg.
1 lb. per sq. in.	=	0·070307 kg. per cm.^{2}
1 lb. per sq. in.	=	2·0360 in. Hg.
Standard atmosphere	=	760 mm. Hg.
Standard atmosphere	=	29·921 in. of Hg.
Standard atmosphere	=	14·696 lb. per sq. in.
Standard atmosphere	=	1·0333 kg. per sq. cm

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